GEOMETRY AS AN ASPECT OF DYNAMICS

by

A. L. L. VIDEIRA

Physics Department, Pontifícia Universidade Católica,

Rio de Janeiro, Brazil

A. L. ROCHA BARROS

and

N. C. FERNANDES

Institute of Physics, University of São Paulo,

Brazil

1. INTRODUCTION

In the usual manner of building dynamical models of Nature one always begins by postulating a certain space-time and from there proceeds to develop a certain physics. That is, one always starts from a given, preestablished geometry, upon which a consequential dynamics is established, and it is well known

that the choice of the geometry (of the postulated space-time) uniquely determines the physics that can be constructed in that postulated space-time. Thus, just as the only dynamics compatible with the absolute space-time of Newton is precisely Newtonian dynamics, correspondingly, in Minkowski space-time only the dynamics of Special Relativity can be naturally built. Schönberg [1] observed that while the contravariant vectors are the ones which are more intimately related with geometry, the covariant vectors are the ones which are more closely connected with physics. As an immediate example, we recall that the position vector x is essentially contravariant, whereas the momentum p is essentially covariant. This subject will be unfolded in the following sections. Before turning over to the point of view to be developped here, let us present the Special Relativistic and Newtonian cases, as they are usually stated.

The four-dimensional space-time manifold of Minkowski consists of a 3-dimensional spatial hypercone with time pointing along its symmetry axis. The geometry of this manifold has as its invariance group the full Lorentz group (or group of Poincaré) :

$$x'^{\mu} = L^{\mu}_{\ \nu}\, x^{\nu} + a^{\mu} \qquad (1.1)$$

with greek indices running from 1 to 4. Here, ($L^{\mu}_{\ \nu}$) is a (4 x 4) orthogonal matrix and a^{μ} is an arbitrary (constant) 4-vector.

In the Newtonian case, the 4-dimensional space-time manifold was first introduced by E. Cartan [2] and is taken to consist of a 3-dimensional space-like hypersurface, orthogonal to the absolute time axis. This geometry fixes the group of symmetry (invariance or relativistic group)

$$x'^{\alpha} = G^{\alpha}_{\ \beta}\, x^{\beta} + k^{\alpha} \qquad (1.2)$$

Here, greek indices run from 1 to 4 and the matrix ($G^{\alpha}_{\ \beta}$) has the (3+1) x (3+1) block form :

$$\left(G^{\alpha}_{\ \beta} \right) = \begin{pmatrix} G & w \\ 0 & 1 \end{pmatrix} \qquad (1.3)$$

where G is a (3 x 3) orthogonal matrix and the (3 x 1) column vector w is arbitrary. This geometry (and its related symmetry group) determines both the absolute kynematical and dynamical entities, that is, those entities which are left invariant by the transformations (1.2).

Let us take Newtonian 4-dimensional space-time as an affine manifold, E_4.

The matrix ($G^{\alpha}_{\ \beta}$) can be diagonalized and put in the form

$$\begin{pmatrix} G\,G^{T} & w \\ 0 & 0 \end{pmatrix}.$$

From this, it is seen that the metric (or fundamental) tensor ${}^{(N)}g_{\alpha\beta} = {}^{(N)}\eta_{\alpha\beta} = \left(\zeta_{\alpha\beta} \right)$ of the affine Newtonian space-time E_4 is <u>singular</u>. This fact immediately distinguishes Newtonian space-time from its special-relativistic counterpart. In fact, while in this latter case one can introduce dual metric tensors ${}^{(o)}g_{\alpha\beta}$ and ${}^{(o)}g^{\alpha\beta}$, one being the inverse of the other, this cannot be done in E_4, since there the inverse $\eta^{\alpha\beta}$ does not exist. Therefore, it is precisely in E_4 where the distinction between covariant and contravariant 4-vectors will be expected to be more fundamental than in the special relativistic case, where there exists a complete transposition between contravariant and covariant quantities. This, of course, should not be taken as meaning that in the 3-dimensional space-like hypersurface E_3 of E_4 this raising or lowering of indices is not fully justified , since that submanifold E_3 is Euclidean. This last fact leads to the consideration made a long time ago by E. Cartan[3] , that E_4 is <u>not</u> an Euclidean manifold, but its affine connection, ${}^{(N)}V_4$ is Euclidean, which is just another way of seeing that the metric tensor of E_4 is singular.

