CATASTROPHE GEOMETRY IN PHYSICS AND BIOLOGY*

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1. Introduction

In contrast to mathematics, the biological and human sciences are interested in life and the course it takes. If, up to now, these sciences are so little mathematized and remain inexact, this is because our immediate intuitive and qualitative comprehension of biological, social and economic facts is entirely sufficient to cope with the necessities of life. Even in the physical sciences an intuitive, phenomenological approach to complex problems generally is often superior to, and more effective than, precise quantitative methods and, indeed, is always at the root of new fundamental conceptions.

Relying on a qualitative understanding of processes that have proved themselves too complex for rational quantitative analysis is a basic human instinct. Whenever we run into difficulties or are overwhelmed by a mass of data and contradictions -- as in biology and economics today -- it provides us with conceptual guidance to single out the most significant phenomena and to simplify matters to the point where we can come to terms with them. This may be a matter of common sense but, then, finding out just how this structural approach to complex phenomena comes about would add a new dimension to man's knowledge.

Attempting a qualitative description of, and gaining global geometric insight into complex systems dates back to H. Poincaré toward the end of the nineteenth century [1]. Out of it topology was born. It soon followed quite entangled paths in the hands of increasingly abstract mathematicians. Thus it took a hundred years until R. Thom returned to topology's intuitive geometric roots, making it into a fundamental and universally valid basis for a qualitative, structural understanding of complex phenomena in the animate and inanimate world [2]. Truly enough, ever since Pythagorean times the Gods have made geometry. But it took Thom's genius to recognize that growth and change of forms must be attributed to a genericity hypothesis that, in every circumstance, nature realizes that morphology which locally is the simplest and the least fragile compatible with given local initial data. The ensuing concept of structural stability (insensitivity of a system against slight perturbations) leaves but a few possible modes of sudden qualitative changes of structure and behavior -- Thom's elementary catastrophes --, at least as long as nature behaves reasonably smooth. This discovery has, of course, far-reaching consequences, in particular for those inherently qualitative sciences which, justifiably, rely more upon intuition than upon precise quantitative mathematical description, notably biology and the human and behavioral sciences, economics and all that.

However, the observed accordance between empiric morphologies and topology -- this preestablished harmony between nature and mathematics -- has both a geometric and a physical origin. Catastrophe theory by itself can only explain the local geometry and not the forces that are shaping it. On the other hand, looking at local forces alone cannot explain the geometry unless structure formation can be attributed to local minima of entropy production. This presents one of the most challenging problems at this time, viz., to unify catastrophe geometry with the thermodynamics of dissipative structures [3].

The reader may very well pause at this point and ask what on earth we are talking about and why, for that matter, bother with catastrophes in biology. Let me try to answer this question by giving an example from motivation analysis. Rational behavior is a rather rare activity of our nervous system, especially when we are angry and frightened at the same time and then jump from attack to flight and vice versa. Attack and flight are two extreme forms of aggressive behavior caused by the two conflicting drives rage and fear. Rage-only causes attack, fear-only causes flight, neither causes neutral behavior, but both competing drives mixed together cause either attack or flight, depending on chance if equally strong. The catastrophes are the sudden qualitative changes of behavior. It is precisely this story of drives and behavior, causes and effect, K. Lorenz's dogs and N. Tinbergen's seagulls are telling us [4], e.g., the dog in Fig. 1a that faces a rival. It is a topological theorem due to Thom and elucidated by E.C. Zeeman [5], that this elementary behavior catastrophe has to be visualized as the surface of an overhanging cliff whose middle sheet represents the unstable situation and whose two edges, when projected onto the control plane made up by the two drives, form a cusp inside of which the two drives are in conflict (Fig. 1b).

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But what, if, as time goes on, a third competing drive -- say, love, w -- comes into play? Then the single cusp eventually bifurcates into three cusps enclosing a pocket with three conflicting regimes as a new organizing center around the point O in Fig. 1c which marks the beginning of a terrace inside the cliff.

