BOSON-FERMION SYMMETRY AND DIRAC KÄHLER FORMS

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Super-symmetry is studied with help of the Atiyah-Kähler Algebra of differential forms.

More than 25 years ago, Mario Schenberg has investigated in a series of papers[1] on "Quantum Mechanics and Geometry" the significance of geometric algebras for the formulation of the basic laws of particle physics. Recently, the problem of lattice approximation of QCD led us to the attempt[2,3,4] to describe matter fields by a differential geometric generalization of the Dirac equation due to E. Kähler[5]. This description of fermion fields in terms of a geometric algebra is very close to Schenberg's ideas. No wonder that the vivid memories of my discussions with Prof. M. Schenberg greatly inspired my work on this topic. In this spirit I want to dedicate to Prof. M. Schenberg on the occasion of his 70th birthday this short note on geometric superalgebra. It should open a particular view on the geometric meaning of supersymmetry[6], which in these days plays such an important role in the discussion on a universal theory of matter[7]. I want to thank Prof. M. Schenberg for a long lasting inspiration.

1. ATIYAH-KÄHLER ALGEBRA AND DIRAC-KÄHLER EQUATION

E. Kähler enlarged the structure of the wellknown Grassmann algebra of differential forms by the introduction of an additional associative product ("Clifford product") defined by

$$dx^\mu \vee dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu} \tag{1.1}$$

$\wedge$ denotes the usual antisymmetric product of the Grassmann algebra,

$g^{\mu\nu}$ is the metric tensor of the underlying Riemannian manifold. A space of inhomogeneous forms $\{\phi\}$ in D dimensions (D even)

$$\phi = \sum_{n=0}^{D} \frac{1}{n!} \Phi_{\mu_1 \ldots \mu_n}(x)\, dx^{\mu_1} \wedge \ldots dx^{\mu_n} \equiv \sum_H \Phi_H(x)\, dx^H \tag{1.2}$$

equipped with wedge product and Clifford product with properties following naturally from Eq. (1.1) is called an Atiyah-Kähler algebra[8,9]. In the notation (1.2) H runs over all ordered index sets $H = \{\mu_1, \ldots, \mu_n\}$ with $0 \leq \mu_1 < \mu_2 \ldots < \mu_n \leq D-1$.

Within the framework of an Atiyah-Kähler algebra it can easily be shown that the Dirac-Kähler equation (DKE) in flat space (f.i. with Minkowski metric $-g_{00} = g_{11} = \ldots = g_{D-1,D-1} = 1$):

$$(d - \delta - m)\,\phi = 0 \tag{1.3}$$

where d means exterior differentiation, δ is the generalized divergence, is equivalent to $2^{D/2}$ uncoupled Dirac equations[2,9]. For this we reduce the left regular representation of the Clifford algebra with help of the Dirac matrices γ^μ by introducing a "Dirac basis" $Z_a^{(b)}$

$$\phi = \sum_{a,b} \Phi_a^{(b)}(x)\, Z_a^{(b)} \tag{1.4}$$

with the property of the matrix $Z = (Z_a^{(b)})$

$$dx^\mu \vee Z = \gamma^{\mu T} Z\,, \qquad Z \vee dx^\mu = Z\gamma^{\mu T} \tag{1.5}$$

The explicit construction of Z follows the standard methods of group representations. For this one considers the elements $\pm\, dx^H$ with v-multiplication as a group C_D of order 2^{D+1}. The inverse is $(dx^H)^{-1} = (-1)^{\binom{h}{2}}\, dx_H$, $dx_H = G_{HK} dx^K$, with G^{HK} the metric tensor extended to the forms of higher degree. Then one gets the result

$$\left(Z_a^{(b)}\right) = \sum_H (-1)^{\binom{h}{2}} \gamma_H^T \, dx^H ,$$

$$\gamma_H = \gamma_{\mu_1} \ldots \gamma_{\mu_n} \qquad (1.6)$$

which leads to the transformation formulas

$$\Phi_H(x) = Tr(\Phi\gamma^{H+}) , \qquad \left(\Phi_a^{(b)}(x)\right) = 2^{-D/2} \sum \Phi_H(x) \, \gamma^H \qquad (1.7)$$