In the present work we intend to fix the geometry of the space-time by introducing a certain minimal number of fundamental dynamical entities. This view is contrary to the usual one and this epistemological difference will be scrutinized in detail.

[redacted].

2. CONTRAVARIANT AND COVARIANT VECTORS

When examining the interconnection between physics and geometry is of paramount importance to establish the essential distinction that exists between contravariant and covariant entities. In this section, we discuss some aspects which manifest this distinction.

Given the vector affine space E_n, the linear mapping $\omega : E_n \longrightarrow R$ of E_n over R defines a linear form over E_n. The vectors of E_n are the <u>contravariant</u> vectors $\boldsymbol{x}$ which, in a given basis $\{ e_i \}$ are written as

$$\boldsymbol{x} = x^i \; e_i \tag{2.1}$$

The linear forms over E_n belong to another vector affine space E_n^*, dual of E_n. The vectors $x^* \in E_n^*$ are the <u>covariant</u> vectors $\omega (\boldsymbol{x})$:

$$\boldsymbol{x}^* = \omega (\boldsymbol{x}) = x^i \; \omega (e_i) = x^i \; a_i \tag{2.2}$$

$$\boldsymbol{x}^* = \omega (\boldsymbol{x}) = \omega (e_i) \; x^i = a_i \; x^i \tag{2.2}$$

where we can consider the $a_i \equiv \omega(e_i)$ as the components of the covariant vector ω in the dual basis $\{x^i\} = \{e^i\}$, i.e., we may write a covariant vector $x^* \in E_n^*$ as

$$x^* = x_i e^i, \text{ where } x_i \equiv a_i \qquad (2.3)$$

The geometrical meaning of the contravariant and covariant vectors is obtained through the introduction of an affine space $(0, E_n) \equiv \mathcal{E}_n$, which is a space of points having a structure of a vector space depending of the point 0, taken as the origin[4]. It should be noticed that neither a metric was defined in E_n, nor a distance in $\mathcal{E}_n$.

The contravariant vector $x = x^i e_i \in E_n$ is represented geometrically by an oriented line, whereas the covariant vector $x^* = x_i e^i \in E_n^*$ is represented by two parallel hyperplanes, since we have a family $x^* = x_i e^i = \omega(x) = a_i x^i = k$ of parallel hyperplanes, depending on the parameter k. Since the coordinate axis are intercepted at $x^i = k / a_i$, the components of a contravariant vector have dimensions of lenght - an <u>extensive</u> quantity - while the covariant vector components have dimensions of the inverse of a lenght - an <u>intensive</u> quantity.

As an example, we recall that the position vector x is contravariant, while the gradient $\partial\phi/\partial x$ of a scalar function $\phi(x)$ of position is a covariant vector. We recall that in physics the dynamical quantity <u>momentum</u> p is defined as $\propto \partial\phi/\partial x$. This definition makes momentum a covariant vector, and hence it is much more appropriate to write down the fundamental equation of Newtonian dynamics as $f = - dp / dt$, than in the form $f = m\, d^2 x / dt^2$.

With contravariant and covariant vectors many different kinds of algebras can be built [5]. Thus, let the contravariant vector $V = V^j I_j$ and the covariant vector $U = U_j I^j$ be written in the reciprocal basis I_j and I^j of a certain n-dimensional affine space. The invariant $U_j V^j$ is denoted here by $\langle U, V \rangle$. Introducing the symbols (V) and (U) associated to the vectors V and U by the anticommutation rules

$$\begin{aligned} [(V),(V')]_+ &= 0 \\ [(U),(U')]_+ &= 0 \\ [(V),(U)]_+ &= \langle V, U \rangle 1_{G_n} \end{aligned} \qquad (2.4)$$

we obtain the <u>Grassmann algebra</u> G_n (1_{G_n} is the unit of G_n). This algebra is generated by the elements (I_j) and (I^j) through the anticommutation rules

$$[(I_j),(I_k)]_+ = 0 \qquad [(I^j),(I^k)]_+ = 0$$

$$[(I_j),(I^k)]_+ = \delta_j^k \, 1_{G_n} \quad , \quad \langle I_j , I^k \rangle = \delta_j^k \qquad (2.5)$$