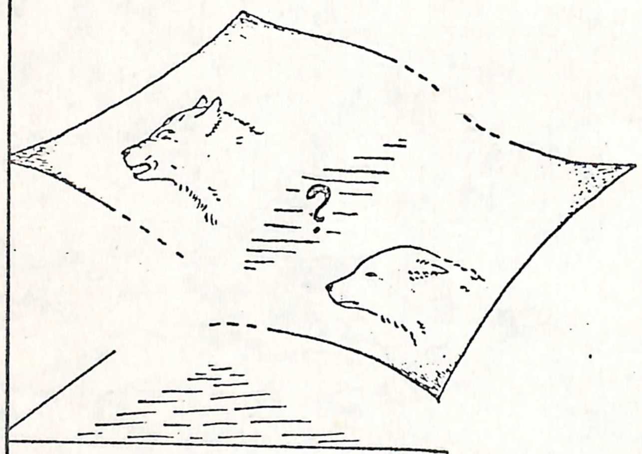


Figure 1a

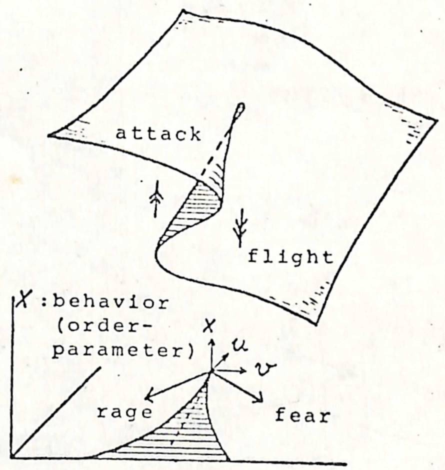


Figure 1b

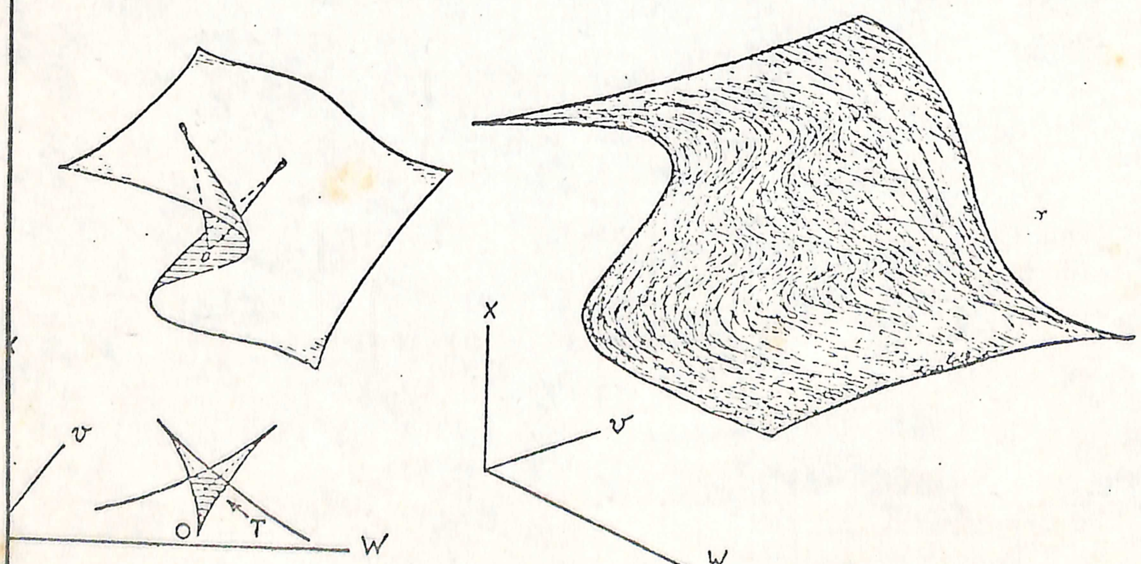


Figure 1c

As another example, these graphs gain immediate physical significance as phase transition surfaces (properly amputated according to the 2nd law of thermodynamics) when Fig. 1b is viewed as van der Waals' diagram for liquid-gas transition [6,11] and T in Fig. 1c is recognized as tricritical point in ternary fluid mixtures, metamagnets, etc. [7,11].

Throughout nature we observe such fascinatingly similar critical phenomena: A laminar flow suddenly turns turbulent, rain turns suddenly into snow crystals, plasmas exhibit shock waves and nerve cells suddenly fire. A gradual relaxation after an economic squeeze causes an inflationary burst, slow changes in environment cause sudden evolutionary or cultural breakdowns, and nations suddenly go to war. Their common characteristics is that one or more significant behavior variables or order parameters (in Landau's terminology) x,y,... undergo sudden large, discontinuous or catastrophic changes if their slow, competing but continuous driving controls u,v,... cross a bifurcation set (the cusps in Figs. 1b,c) and enter conflicting regimes causing instabilities in the behavior variables. The latter measure the degree and kind of ordering or structure which is built up (or destroyed) near the bifurcation set. Examples of behavior variables (x in Fig. 1b) are aggressiveness, particle density and magnetization. Typical control variables (u,v,...) include rage and fear, pressure, temperature, and magnetic field in ferromagnets. One or more controls (v in Fig. 1b) drive the system from one "phase" to another. Other control variables (u in Fig. 1b) drive the "orthogonal" changes, i.e., toward or away from the cusp vertices, and thus act as splitting factors because they separate the behavior pattern into its extremes.

Common to all these catastrophe phenomena (besides their qualitative similarity) is their universality expressed by the fact that the details of the system undergoing sudden transitions are almost irrelevant. While this empirical evidence is at the root of Landau's phenomenological theory of phase transitions [8] and thus hints at thermodynamic principles, already the dimensional analysis underlying familiar scaling laws [9] points to a qualitatively invariant description of the phenomena under consideration. This means that we have to disregard algebraic structures for a while (we cannot add two phases to yield a third one), and confine ourselves to precisely those two concepts which alone appear sufficient to allow qualitative conclusions, viz. order structures and topology. It amounts to saying that the qualitative laws of nature are written in the language of thermodynamics and geometry. This I shall now explain.

2. Dynamical Systems

To explore the topological features of an autonomous dynamical system we describe its temporal evolution by the solutions of the first order differential system

$$dx_i/dt = X_i(x_j,\mu) \quad \text{or} \quad \dot{x} = X(x,\mu) \ . \qquad (1)$$

The state variables $x_i(t)$, depending on time t and forming a vector $x \in R^n$, may represent concentrations of interacting chemical substances, particle positions, etc. The X_i, depending on parameters μ describing external factors, define a vector field X in R^n which, by the existence theorem, determines a unique flow ϕ in R^n. Given a set of initial data, there is a unique trajectory (parametrized by t) or solution curve $x = \phi(t,\mu)$ through each point of R^n, i.e., a curve to which the vector field is everywhere tangent, X giving the speed of the motion at x [10].

The linear harmonic oscillator $\ddot{z} + \gamma\dot{z} + z = 0$ provides the simplest example when written as a first order system $\dot{x} = y$, $\dot{y} = -x-\gamma y$ in position-velocity space R^2 of points $x = z$, $y = \dot{z}$. Its trajectories around the stationary point O, $x = y = 0$, which is reached asymptotically for $t\to\pm\infty$, are shown in Fig. 2.