It follows immediately from the form of the DK-operator $d-\delta = dx^\mu \vee \partial_\mu$ and these formulas, that the Dirac components $\Phi_a^{(b)}(x)$ of a solution of the DKE satisfy the Dirac equation

$$(\gamma^\mu\partial_\mu - m) \, \Phi^{(b)}(x) = 0 \qquad b = 1,2, \ldots 2^{D/2} \qquad (1.8)$$

The consideration of the γ-matrices as representation of the Clifford group C_D leads to the following convolution formula

$$Z_a^{(b)} \vee Z_{(\bar{b})}^{\;(\bar{a})} = 2^{D/2} \, Z_a^{(\bar{a})} \, \delta_{\bar{b}}^{\;b} \qquad (1.9)$$

It will turn out that this relation between differential forms is the key of a special realization of supersymmetry.

We conclude this short glossary of the Atiyah-Kähler algebra by listing some properties of the local scalar product $(\phi,\psi)_x$ for later use:

$$(\phi,\psi)_x = \sum_H \Phi_H(x) \, \psi^H(x) \qquad (1.10)$$

If the context allows it, we omit the index x. It is

$$(\phi,\psi) = (\psi,\phi) = (A\phi,A\psi) = (B\phi,B\psi) \qquad (1.11)$$

A is the isomorphism of the Atiyah-Kähler algebra $A\,dx^H = (-1)^h \, dx^H$; B the anti-morphism $B\,dx^H = (-1)^{\binom{h}{2}} \, dx^H$, $B(dx^H \vee dx^K) = Bdx^K \vee Bdx^H$, etc. With respect to $\vee$-multiplication (ϕ,ψ) has the following symmetry:

$$(\phi \vee w,\psi) = (\phi,\psi \vee Bw)\ , \qquad (v \vee \phi,\psi) = (\phi,Bv \vee \psi) \quad (1.12)$$

The scalar product allows the formulation of the following completeness relation:

$$\phi = \sum_H (\phi,dx^H)\ dx_H \quad (1.13)$$

Finally we want to mention the formula[10]

$$-\delta(\phi,dx^\mu \vee \psi).dx_\mu = (\phi,(d-\delta)\psi) + ((d-\delta)\phi,\psi) \quad (1.14)$$

which is very important for the construction of conserved currents from solutions of the DKE.

2. BOSON-FERMION CURRENTS

The Kähler formalism represents Dirac fields, i.e. fermions, by coherent superpositions of inhomogeneous differential forms. On the other hand, forms describe in a natural way tensor fields which are bosonic. This similarity strongly suggests the investigation of supersymmetry between bosons and fermions in terms of Dirac-Kähler forms.

In order to pursue this idea, we consider solutions of the DKE

$$(d-\delta-m)\ \phi = 0\ , \qquad (d-\delta-m)\ \psi = 0 \quad (2.1)$$

where ψ describes Dirac fields according to Eq. (1.7,8). The homogeneous components $\Phi_{\mu_1 \ldots \mu_n}(x)$ of ϕ, Eq. (1.1), we treat as tensor fields, and therefore as bosons. The associativity of Clifford-multiplication and the formula $d-\delta = dx^\mu \vee \partial_\mu$ implies that by right $\vee$-multiplication with constant forms C solutions of the DKE get transformed into solutions. Thus we may use the formula (1.14) for the construction of conserved boson-fermion currents from the solutions of Eq. (2.1):

$$j_\mu^C(x) = \left(\psi \vee C , dx_\mu \vee \overline{\phi}\right) , \qquad \overline{\phi} = A\phi$$

$$(d - \delta + m)\, \overline{\phi} = 0 \tag{2.2}$$

Are the charges related to these conserved currents the generators of supersymmetry? We approach this question by analyzing the covariance properties of the boson-fermion currents j_μ^C with respect to Lorentz transformations. The DKE is invariant under the Lorentz group, which transforms the components of ϕ as antisymmetric tensor fields. The infinitesimal rotation in the μ-ν-plane may be written as

$$L^{\mu\nu}\,\phi = \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\phi + \frac{1}{2}\left(S^{\mu\nu} \vee \phi - \phi \vee S^{\mu\nu}\right) \tag{2.3}$$

with $S^{\mu\nu} = dx^\mu \vee dx^\nu$. How is this tensorial transformation law compatible with the spinorial transformation of the Dirac fields contained in ψ? According to Eq. (1.5), $\frac{1}{2} S^{\mu\nu} \vee \psi$ represents an infinitesimal spinor transformation of the Dirac components: $\frac{1}{2} S^{\mu\nu} \vee \psi \sim \frac{1}{2} \gamma^\mu\gamma^\nu \psi^{(b)}$, $\mu \neq \nu$. Hence the Lorentz transformation of fermion forms is represented by

$$L^{\mu\nu}\,\psi = \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\psi + \frac{1}{2} S^{\mu\nu} \vee \psi \tag{2.5}$$