Equations (2.5) show that, although G_n is an algebra of a n-dimensional space, it has the structure of a <u>Clifford algebra</u> C_{2n} of a 2n-dimensional space. The theory of G_n is, essentially, that of the spinors of S_{2n}. The Grassmann algebra G_n, taken over the complex numbers, is equivalent to a n-dimensional Jordan-Wigner algebra. Taking the adjoint $(I^j) = (I_j)^{\dagger}$, the anticommutation rules (2.5) become the n-dimensional equivalent to emission and absorption of the second quantization for fermions[6].

Similarly, one can define an associative algebra L_n , with elements denoted by $\{ v \}$ and $\{ u \}$, satisfying the commutation rules :

$$[\{v\},\{v'\}] = 0 \qquad [\{u\},\{u'\}] = 0 \qquad (2.6)$$

$$[\{v\},\{u\}] = \langle v , u \rangle \, 1_{L_n}$$

(1_{L_n} being the unit element of L_n), and the generators of L_n satisfying the commutation rules :

$$[\{I_j\},\{I_k\}] = 0 \quad , \quad [\{I^j\},\{I^k\}] = 0 \qquad (2.7)$$

$$[\{I_j\},\{I^k\}] = \delta_j^k \, 1_{L_n}$$

Equations (2.7) provide the Heisenberg commutation rules for the coordinate $Q = Q^j q_j$ and momentum operators $P = P_j p^j$, the generators of which are given by $q_j = \{ I^j \}$ and $p^j = (2 \pi i) \, h^{-1} \{ I^j \}$, where h is Planck's constant. Thus, L_n over the complex numbers is equivalent to <u>Heisenberg algebra</u> for the operators Q and P of a quantum system with n degrees of freedom. It can also be shown that quantum kynematics is related to the <u>symplectic geometry</u> of the phase space of Hamiltonian classical mechanics through its symplectic algebra L_n [7]. Besides, the algebra L_n over the complex numbers provides the n-dimensional equivalent to the <u>Dirac-Jordan-Klein</u> algebra for the emission and absorption operators of the second quantization for bosons. In 4-dimensional space, the action algebra, obtained from $dV = p_i \, dx^i$, $i = 1,2,3,4$, provides a quadratic form in 8 variables. This is the only instance in which there is a triality : one vector and two half-spinors, all with 8 components and all with similar properties[8] .

3. BASIC POSTULATES

Having presented the above considerations upon the different algebraic strutures generated by covariant and contravariant vectors, we may begin to assign a dynamical meaning to some of these vectors.

As we already said, the usual way of building physical models and / or theories consists in postulating a given space-time manifold, which is almost always metric (It can be shown that a differentiable manifold always admits a Riemannian metric[(9), (10)]), and where that metric is always fixed <u>ab initio</u>. This is the fixed space-time framework upon which a certain theory is built.

Our starting point here is epistemologically just the opposite : We try to determine the geometry by means of the introduction of a certain minimal number of fundamental dynamical objects. This point of view opposes the usual epistemological stand, which begins with the notion of space-time (of Aristotle, Newton, Minkowski, Riemann, Weyl, etc.) as the basic entity in Nature (and of human perception).

With this aim in mind, of trying to determine a certain geometry (i.e., a certain metric) starting from a minimal

number of dynamical objects, we begin by postulating the existence of a space-time manifold, the most general possible, with the least number of predetermined geometrical properties. Next, we shall populate the naked manifold with certain dynamical objects, taken as fundamental, trying then to determine what kind of manifold is compatible with these dynamical objects.

The only way that a physicist has of interacting with Nature is by means of measuring processes (observations transmited first to his senses and from those to the brain). The only way of an interaction reaching the senses (and thence the brain) is by means of a signal which transfers information from the system to the observer. For this, a physical field is needed, to which a certain energy and momentum densities may be ascribed, and which are the physical agents for the transmission of the signal. Therefore , it is only through the transfer of energy and momentum that a certain knowledge of the World, that is, of natural phenomena, may be obtained; in particular, a certain knowledge of its space-time features. In other words, the very the notion of space-time is strictly dependent of the notion of energy-momentum. In the very cosmological model most widely accepted nowadays - the big-bang model - the creation (expan- sion) of space-time is inextricably associated to the total initial energy-momentum density of the universe. That is, the initial dynamical content is the only determinant on how the geometric structure unfolds.