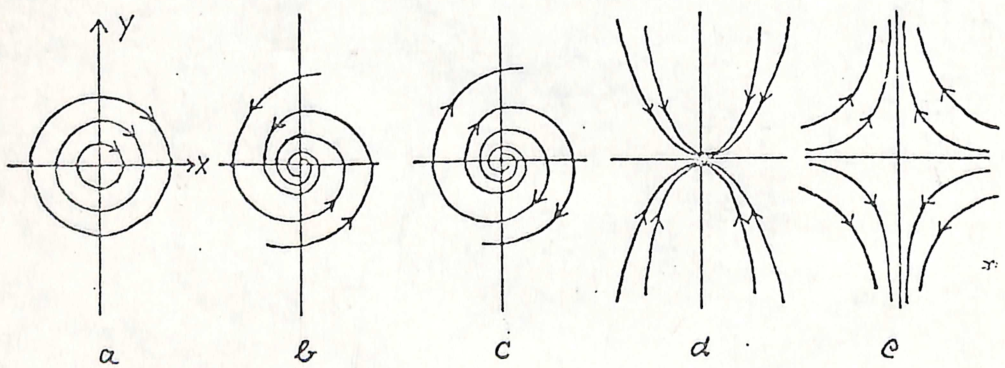


Figure 2

For $\gamma = 0$ the orbits are concentric circles around the origin O which is a center (Fig. 2a). For $\gamma > 0$, O is an attractor with everything outside spiraling toward it (Fig. 2b), while for $\gamma < 0$, O is a repellor (Fig. 2c). In the aperiodic case (Fig. 2d) O is a stable node. O becomes a saddle point (Fig. 2e) if the last term z in the oscillator equation is replaced by $-z$. New phenomena are springing up when nonlinearities come into play, as in van der Pol's self-sustained relaxation oscillator obtained by replacing the constant damping γ by an amplitude-dependent one, $\gamma = 3z^2-\mu$, μ real, to yield $\ddot{z} + (3z^2-\mu)\dot{z} + z = 0$ or the equivalent system $\dot{x} = -x^3 + \mu x - y$, $\dot{y} = x$. For $\mu < 0$, the stationary point O, $x = y = 0$, is a stable attractor as in Fig. 2d. If μ crosses the value $\mu_c = 0$ and becomes positive, O changes into a repellor and there appears a periodic, self-sustained oscillation or limit cycle Γ (Fig. 3a) which itself is a stable attractor: Everything inside spirals out toward the closed orbit Γ and everything outside spirals in. We call $R^2 - O$ the basin of attraction of Γ. It is no accident that a plane u = const. through the cliff of Figs. 1b,8 gives the dotted Z-shaped curve in Fig. 3a.

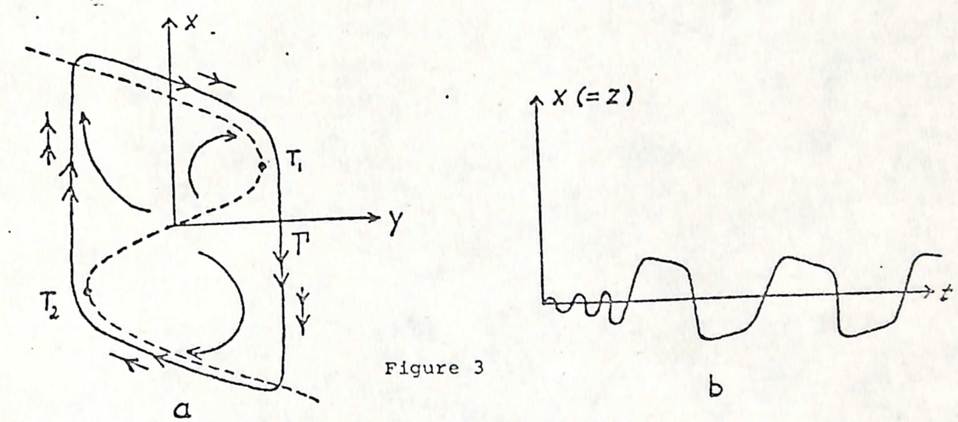


Figure 3

Indeed, for $\mu > 0$ and z small (when $\gamma < 0$) the system absorbs energy from its environment and the amplitude increases, but when it becomes too large ($\gamma > 0$) energy will be dissipated and the amplitude decreases again. Then the game starts anew (Fig. 3b). Absorption and dissipation of energy tend to balance each other throughout the motion ($\oint_\Gamma dE = 0$) and, whatever the initial state of the system, it tends to the periodic oscillation Γ. Along nearly horizontal sections of the trajectories the motion is very slow (single arrows) whereas the representative phase points move very fast along nearly vertical sections of the orbits (double arrows): slow and fast motions are transversal except for the thresholds T_1, T_2 where they undergo sudden transitions (catastrophes). Identifying the hysteresis phenomenon underlying van der Pol's relaxation oscillator with the behavior of transistors and ferromagnets [11], circadian clocks [12], the heartbeat [13] and other oscillatory systems nowadays is a familiar exercise.

A singular or stationary point $x_o = x_o(\mu)$ of the flow described by Equ. (1) is defined by $X(x_o,\mu) = 0$. The behavior of a flow near, and the stability properties of, a singular point are determined by the location of the eigenvalues $\lambda = \lambda(\mu)$ of the Jacobian matrix $J = (\partial X_i/\partial x_k)_{x_o} = J(x_o)$ in the complex λ-plane. This is because in the linear approximation to Equ. (1) the solutions vary in time as $\exp(\lambda t)$. The point x_o is an attractor (stable equilibrium) of the system if $\mathrm{Re}\lambda(\mu) < 0$ for all λ. Suppose now that, as μ increases beyond a critical value μ_c, a pair of complex eigenvalues cross the imaginary axis. Then, for $\mu > \mu_c$, the point x_o is no longer attracting but usually changes into a repellor and there appears a new, isolated attracting periodic solution Γ_μ of (1) (limit cycle) close to x_o (Fig. 4) whose amplitude is proportional to $\sqrt{\mu-\mu_c}$ in first approximation.