Right multiplication of ψ by $S^{\mu\nu}$:

$$\frac{1}{2}\, \psi \vee S^{\mu\nu} \sim \sum_{b'} \psi^{(b')} (\gamma^{\mu\nu})_{b'}{}^{b}$$

mixes the different Dirac fields. Such transformations induced by Clifford right multiplications are called sometimes "Susskind flavour transformations". Therefore a tensor transformation can be considered as a product of a spinor transformation and a corresponding $SL(2,C)$ S-flavour transformation. The transformation properties of currents follow from those of the forms. Tensor-transformations leave the inner product (1.10) invariant. Hence currents constructed from bosonic forms $j_\mu(x) = (\phi', dx_\mu \vee \bar{\phi})_x$ transform as tensors

$$L^{\mu\nu} j_\rho(x) = (x^\mu \partial^\nu - x^\nu \partial^\mu)\, j_\rho(x) + \delta^\mu_\rho\, j^\nu(x) - \delta^\nu_\rho\, j^\mu(x) \qquad (2.5)$$

This might be confirmed with help of Eqs. (1.12) and (2.3). For the boson-fermion currents we are interested in expressions which transform like super-currents:

$$L^{\mu\nu} j_{\rho,a}(x) = (x^\mu \partial^\nu - x^\nu \partial^\mu)\, j_{\rho,a}(x) + \delta^\mu_\rho\, j^\nu_{,a}(x) - \delta^\nu_\rho\, j^\mu_{,a}(x) +$$

$$+ \frac{1}{2} (\gamma^{\mu\nu})^{\bar{a}}_a\, j_{\rho,\bar{a}}(x) \qquad (2.6)$$

Writing for C in Eq. (2.2) a constant form W_a, then a straightforward calculation using (1.12), (2.3,4) leads to the requirement

$$W_a \vee S^{\mu\nu} = (\gamma^{\mu\nu})^{\bar{a}}_a\, W_{\bar{a}} \qquad (2.7)$$

It follows immediately from (1.5) that this is satisfied by

$$W^{(b)}_a = B\, T_{a\bar{a}}\, Z^{(b)}_{\bar{a}} \qquad (2.8)$$

where T is defined by $T \gamma^\mu T^{-1} = \gamma^{\mu T}$.

Now we can formulate more precisely our program. We want to know if there are generators of supersymmetry which are linear combinations of boson-fermion charges of the form

$$Q^L_a = \int d\vec{x} \left(\psi \vee W_a\, , dx_o \vee \bar{\phi}^L \right) \qquad (2.9)$$

where ψ is a fermionic DK form satisfying the DKE. ϕ^L are bosonic solutions of the DKE like

(a) $\phi^S = mA + dA \qquad \partial_\mu \partial^\mu A = m^2 A$

(b) $\phi^P = mB\, dx^{0 \ldots D-1} - \delta(B\, dx^{0 \ldots D-1})$

(c) $\phi^V = (d - \delta + m)\, A_{\mu\nu}\, dx^\mu \wedge dx^\nu$ etc. $\qquad (2.10)$

In two dimensions the forms (a) and (b) span all solutions. Unfortunately we can give in this note an answer to this question only for the most simple case of real DK forms in 2 dimensions.