Thus, let us consider the antisymmetrical bilinear form $dV = dp_\mu \ dx^\mu$, built up with the covariant momentum four-vector p_μ with the contravariant position four-vector x^μ . The hypervolume dV (physically, the <u>action</u>) is constant with respect to a variation of a parameter λ (which may be identified with the cosmological time). The universe's initial conditions are such that for $\lambda = 0$, the momentum content was extremely high, whereas the space-time content was extremely low. We have here the most basic and fundamental observation refered above that the covariant vectors characterize the dynamical aspects whereas the contravariant ones characterize the geometrical aspects..

We begin by considering a fundamental field characterized by the 4-momentum p_μ , and with the aid of this dynamical quantity we try to construct a space-time geometry. For this purpose, we insert in the given field a <u>test particle</u> which

will act as our means of probing space-time geometry which we want to determine. The physical characteristics of which this particle will be revested will depend, essentially, on the kind of dynamics initially admitted as being associated with the postulated fundamental field. Thus, if we start from Newtonian dynamics, the fundamental property of that test particle will be its <u>inertia</u> while if we start from Maxwellian dynamics, the property which will enable it to probe the space-time structure will be its <u>charge</u> and so on.

We shall take, then, as basic postulates of all our future considerations the two following ones.

I. FUNDAMENTAL DYNAMICAL POSTULATE . The covariant 4-vector <u>momentum</u>, p_μ , $\mu = 1,2,3,4$ is the fundamental dynamical object.

II. FUNDAMENTAL GEOMETRICAL POSTULATE : The contravariant 4-vector <u>position</u> , x^μ , $\mu = 1,2,3,4$,is the fundamental geometrical object.

Based on this last one, we further postulate

III. EXISTENCE OF A DIFFERENTIABLE MANIFOLD . There is a 4-dimensional differentiable manifold, $V_2\ (x^{\nu})$, homogeneous in the (contravariant) space-time coordinates x^{ν} .

Following our plan, let us start trying to determine the specific nature of the manifold V_4 by means of the incorporation of specific dynamical entities. We shall analyse separately the cases of relativistic mechanics (both of the General and of the Special theories) and of Newtonian mechanics.

4. RELATIVISTIC MECHANICS AND RELATED GEOMETRIES

Here, we shall take as our differentiable manifold any four-dimensional space-time ${}^{(r)}V\ (x^{\nu})$. This manifold is endowed with the covariant dynamical momentum p_{ν} , according to the Fundamental Dynamical Postulate.

Let us introduce in our " naked "* manifold a test particle of momentum p_{ν} , describing a world-line characterized by the coordinates x^{ν} . Following Schönberg(11), we may associate to this particle a mass-energy function $E^2\ (p_{\nu}, x^{\nu})$ which allows the definition of a contravariant vector, p^{ν} , tangent to the particle's world-line

---

* The manifold ${}^{(r)}V\ (x^{\nu})$ is " naked ", ab initio, due to the absence of dynamical objects (besides the momentum p_{ν} , Post. I) and to the absence of any geometrical structure (besides the existence of coordinates - Post. II)

$$p^{\mu} \equiv \frac{\partial E^2(p_{\mu}, x^{\mu})}{\partial p_{\mu}} \qquad (4.1)$$

Although the quantity defined above p^{μ} has the same dimensions as the covariant momentum p_{μ}, no dynamical meaning is assigned, a priori, to this contravariant vector. At any rate, the introduction of this quantity forces the mass-energy function E^2 to be written down in the usual way, obtained from (4.1) by integration

$$E^2 = \int p^{\mu}\, dp_{\mu} = p^{\mu} p_{\mu} + \text{const.} \qquad (4.2)$$

For the function E^2 to have the usual meaning of energy-momentum, the bilinear form in (4.1) must represent an inner product, i.e.,

$$E^2 = p^{\mu} p_{\mu} = p_{\mu} p^{\mu} \qquad (4.3)$$

(where the constant in (4.2) was put equal to zero). Condition (4.3) is equivalent to ${}^{(r)}V_4$ being a metric manifold:

$$p^{\mu} = g^{\mu\nu}(x^{\lambda})\, p_{\nu} \qquad (4.4)$$

where $g^{\mu\nu}$ is the contravariant metric tensor of ${}^{(r)}V_4$, satisfying the orthogonality relation