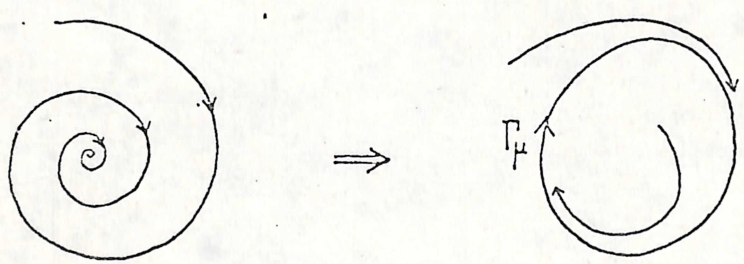


Figure 4

This phenomenon, where out of a stable structure a qualitatively new structure emerges, is called (Hopf-) bifurcation. The limit cycle Γ_μ may lose its stability as μ increases further past another critical value μ'_c where the periodic oscillation may bifurcate into one of longer period or into a nonperiodic, "turbulent" one. Bifurcation phenomena are typical of nonlinear dissipative systems and have been observed in various chemical reactions [14], hydrodynamics (turbulence) [15], etc. but the theory is still in its infancy [16] with the exception of gradient systems [2]. The onset of bifurcation may be understood geometrically by visualizing singular points x_o as loci of intersections of hyperplanes defined by the equation system $X(x_o,\mu) = 0$. For example, in $R^2(x_1,x_2)$ a singular point x_o is given by the intersection of the two solution curves C_1, C_2 of the equations $X_1(x_1,x_2,\mu) = 0$, $X_2(x_1,x_2,\mu) = 0$, respectively. For a given μ, C_1 and C_2 may intersect transversally (i.e., "without contact") as in Fig. 5a (x_o non-degenerate: $\det J \neq 0$) or with a common tangent (i.e., "with contact") as in Figs. 5b,c (x_o degenerate: $\det J = 0$). Changing μ results in a deformation of the curves, say, $C_2 \to C'_2 = C_2 + \delta C_2$, which in turn gives rise to new intersections near x_o and new types of singularities (if $\det J = 0$) as shown in Figs. 5d,e, and, consequently, to new topological situations and modes of behavior. We remark that the distinction between intersections with and without contact underlies Ehrenfest's original classification of phase transitions.

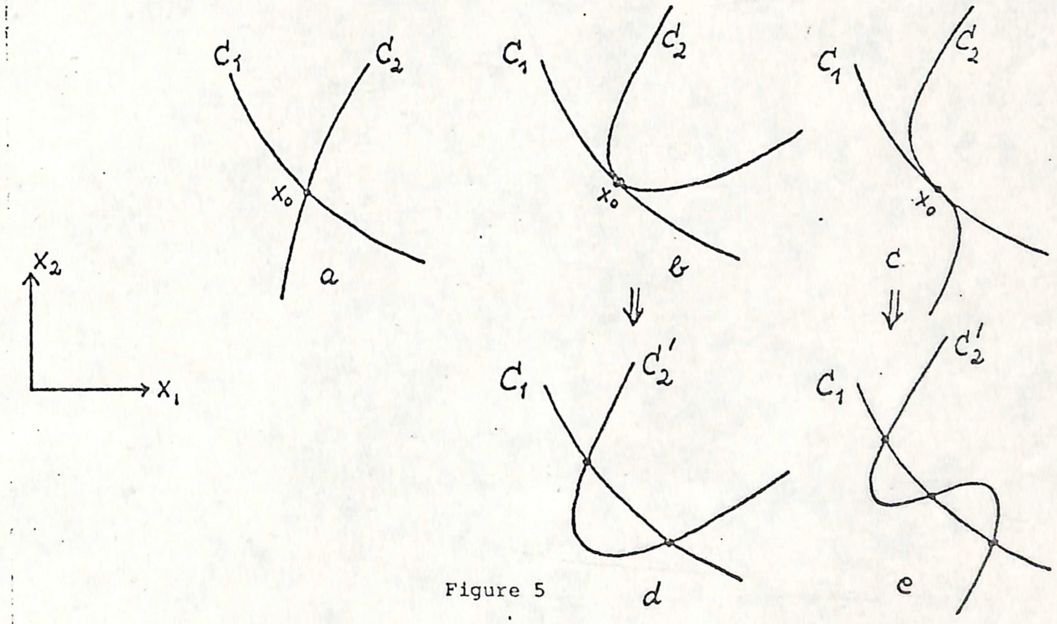


Figure 5

After all that has been said above, physicists who are used anyway to believing that the world is made up of oscillators will have no problems in accepting the following generalizations, though their differential-topological formulation is difficult. We define a dynamical system (M,X) as a smooth vector field X on a smooth (and compact) manifold $M \subset R^n$. Smooth means sufficiently differentiable. We call "X-orbit" a curve in M to which X is everywhere tangent. X determines a unique flow ϕ on M (i.e. a smooth map ϕ: $R \times M \to M$ such that $\forall t \in R$, ϕ: $t \times M \to M$ is a diffeomorphism). The basic geometric units making up the system are (i) a set of fixed points: attractors (sinks), repellors (sources) and saddle points, (ii) a set of periodic orbits or limit cycles (attractors and repellors) [adding saddle-type "horseshoes", the system is called a

Smale flow if the number of units is finite]. Besides, there may be centers which, however, we shall soon eliminate. We might further expect that almost all of the space M may be partitioned into the basins of attractors (towards which almost all orbits eventually flow) and that this geometric structure characterizes entirely the qualitative behavior of the system.

The aim of this construction is not to study individual trajectories (or solutions to Equ. (1)) but to describe qualitatively the family of all the orbits as a whole from a global geometric point of view, and to find out if and how the system undergoes drastic qualitative changes of shape and form under perturbations.

3. Structural Stability

Many systems and structures encountered in nature enjoy an inherent stability property: they preserve their quality under slight distortions. Otherwise we could hardly think about or describe them, and today's experiment would not reproduce yesterday's result. We give the scheme (M,X) some content by this hypothesis of structural stability. We call the dynamical system (M,X) structurally stable if, for a sufficiently small perturbation δX of the vector field X, the perturbed system $(M, X+\delta X)$ is, roughly speaking, topologically equivalent to the unperturbed system. Thus the system (1), $\dot{x} = X$, is structurally stable if there exists a continuous one-to-one map that carries each trajectory of the perturbed system (with $|\delta X| < \varepsilon \quad |\partial \delta X| < \varepsilon$)

$$\dot{x} = X + \delta X \qquad (2)$$

onto a trajectory of the unperturbed one.

By tilting all arrows (X) in the flow of Fig. 2a of the undamped harmonic oscillator slightly inwards (by δX), all the circular orbits change their quality basically and become infinite spirals towards the center, as in the damped oscillator (Fig. 2b). Thus, the undamped oscillator is structurally unstable, and so are all conservative Hamiltonian systems: adding a small damping term results in energy dissipation, the trajectories spiral down to energy minima and the dissipative flow is no longer equivalent to the conservative flow. On the other hand, dissipative systems such as the damped oscillator (Fig. 2b) and van der Pol's oscillator (Fig. 3) are structurally stable (for $\gamma > 0$ and $\mu > 0$, respectively): tilting the vectors by δX turns spirals into spirals. Considering a system (1) controlled by parameters μ, we may interpret a small change in μ as a perturbation δX. Then, according to what has been said in the context of Fig. 5, the system may suddenly change in quality, attractors might bifurcate or coalesce if singular points degenerate and μ crosses a critical set. The study of such changes is bifurcation and catastrophe theory.