3. N=2 EXTENDED SUPERSYMMETRY IN 2-DIMENSIONS[11]

In this section we want to demonstrate that our conjecture of a close connection between the boson-fermion currents defined above and supersymmetry is true for 3-dimensional real Dirac-Kähler forms. For this we consider the scalar (pseudoscalar) boson-fermion charges $Q_a^{(i)s}$ $(Q_b^{(k)p})$ of the form (2.9)

$$Q_a^{(i)L} = \frac{1}{\sqrt{8}} \int dx^\Lambda \left(\psi \vee v_a^{(i)}, dx_o \vee \bar{\phi}^L\right)_x \qquad L = S,P. \tag{3.1}$$

which are derived from the conserved boson-fermion currents of type (2.2). The constant forms $v_a^{(i)}$ are defined similar to Eq. (2.8)

$$v_a^{(i)} = T_{a\bar{a}} B Z_{\bar{a}}^{(k)} T_{ki} \tag{3.2}$$

The γ-matrices in the definition of Z, Eq. (1.6), are a real unitary representation of the Clifford algebra of 2-dimensional Minkowski space. Our treatment is therefore covariant with respect to real, orthogonal equivalence transformations. The solutions of the bosonic DKE in 2 dimensions are

$$\phi^s = mA^s + dA^s , \qquad \phi^p = -\delta(A^p.dx^{01}) + mA^p.dx^{01} \tag{3.3}$$

where A^L are free fields: $\partial_\mu\partial^\mu A^L - m^2 A^L = 0$.

We shall show that the

$$Q_a^{(i)} = Q_a^{(i)s} + \gamma_k^{0,i} Q_a^{(k)p} \tag{3.4}$$

satisfy the anticommutation relations of the $N = 2$ extended supersymmetry

$$\{Q_a^{(i)}, Q_b^{(k)}\} = \left(H\delta_{ab} + P\gamma_{ab}^{01}\right) \delta^{ik} \tag{3.5}$$

H and P are the Hamiltonian and momentum operator of the system consisting of two real Dirac fields: a scalar and a pseudoscalar field described by the DK-forms ψ, $\phi = \phi^s + \phi^p$. This is our main result.

The anti-commutators of the boson-fermion charges (3.1) follow from the free field commutation relations of bosons ϕ and fermions ψ. These can be written as bi-differential forms[3,12,13]

$$\{\psi(x),\psi(x)\} = 2(d-\delta+m)_y\ \Delta(x-y)\ A_x \left[\sum_H dx^H . dy_H \right] \vee dy^0$$

$$[\phi^s(x),\phi^s(y)] = \frac{1}{i}\ (d-\delta+m)_x\ (d-\delta+m)_y\ \Delta(x-y) \qquad (3.6)$$

$$[\phi^p(x),\phi^p(y)] = \frac{1}{i}\ (d-\delta+m)_x\ (d-\delta+m)_y\ \Delta(x-y).dx^{01}.dy^{01}$$

$$[\phi^p(x),\phi^s(y)] = 0$$

Here $dx^H.dy^K$ denotes the basis elements of a direct product of Atiyah-Kähler algebras at different space-time points. The canonical equal time commutation relations contained in (3.6) lead to anticommutators, (commutator), for scalar products of fermion (boson) forms with C-number forms:

$$\left\{ \int dx^1\ (\psi,F)_x ,\ \int dy^1\ (\psi,G)_y \right\} = -2 \int dx^1 \left(dx^0 \vee \bar{G} \vee dx^0\ ,\ F\right)_x$$

$$\left[\int dx^1 (F,\bar{\phi}^{\overset{s}{p}})_x ,\ \int dy^1 (G,\bar{\phi}^{\overset{s}{p}})_y \right] = i \int dx^1 \left(\partial_1 F_{\underset{0}{0}}(x) G_{\underset{1}{1}}(x) + \partial_1 F_{\underset{1}{1}}(x) G_{\underset{0}{0}}(x)\right)$$

$$+\ im \int dx^1 \left(F_{\underset{1}{\phi}}(x) G_{\underset{01}{0}}(x) - F_{\underset{01}{0}}(x) G_{\underset{1}{\phi}}(x)\right) \qquad (3.7)$$

These formulas are the main tool for the calculation of the anticommutators of the $Q_a^{(i)L}$.

According to Eq. (3.1), the boson-fermion charges $Q_a^{(i)L}$ are also integrals over scalar products of two forms, like the expressions in (3.7). However, both forms ϕ and ψ are quantized. We have to use the operator identity

$$\{\psi A, \phi B\} = \{\psi,\phi\}\, AB + \phi\psi\, [B,A] = \psi\phi\, [A,B] + \{\psi,\phi\}\, BA$$

$$\text{for} \quad [\psi,A] = [\phi,A] = [\psi,B] = [\phi,B] = 0 \tag{3.8}$$

Then Eq. (3.7) permits a straightforward calculation of the $\{Q_a^{(i)L}, Q_b^{k\,L'}\}$. Before we make some comments on the somewhat lenghty calculation, we state the result