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu} \qquad (4.5)$$

Differentiating now eq. (4.3) twice with respect to p. :

$$\frac{\partial^2 E^2}{\partial p_{\nu} \partial p_{\mu}} = \frac{\partial^2}{\partial p_{\nu} \partial p_{\mu}} \left(g^{\rho\sigma} p_{\rho} p_{\sigma} \right) = g^{\rho\sigma} \delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} = g^{\mu\nu}(x^{\lambda}) \qquad (4.6)$$

Since ${}^{(r)}V_4$ is differentiable (Postulate III) the Schwarz-Young conditions hold ($\partial^2 E^2 / \partial p_{\nu} \partial p_{\mu} = \partial^2 E^2 / \partial p_{\mu} \partial p_{\nu}$), that is, the metric tensor $g^{\mu\nu}$ (x^{λ}) is symmetric

$$g^{\mu\nu}(x^{\lambda}) = g^{\nu\mu}(x^{\lambda}) \qquad (4.7)$$

which characterizes ${}^{(r)}V_4$ as a (pseudo) Riemannian metric. We conclude, therefore, that resorting to the dynamical momentum p_{μ} (Dynamical Postulate), and ascribing to the dynamical function E^2 (p_{μ}, x^{μ}). the usual meaning of energy, it is possible to endow the manifold ${}^{(r)}V_4$ with a Riemannian metric (General Relativity Theory case).[11].

Next, we observe that in (4.1) E^2 (p_{μ}, x^{μ}) was differentiated with respect to its covariant variables p_{μ}, defining thus a contravariant vector p^{μ}. Obviously, the energy function may also be differentiated with respect to its contravariant variables x^{μ}, defining, then, a covariant vector ϕ_{μ} :

$$\phi_{\mu} \equiv \frac{\partial E^2(p_{\mu}, x^{\mu})}{\partial x^{\mu}} \qquad (4.8)$$

And taking (4.3) and (4.4) into (4.8), we get

$$\phi_\mu = \frac{\partial}{\partial x^\nu}\left(g^{\rho\sigma} p_\rho p_\sigma\right) = \left(g^{\rho\sigma}{}_{,\mu}\right) p_\rho p_\sigma \qquad (4.9)$$

If we take the potential function $\phi_\mu(x^\lambda)$ as constant and equal to zero along all the test particle's world-line, then $g^{\mu\nu}{}_{,\rho} = 0$, that is, $g^{\mu\nu}$ = constant along all the world-line. Since that world-line is arbitrary, this is equivalent to saying that $g^{\mu\nu}$ is constant over all the manifold ${}^{(r)}V_4$, that is, putting $\phi_\mu(x^\lambda) = 0$ over all of ${}^{(r)}V_4$ makes the manifold flat.

At this point, it seems that the metric of this flat manifold is not fixed yet, since we apparently still have the freedom to choose between the different possibilities of a manifold with indefinite metric (signature of absolute value 2) and of a manifold with definite metric (signature of absolute value 4). As we have already said before, it is only through the transfer of energy and momentum that we may obtain a knowledge of the space-time (geometric) aspects of the manifold. It is easily seen(12), (13), (14) that if the flat manifold has a positive definite metric of the kind $(dx^1)^2 + (dt)^2$, this metric demands that an infinite velocity be physically realizable.

This, in turn, is equivalent to admitting that space and time are entirely interchangeable, a possibility which is in complete disagreement with our experience. Therefore, we must impose the dynamical principle that there is a limiting velocity for the propagation of physical signals. Taking this limiting velocity as being the velocity of propagation of the eletromagnetic field, c, we immediately conclude that the geometry of our flat manifold ${}^{(r)}V_4$ is Minkowskian. Or, in other words : considering $\phi_\mu(x^\lambda) = 0$ over all of ${}^{(r)}V_4$ is equivalent to stating that we can build a global inertial frame over all the manifold. On the other hand, this hypothesis of considering $\phi_\mu(x^\lambda) = 0$ along all the particle's world-line corresponds to having $E^2(p_\mu, x^\mu)$ constant along that world-line; this means that we can define that function E^2 over all the manifold.