4. Catastrophe Theory

To characterize structurally stable systems we must consider the whole set $\mathbb{D}$ of vector fields on M, i.e., a space $\mathbb{D}$ of dynamical system: and choose an equivalence relation and a topology on the set $\mathbb{D}$. We ma call $X, X' \in \mathbb{D}$ equivalent if a homeomorphism of M exists throwing X-orb. on X'-orbits, and choose a C^0- (or C^1-) topology on $\mathbb{D}$, i.e., define X,X ($X' = X+\delta X$) to be close if the vectors (and their derivatives) are clo: Then define $X \in \mathbb{D}$ to be structurally stable if it has a neighborhood of equivalents in $\mathbb{D}$. Besides classifying structurally stable systems, th outstanding question is to know if they are dense, that is, if any sys tem can be approximated by structurally stable ones, so that almost al systems are structurally stable and one can ignore the unstable residu This difficult problem gave rise to deep mathematical studies [10, 18] showing for example, that Smale flows (Sec. 2) are structurally stable and dense in the C^0-topology. Suppose we had developed a good stabili theory and could decompose $\mathbb{D}$ into an open-dense set $\mathbb{S} \subset \mathbb{D}$ of structural stable (stable, for short) systems and the complementary bifurcation set $\Sigma = \mathbb{D} - \mathbb{S}$ of unstable systems, $\mathbb{D} = \mathbb{S} \cup \Sigma$ (Fig. 6).

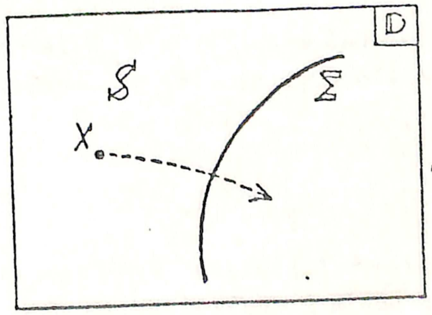


Figure 6

Consider then a system (X in Fig. 6) controlled by a parameter. The system's change can be represented by an arc in $\mathbb{D}$. If the arc crosses Σ, the system changes its quality. The crossing points are called catastrophes.

Naturally enough, the geometric structure of the bifurcation set of vector fields may be very wild, and the only category for which the program has thus far proved feasible is that of a local gradient dynamics $V: M \to R$, $X = -\text{grad } V$. Its singularities make up those of the map V, and perturbing the system is related to the problem of stability of differentiable maps between two manifolds (cf., however [17]). Notwithstanding its limitations, gradient systems offer an exceptionally rich and beautiful approach to qualitative dynamics.

Let the vector field $X = X(x,\mu)$ be derived from a real-valued, differentiable potential $V = V_\mu(x)$, depending on a parameter set μ,

$$X(x,\mu) = -\underset{x}{\text{grad}}\, V_\mu(x), \tag{3}$$

so that Equ. (1) becomes the gradient system

$$\dot{x} = -\underset{x}{\text{grad}}\, V_\mu(x) \quad \text{or} \quad \dot{x}_i = -\partial V_\mu(x)/\partial x_i \ . \tag{4}$$

The singular points x_o (attractors, repellors, saddles) of the flow $(X(x_o) = 0)$ are then just the local extrema (minima, maxima, inflexion points) of the potential V, grad $V(x_o) = 0$, i.e., its stationary or equilibrium points. We call x_o structurally stable (or a non-degenerate quadratic point of the map $x \to \nabla V$) if the Hessian $H(x_o) := |\partial^2 V/\partial x_i \partial x_k|_{x=x_o} \neq 0$ (i.e., if $\det J \neq 0$, cf. Sec. 2). Around such a point, V is a non-degenerate quadratic form and perturbing V by δV ($\partial^\nu \delta V/\partial x_i^\nu$ small for all ν and i) does not change its shape. For example, the potential $V = x^2$ $(x \in R^1)$ is invariant against small perturbations $\delta V = ax + b$ near its stationary point $x_o = 0$: $V + \delta V$ has the same form as V because $H(0) = V''(0) \neq 0$ (Fig. 7a). If, however, $H(x_o) = 0$, i.e. if x_o is not structurally stable but is a degenerate or inflexion point, then a perturbation of V by δV results in a number (assumed finite) of topologically and physically different types of potentials $V + \delta V$, called universal unfoldings of V (or of x_o). For example, a perturbation of $V = V_o = x^3/3$ $(x \in R^1)$ around its stationary point $x_o = 0$ by $\delta V = ax^2 + bx + c$ gives $V_b := V + \delta V = x^3/3 + bx$ (since the quadratic and constant terms can be removed by translations). This perturbed potential $V_b(x)$ is quite different from $V_o = V_{b=0} = V$ in shape for $b < 0$ (where it has two extrema) and $b > 0$ (where it has none): Fig.7b. The point $x_o = 0$ $(H(0) = V''(0) = 0)$ is structurally unstable and so is the form of $V = V_o$, in that a small change of the parameter b in V_b around $b = 0$ changes the shape of V_b completely. On the other hand, changing b a bit within the domains $b > 0$ or $b < 0$ leaves the form of V_b qualitatively unaffected. For $b > 0$ or $b < 0$, $V_b(x)$ remains geometrically the same, i.e., V_b is structurally stable under small parameter variations in these domains. We call V_b the "universal unfolding" of $V = V_o$ and $b = 0$ a "catastrophe point" of the family of potentials $V_b(x)$ that separates two stable regions $b \lessgtr 0$. V_b determines a fold curve. The singularity $x=0$, or V_o, acts as an "organizing center".