$$\{Q_a^{(i)L}, Q_b^{(k)L}\} = H[A^L]\, \delta_{ab}\delta^{ik} + H[\psi^{(i)},\psi^{(k)}]\, \delta_{ab} \pm M(\psi^{(i)},\psi^{(k)})\gamma^0_{ab}$$

$$+ P[A^L]\, \delta^{01}_{ab}\delta^{ik} + P[\psi^{(i)},\psi^{(k)}]\, \gamma^{01}_{ab} \qquad L = S,P$$

$$\{Q_a^{(i)S}, Q_b^{(k)P}\} = F_{ab}[A^S,A^P]\, \delta^{ik}, \qquad F_{ab} = F_{ba} \tag{3.9}$$

with

$H[A] = \frac{1}{2}\int dx^1 \left(\pi_A^2 + (\partial_1 A)^2 + m^2A^2\right)$: energy of a scalar field A

$H[\psi^{(i)},\psi^{(k)}] = \frac{1}{4i}\int dx^1 \left(\psi^{(i)}(-\gamma^{01}\partial_1 + m\gamma^0)\psi^{(k)} + (i \leftrightarrow k)\right)$:

"mixed" Hamiltonian of two Dirac fields contained in ψ.

$P[A] = \int \pi_A\, \partial_1 A\, dx^1$: momentum of a scalar field A.

$$P[\psi^{(i)},\psi^{(k)}] = \frac{i}{4}\int dx^1 \left(\psi^{(i)}(x)\, \partial_1\, \psi^{(k)}(x) + (i \leftrightarrow k)\right)$$

$$M[\psi^{(i)},\psi^{(k)}] = \frac{mi}{2}\int dx^1\, \psi^{(i)}(x)\, \psi^{(k)}(x) \tag{3.10}$$

Our main result, namely that $Q_a^{(i)}$ defined in (3.4) are generators of supersymmetry satisfying (3.5), follows immediately from (3.9). The following facts must be used in the simple calculation: (a) $\gamma^0 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in a real representation of 2-dimensional γ-matrices,

(b) $A^T = A$ implies $A - \gamma^0 A \gamma^0 = \text{trace } A.\underline{1}$, (c) $A^T = -A$ implies $A + \gamma^0 A \gamma^0 = 0$, and finally

$$H = H[A^S] + H[A^P] + H[\psi^{(1)},\psi^{(1)}] + H[\psi^{(2)},\psi^{(2)}] \qquad (3.11)$$

and similar for the momentum P.

In order to illustrate the algebra contained in the derivation of Eqs. (3.9), we include in this note a small part of the explicit calculation. Eq. (3.8) combined with Eq. (2.7) yield

$$\{Q_a^{(i)S}, Q_b^{(k)S}\} = + \frac{1}{4} \int dx^1 \left[\phi^S \vee BA\, v_a^{(i)} \vee dx^0,\; dx_o \vee \bar{\phi}^S \vee B\, v_b^{(k)}\right]_x \qquad \text{(I)}$$

$$+ \frac{1}{16i} \int dx^1 \left[dx^{01} \vee \psi \vee v_b^{(k)} \vee dx^0, (1-A)\; dx_o \vee \partial_1 \psi \vee v_a^{(i)}\right]_x \qquad \text{(II)}$$

$$+ \frac{mi}{8} \int dx^1 \left(\left[dx^0 \vee \psi \vee v_a^{(i)}\right](x,\phi) \left[dx^0 \vee \psi \vee v_a^{(k)}\right](x,0) - ((i,a) \leftrightarrow (k,b))\right) \qquad \text{(III)}$$

With the different rules for calculations with the scalar product (1.11), (1.12) we get

$$(I) = \frac{1}{4} \int dx^1 \left[\phi^S \vee BA\, v_a^{(i)} \vee dx^0 \vee v_b^{(k)},\; dx_o \vee \bar{\phi}^S\right]_x$$

Using the convolution formula (1.9) in the form

$$BA\, v_a^{(i)} \vee dx^0 \vee v_b^{(k)} = 2\, dx^0 \vee Z_a^{(b)}\, \delta^{ik} \qquad (3.12)$$

this becomes

$$(I) = \frac{1}{2} \int dx^1 \left[\phi^S \vee dx^0 \vee Z_a^{(b)},\; dx_o \vee \bar{\phi}^S\right] \delta^{ik}$$

