A comment on

EINSTEIN - A NATURAL COMPLETION OF NEWTON

In the discussion following the inaugural address of Professor Alladi Ramakrishnan a point was raised regarding the uniqueness of the velocity addition formula (see equation 4 above) as following from assumptions stated in the address. The answer of course is yes if we restrict ourselves to only those functions which are continuous and otherwise well behaved and satisfy the assumptions stated at every point in the closed (or the open) interval between -1 and +1. A rigorous proof of this would involve solution of a functional equation. But since certain group properties are explicit in the assumptions made one can take advantage of this to construct a proof. In the following I present a simple minded proof along these lines which was developed in conversations with Professor Alladi Ramakrishnan.

The object is to obtain Einstein's law of addition of velocities as a natural generalization of the Newtonian law when the assumption regarding the upper limit of velocity is made. Precisely stated, the assumption are

(1) the upper limit to a realisable velocity is in suitable units.

(2) For velocities much smaller than 1 the Newtonian law results.

(3) If v_1 and v_2 are realisable velocities then their composition v_{12} is also realisable.

Now in the Newtonian case the set of all realizable velocities $\{u_i\}$ satisfy the relations.

$$u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$$ distributive

$$u_1 + u_2 = u_2 + u_1$$ commutative

$$u + (-u) = 0$$ has unique inverse

$$u + 0 = u$$ identity

It is clear that any relativistic law of composition will also have these properties, but with one difference; whereas the Newtonian velocities span the entire real line, the Einstein velocities span a bounded interval between -1 and +1. Let us denote by * the sign of composition for relativistic velocities. Then

$$v_1 * v_2 = v_2 * v_1 = v_{12} \leq 1 \qquad \text{if } v_1, v_2 \leq 1$$

$$v_1 * (v_2 * v_3) = (v_1 * v_2) * v_3$$

$$v * (-v) = 0$$
$$v * (0) = (0) * v = v$$
$$v * (\pm 1) = (\pm 1) * v = \pm 1$$

If we compare the relativistic and Newtonian rules we see that '0' of relativistic formula behaves like '0' of Newtonian case but ± 1 of relativistic case behaves as $\pm\infty$ of Newtonian case. Since the Newtonian case corresponds to the real line it means that there exists a one-to-one continuous map from the open interval $-1 < v < +1$ to the real line, $-\infty < \theta < +\infty$ and vice-versa. We can of course also take the closed interval $-1 \leq v \leq +1$ for the relativistic velocities, in which case the mapping is to the extended real line (which includes infinity as a 'number'). Any such map has to be necessarily unique. It is given by

$$v = \tanh \theta \quad , \quad \theta = \tanh^{-1} v . \tag{A}$$

Now observe that on the real line, θ would satisfy all the composition rules of Newtonian velocities. In particular

$$\theta_{12} = \theta_1 + \theta_2 = \theta_2 + \theta_1$$

Therefore if we define $\tanh \theta_1 = v_1$, $\tanh \theta_2 = v_2$

$$\tanh \theta_{12} = v_{12}$$

then

$$v_{12} = \frac{v_1 + v_2}{1 + v_1 v_2} = v_1 * v_2 \tag{B}$$

From uniqueness of the mapping (A) follows the uniqueness of the formula (B).

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