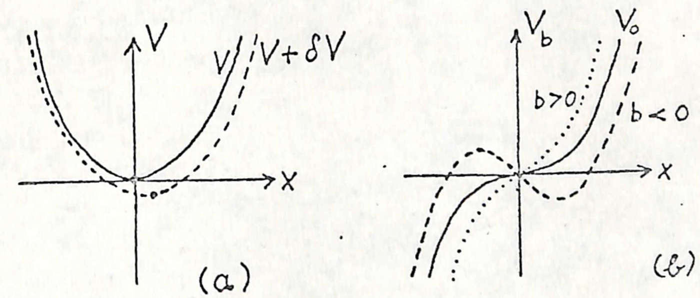


Figure 7

Similarly, the potential $V_{oo} = x^4/4$ has a degenerate minimum at $x = x_o = 0$. The perturbed potential or universal unfolding of V_{oo} is given (near $x = 0$) by

$$V_{uv}(x) = x^4/4 + ux^2/2 + vx \tag{5}$$

with two "unfolding parameters" u,v (cubic and constant terms can be removed by translations, and higher powers of x are negligible in a neighborhood of $x = 0$). The stationary points x_o of $V_{uv}(x)$ are given by the roots of the equation

$$S: \quad \text{grad}V = dV_{uv}(x)/dx = x^3 + ux + v = 0. \qquad (6)$$

The set $S(V_{uv}): x = x_o = x(u,v)$ of these equilibrium points form a surface S in (x,u,v)-space $R^3 = R^1 \times C$, $C = R^2$, showing the beginning (at $x=u=v=0$) of an overhanging cliff made up of two folds (Figs. 8a, 1b), which is best modeled by a rubber sheet. For fixed $(u,v) \in C$ (control space) the points $x = x_o$ $(=x_1, x_2, x_3)$ on S give the local minima (regimes) and maxima of V_{uv} at (u,v), corresponding respectively to the upper or lower and the middle sheet of the surface S. The various forms of $V_{uv}(x)$ when u,v vary in C are shown in Fig. 8b. The cliff of Fig. 8 (and the double cliff of Fig. 1c) may be modeled by rubber sheets. The analogy of (6) to the Van der Pol case is obvious.

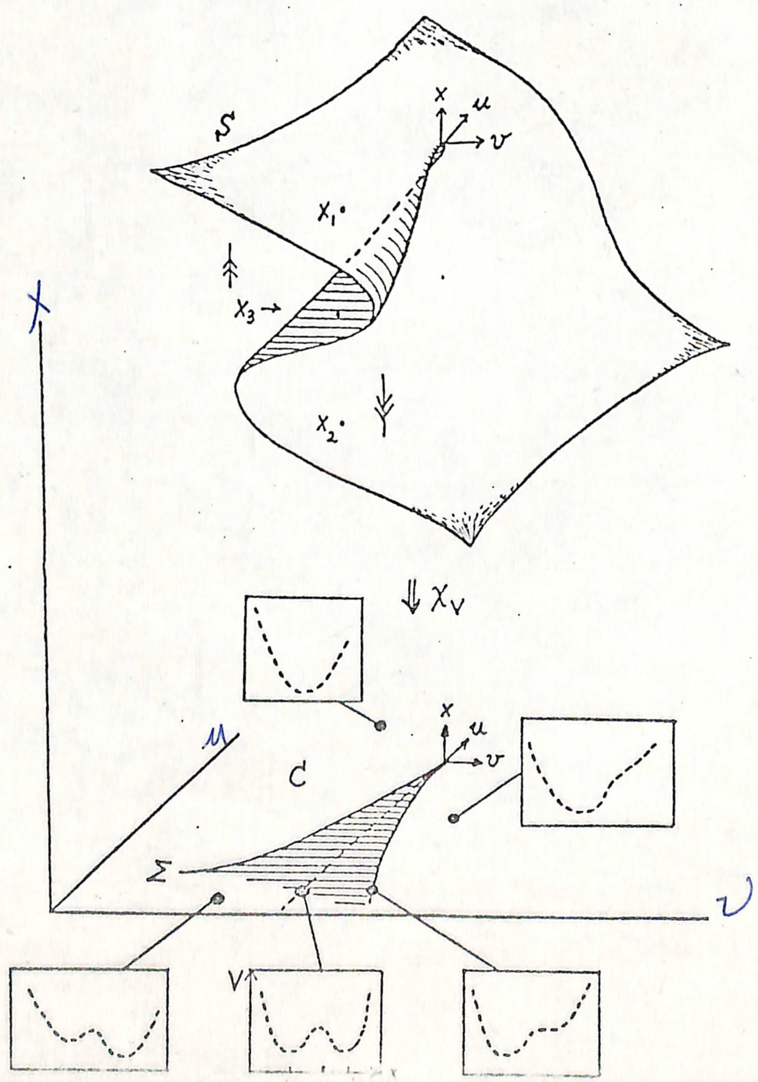


Figure 8a

Figure 8b

The cusp (bifurcation set) Σ in Fig. 8b is the (vertical) projection onto the plane C of the two edges of the cliff S of Fig. 8a (where the tangents to S become vertical: $V''_{uv}(x) = 0$), and thus follows by eliminating x from Equ. (6) and

$$d^2V_{uv}(x)/dx^2 = 3x^2 + u = 0 \qquad (7)$$

to yield the cusp equation

$$\Sigma: \quad 4u^3 + 27v^2 = 0. \qquad (8)$$

Σ is the set of (u,v)-values for which two stationary points (a minimum and a maximum) of V_{uv} coalesce. Imagine V_{uv} to describe locally a geographic landscape varying in shape by forces u,v which change the number, position and relative heights of valleys. Suppose that the stable states accessible to the system are given by the non-degenerate quadratic minima of V_{uv} (the attractors reached by the trajectory of a material point) and corresponding to the upper and lower sheets of the cliff while the maximum represents the unstable state and corresponds to the cliff's middle sheet. Then, crossing the cusp Σ from inside ($4u^3 + 27v^2 >$ 0) to outside ($4u^3 + 27v^2 < 0$) or vice versa annihilates an attractor or generates a new one, and the state must either flow towards a different attractor or decide between two possible minima where to stay or go on to, until one of them disappears when the other branch of the cusp is crossed. If the speed of the flow is large compared with the speed of change of control--which in general is the case and can be formally achieved by multiplying the l.h.s. of Equ. (4) by a small parameter ε--then the system will appear to jump into a qualitatively new state when Σ is crossed. These jumps, from the cliff's upper sheet over its edge down to the lower sheet, or up at the other side, are the "catastrophes" and the cusp Σ is the catastrophe set. The two examples given in Sec. 1 in the context of Fig. 1b illustrate precisely this "cusp catastrophe". Why is it so fundamental?