5. NEWTONIAN MECHANICS AND RELATED GEOMETRY

Here, we shall take as our differentiable manifold a four-dimensional space-time ${}^{(n)}V_4\,(x^i, t)$, separated into two orthogonal sub-manifolds 3 + 1 : a three-dimensional hypersurface $V_3(x^i)$ in the space coordinates x^i, $i = 1,2,3$ and a one-dimensional manifold $V_1(t)$ in the time coordinate $x^4 = t$. As in the previous relativistic case, ${}^{(n)}V_4$ is endowed with the covariant dynamical 3-momentum p_i. However, while in the relativistic case, in the determination of the related geometry, we resorted to the mass-energy function $E^2(p_\mu, x^\mu)$, here, in the Newtonian case, we shall necessarily have to employ separately two dynamical functions : mass and energy.

We shall take as starting point of Newtonian dynamics the Principle of Inertia. Thus, if we introduce a test particle into our naked manifold, its fundamental characteristic for Newtonian physics is the one associated to its inertia, that is, its mass m.

Let the test particle be moving with velocity $\dot{x}^i = dx^i / dt$. Using the concept of mass of Maupertuis [15],

we define the contravariant vector :

$$p^i = m\,\dot{x}^i \tag{5.1}$$

As before in relativistic dynamics, the contravariant object p^i defined above cannot, a priori, be identified with the covariant fundamental dynamical momentum p_i. However, this identification is immediate once the three-dimensional hypersurface ${}^{(n)}V_3\,(x^i)$ is metric. (We notice here that the entire four-dimensional manifold ${}^{(n)}V_4$ is non metric [2]) Thus :

$$p^i = g^{ij}(x^k)\,.\,p_j \quad , \; i,j,k = 1,2,3 \tag{5.2}$$

That is, once we may write

$$m\,\dot{x}^i = g^{ij}(x^k)\,p_j \tag{5.3}$$

where $g^{ij}(x^k)$ is the metric tensor of ${}^{(n)}V_3\,(x^i)$, satisfying the orthogonality condition

$$g^{ij}\,g_{jk} = \delta^i_k \tag{5.4}$$

We notice that here , in the Newtonian case, besides the fundamental dynamical entity - three momentum - , we have two accessory dynamical quantities : the mass and the energy. We just concluded above that the introduction of the dynamical concepts of momentum and mass allowed the manifold to be endowed with a metric. It remains to be seen what is the role played by the second of the accessory dynamical concepts : the energy. Clearly, it would be interesting if that function would be able to fix that metric, thus determining the space-time compatible with Newtonian physics.

We repeat now the process already followed in the relativistic case with the important difference that here the full four-dimensional Newtonian space-time manifold ${}^{(n)}V_4$ cannot be endowed with a metric. In the 3-dimensional metric submanifold ${}^{(n)}V_3$ we can, however, define an inner product and hence the two bilinear forms

$$(2\,m)^{-1}\, p_i\, p^i \quad \text{and} \quad (2\,m)^{-1}\, p^i\, p_i \tag{5.5}$$

both of which still have no physical meaning.. Using exactly the same procedure of the preceding section, we differentiate any of the two bilinear forms (5.5) twice with respect to the covariant momentum :

$$\frac{\partial^2}{\partial p_j \partial p_i}\left((2m)^{-1} p_k p^k\right) = \frac{\partial}{\partial p_j}\left((2m)^{-1} \delta^i_k\, p^k\right) = \tag{5.6}$$
$$= \frac{\partial}{\partial p_j}\left((2m)^{-1} g^{ij} p_j\right) = (2m)^{-1} g^{ij}$$

Since ${}^{(n)}V_3$ is differentiable, the Schwarz-Young relations hold and therefore the symmetry condition $g^{ij} = g^{ji}$, which characterizes this manifold as Riemannian.