Now we insert the definition of Z, Eq. (1.6), which leads to

$$(I) = \frac{1}{2} \int dx^1 \left[B\, \bar{\phi}^S \vee dx^0 \vee \phi^S \vee dx_o,\; dx^H\right] \gamma_{H,ba}\; \delta^{ik}$$

We remark that the components of the Clifford product

$$B\,\overline{\phi}^S \vee dx^0 \vee \phi^S \vee dx$$

$$= (mA - \partial_0 A\, dx^0 - \partial_1 A\, dx^1) \vee (+mA + \partial_0 A\, dx^0 - \partial_1 A\, dx^1)$$

$$= +((\partial_0 A)^2 + (\partial_1 A)^2 + m^2 A^2).\underline{1} - m\,\partial_1 A^2 . dx^1 + (\partial_0 A \partial_1 A + \partial_1 A \partial_0 A)\, dx^{01} \tag{3.13}$$

are essentially the energy and momentum density. Hence

$$(I) = \left(H[A^S]\,\delta_{ab} + [A^S, A^S]\,\gamma^{01} \right) \delta^{ik}$$

The evaluation of II and III follows a similar line. The final result is $\{Q_a^{(i)s}, Q_b^{(k)s}\}$ as given in (3.9). We would like to point at the important parts of the calculation: (i) The use of the convolution formula (3.12), and (ii) the V-factorization of the energy -momentum density. In these steps the significance of the Clifford algebra gets apparent.

In finishing this short description of supersymmetry in terms of Dirac-Kähler forms, we give the supertransformations of ϕ and ψ. With help of Eq. (3.7) we get immediately

$$\{Q_a^{(i)L}, \psi(x)\} = \frac{1}{\sqrt{2}}\,\phi^L(x) \vee AB\, v_a^{(i)} \vee dx^0 \qquad L = S,P$$

$$i[Q_a^{(i)S}, \phi^S(x)] = -\frac{1}{\sqrt{8}}\,(m + d)(\psi, B\, v_a^{(i)})_x$$

$$i[Q_a^{(i)P}, \phi^P(x)] = -\frac{1}{\sqrt{8}}\,(m - \delta)(\psi, dx^{01} \vee B\, v_a^{(i)})_x . dx^{01}$$

$$i[Q_a^{(i)L}, \phi^{L'}(x)] = 0 \qquad \text{for} \qquad L \neq L' \tag{3.14}$$

4. THE SIGNIFICANCE OF THE RESULT

General relativity, the success of gauge theories of strong and electroweak interactions, and the new development in supersymmetry reveal more and more a deep connection between geometry and fundamental dynamics. The recent result that $N = 4$ extended supersymmetry gauge theory in 4-dimensions is ultraviolet finit[14], may be considered as additional evidence in favour of this conjecture.

With this in mind, the investigation of the relation between the geometric algebra of differential forms gets interesting. We showed how $N = 2$ supersymmetric free fields in two dimensions can be naturally embedded in the extension of the calculus of differential forms due to E. Kähler[5] and M. Atiyah[8]. The methods developed here may be used in higher dimensions. Because the content of independent Dirac fields in a 4 dimensional Dirac-Kähler form is the same as that of an $N = 4$ - supermultiplet, there is even a superficial similarity to the 2 dimensional case discussed here. This case is under investigation. Another relation between a special supersymmetry and Dirac-Kähler forms is described by I.M. Benn and R.W. Tucker[15].

The problems of lattice approximation of quark fields in QCD did rise the interest in the Dirac-Kähler formalism. The geometric lattice continuum correspondence developed in this context should also become a systematic base for the different attempts[16,17] of a lattice approximation of supersymmetry. In this respect the work by H. Aratyn and A.H. Zimerman on the 2-dimensional Wess-Zumino model comes close to the intensions presented in this note. It is possible to get their ad hoc ansatz by specializing some of our formulas. In two very important points their paper exceed the scope of our more systematic considerations. It includes interacting theories as well as a special lattice formulation.

Thus the shortcoming of our paper becomes evident: it is incomplete, it only marks the beginning of a program. In spite of that, I hope that I succeeded in demonstrating how some of the basic ideas of Prof. Schenberg have developed in the last 25 years.

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