We have embedded the potential $V = V_{oo}(x)$ (a germ) into the two-parameter family of potentials $V_{uv}(x)$ (the unfolding of V_{oo}). While V_{oo} is a very fragile object (because its singular point is degenerate) the forms of V_{uv} remain basically the same inside or outside the cusp, respectively: $V_{uv}(x)$ is stable under small perturbations not crossing Σ. The cusp is a singularity of the map $\chi = \chi_V$ of the surface S onto the plane C, $\chi: S \to C$, induced by the projection $R^3 = R^1 \times C \to C$. Intuitively, bending S (or V_{uv}) smoothly without creasing the rubber sheet (by a diffeomorphism h) into a surface S', and deforming C smoothly into another plane surface C' by a change of coordinates (diffeomorphism k) does not disturb the qualities of the cusp catastrophe: the ensuing map χ' of S' onto C', χ':S'→C', remains equivalent to χ (i.e., equal up to smooth transformations). Consider then the space **V** of all 2-parameter families of potentials $V = V(x;u,v)$ and their surfaces S_V of stationary points defined by $dV/dx = 0$. Let us call $V \in \mathbf{V}$ structurally stable or generic if it has a neighborhood of equivalents, i.e., if χ_V is equivalent to $\chi_{V'}$ for all $V' = V + \delta V \in \mathbf{V}$ close to V. Then, and this is the content of the Thom-Whitney theorem, the only singularities (defined by $dV/dx = d^2V/dx^2 = 0$) of the projection $S_V \to C$ are folds and cusps. This means that the most complicated behavior that can happen locally is the cusp catastrophe of Fig. 8 and that any $V(x;u,v)$ (u,v playing the role of μ in Equs. (3), (4)) is locally equivalent to $V_{uv}(x)$, with the cusp singularity characterizing the breakdown of stability. While this is a geometric fact, its physical meaning is obvious: if a system with order parameter x is slowly driven from one phase to another by a control v, and an orthogonal drive u sets in to split the quality of the phases (different states of order or symmetry, etc.) and if a phase may persist for a while with the transition to the other delayed, the cusp catastrophe is intuitively the simplest model and, as we have seen, the least fragile.

Extending these considerations to potentials $V_\mu(x)$ with n behavior variables or order parameters $x \in R^n$, and k control parameters $\mu \in R^k$, and classifying the elementary modes of sudden qualitative changes of behavior and structure -- the elementary catastrophes -- which are locally possible in structurally stable (gradient) systems, leads to the following adaptation of Thom's theorem (generalized by Mather, Siersma and others [18]):

Theorem. Let R^n be the space of states x (behavior variables, order parameters etc., with coordinates x_i $(i=1,2,\ldots,n)$ denoted by $x,y,z,\ldots$) and let $C = R^k$ $(k \leq 6)$ be the control space of parameters μ (with coordinates $u,v,w,\ldots$). Let **V** be the space of generic potentials $V = V_\mu(x)$, i.e. of smooth (C^∞) maps $V: R^n \times C \to R$. Then there exists an open-dense subset of **V** such that for almost every $V \in \mathbf{V}$: (i) the surface S of stationary points of V, defined by $\mathrm{grad}_x V = 0$, forms a smooth manifold $S \subset R^n \times C$; (ii) any singularity ($\mathrm{grad}_x V = 0$, $H = |\partial^2 V/\partial x_i \partial x_k| = 0$) of the map χ_V: S→C is equivalent to one of the elementary catastrophes supposed to be at $x = \mu = 0$ and listed below; and (iii) χ_V is stable under small perturbations of V.

if this k was greater than 6, the number of critical points becomes infinite, Proof in complex and in topology

The dimension k of C determines the number (m) of basically different catastrophes: k(m) = 1(1), 2(2), 3(5), 4(7), 5(11), 6(14),... . It is important that the number n of state variables x_i does not enter into the classification (at least for k < 7). Thus, when $k \leq 6$ and there are $n = 10^3$ or more state variables x (concentrations, spins, etc.) in a system, its local behavior is still governed by $m \leq 14$ elementary, irreducible modes. They all exhibit striking sudden changes or catastrophes (though smoothed out by noise) when μ crosses the image in C of the set of singularities of χ_V, the bifurcation or catastrophe set $\Sigma \subset C$, determined by the map χ_V of S onto the parameter (μ-) space C, viz., by eliminating x from $\mathrm{grad}\, V_\mu = 0$, $|\partial^2 V_\mu(x)/\partial x_i \partial x_k| = 0$.

List of elementary catastrophes:

Name	k (dim C)	n' (corang R^n)	Universal Unfolding $V_\mu(x)$	Fig.
Fold	1	1	$x^3/3 + bx$	9
Cusp	2	1	$x^4/4 + ux^2/2 + vx$	8
Swallowtail	3	1	$x^5/5 + ux^3/3 + vx^2/2 + wx$	10
Butterfly	4	1	$x^6/6 + tx^4/4 + ux^3/3 + vx^2/2 + wx$	11 & 1c
Hyperbolic Umbilic	3	2	$x^3 + y^3 + wxy + ux + vy$	12a
Elliptic Umbilic	3	2	$x^3 - 3xy^2 + w(x^2 + y^2) + ux + vy$	12b
Parabolic Umbilic	4	2	$x^2y + y^4 + wx^2 + ty^2 + ux + vy$	12c
Higher Order catastrophes:				
Wigwam	5	1	$x^7/7 + px^5/5 + tx^4/4 + \ldots$ (as in Butterfly)	13a
Star	6	1	$x^8/8 + qx^6/6 + \ldots$ (as in Wigwam)	13b
Double Cusp	7	2	$x^4 + y^4 + \ldots$	14
....	..	..	......	