To introduce the energy concept into our manifold we impose the condition that the two bilinear forms (5.5) be identical

$$(2\,m)^{-1}\, p_i\, p^i = (2\,m)^{-1}\, p^i\, p_i = (2m)^{-1}\, p^2 = T \tag{5.7}$$

where T represents now the kynetic energy of a particle of mass m. Hence, the introduction of the kynetic energy implies, necessarily, that

$$p^i = \pm\ p_i \tag{5.8}$$

that is, that the contravariant vector introduced in (5.1) by means of the concept of mass is identical, up to a sign, to the fundamental dynamical momentum. This also implies that

$$g_{ij} = \pm\ \delta_{ij} \quad \text{and} \quad g^{ij} = \pm\ \delta^{ij} \tag{5.9}$$

which means that the 3-dimensional space submanifold ${}^{(n)}V_3(x^i)$ is plane. Consequently, since the fourth axis x^4 of ${}^{(n)}V_4(x^i, x^4)$ is orthogonal to this hyperplane, this time axis is unique, that is, we must have here an absolute time which as we just saw is necessarily and intimately related to the Newtonian dynamical concepts of mass (of Maupertuis) and of (kynetic) energy. As it is well known it immediately follows that in ${}^{(n)}V_4$ the concept of simultaneity is absolute : we have a physics of action at a distance.

6. CONCLUSIONS

Contrary to the costumary way of doing physics, we were presently able to show that starting from a few given dynamical quantities we can uniquely determine a certain geometry. Thus, general relativistic physics implies general Riemannian geometry, while the physics of the special theory of relativity is tied up with a flat Riemann space-time (Minkowski space-time). Finally, Newtonian dynamics is unambiguously bounded to Newtonian space-time.

What this clearly seems to indicate is that the connection between physics and geometry is even more profound than is commonly thought. By this we mean that not only a certain dynamics and a certain space-time are inextricably and uniquely bounded together, as stated above, but, also more important, that maybe the point of view taken here is perhaps the most fundamental. Namely, that instead of departing from a given postulated space-time and then infer the associated dynamics, we should start by postulating a certain physics and then try to determine its related geometry. In other words : geometry should be considered as an aspect of dynamics.

Instead of thinking, as in geometrodynamics, that geometry is everything[16], here, in dynamicgeometry, we take the conjugate point of view : dynamics is everything. This point of view reminds us of Leibniz[17].

REFERENCES

(1) Schönberg, M. - Private communication

(2) Cartan, E. - Ann. Ecole Norm. Sup., 40, 325 (1923) and 41, 1 (1924)

(3) Cartan, E. - Bull. Math. Soc. Roum. des Sciences, 35, 69 (1933)

(4) Schouten, J. A. - " Tensor Analysis for Physicists ", Clarendon Press, Oxford (1951)

(5) Schönberg, M. - An. Acad. Bras. Ci., 28, 11 (1956)

(6) Schönberg, M. and Rocha Barros, A. L. - Rev. Union Mat. Argentina, 20, 239 (1960)

(7) Arnold, V. - " Les methodes mathematiques de la mecanique classique " - Ed. Mir Moscou (1976)

(8) Chevalley, C. C. - " The algebraic theory of spinors " Columbia Univ. Press., N. York (1954)

(9) Thomas, G. H. - Riv. del Nuovo Cim., 3, 4 (1980)

(10) Steenrod, N. - " The topology of fibre bundles ", Princeton New Jersey, (1951)

(11) Schönberg, M. - Acta Phys. Austriaca, 38, 168 (1973)

(12) Trautman, A. - Brandeis Summer Institute in Theoretical Physics - Lectures on General Relativity - Prentice Hall (1964)

13) Dixon, W. G. - " Special Relativity - The Foundation of Macroscopic Physics " -Cambridge University Press (1978)

15) Langevin, P. - " La Physique Depuis Vingt Ans "
Doin - PARIS (1923)

14) Videira, A. L. L., Rocha Barros, A. L. and Fernandes, N. C. - to be published

16) Wheeler, J. A. - " Geometrodynamics " - Academic Press, New York, (1962)

17) Costabel, P. - " Leibniz and Dynamics " , Hermann, Paris, (1973)
— "Leibniz et la Dynamique en 1692
Vrin (1981) - Paris