What the above theorem tells a physicist is this: In any gradient-like physical situation where continuously changing controls $\mu \in C$ cause discontinuous changes (possibly smeared out by noise) in a state space R^n of any dimension, one can choose local coordinates (in R^n and $C = R^k$) such that the modes of change which can occur topologically are one of the above list. Thus if, for example, we have a system with n order barameters $x \in R^n$, controlled by, say 4 parameters $\mu = (u,v,w,t) \in R^4$, then its behavior is locally governed by the first 7 catastrophes of the list, and of the n states but n' = 2 (x,y) remain significant order parameters. The challenge lies in singling them out in observed phenomena. Lower-order catastrophes are trivially contained in higer-order ones, and however complicated V may be, they are always equivalent to the polynomials (the universal unfoldings) in the list.

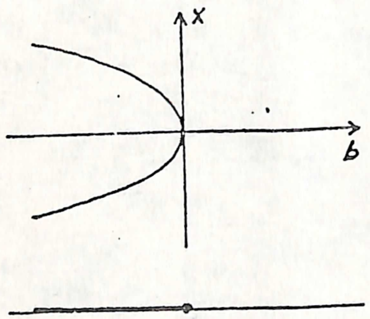


Figure 9

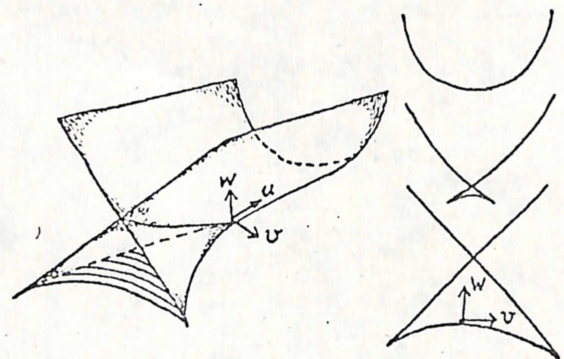


Figure 10a

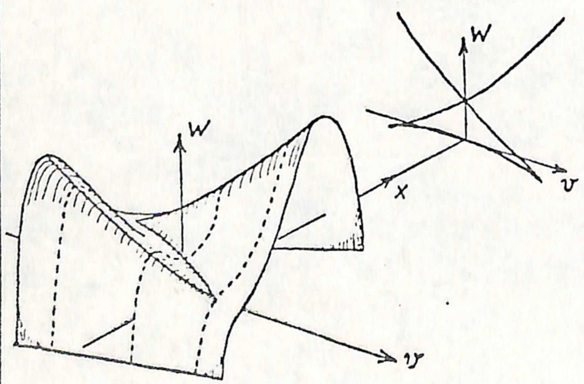


Figure 10b

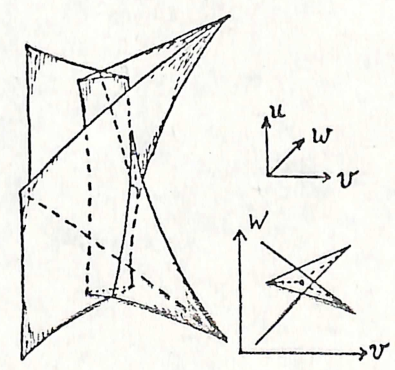
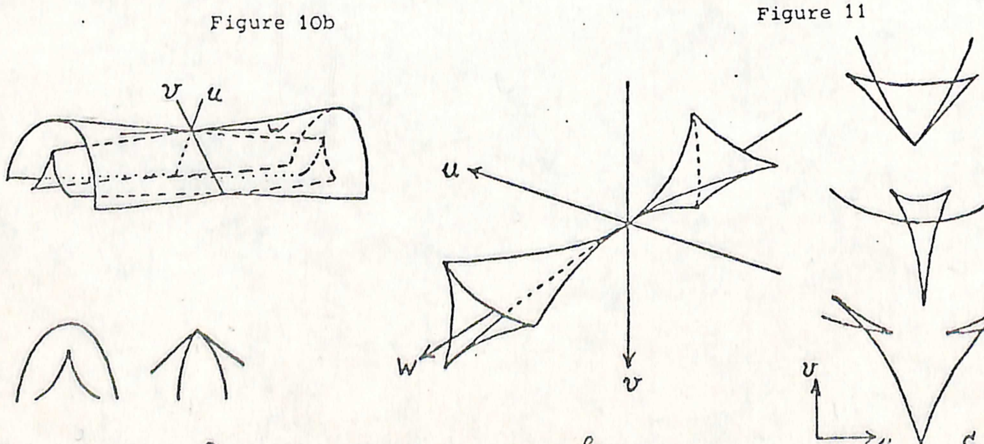


Figure 11



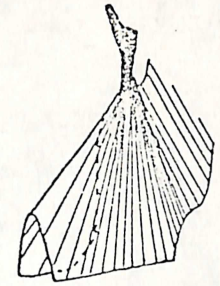


Figure 13a

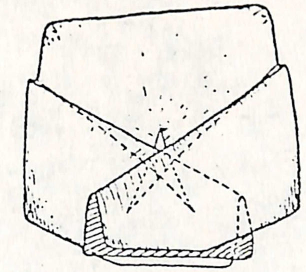


Figure 13b

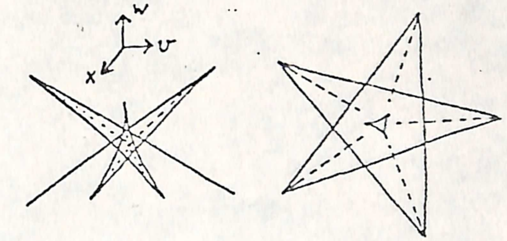


Figure 14

The restriction to gradient systems can be relaxed somewhat by considering any differential system on R^n, parametrized by μ, such that $V_\mu(x)$, V_μ: $R^n \to R$, decreases along trajectories. V may be an energy, cost or Lyapunov function. But, in any case, M must have only isolated fixed points corresponding to attractors (minima of V), repellors (maxima of V) and saddle points of the vector field, so that almost every trajectory eventually reaches a minimum. Recurrences are obviously excluded. Some physical principles must be superimposed on the geometry even at this stage, because the systems considered are basically static. With a terminology borrowed from phase transition phenomena, these principles include: (i) a state may always select the lowest available minimum (Maxwell convention), (ii) the state may remain at a minimum until this disappears (delay or hysteresis convention) or a threshold between another one becomes low enough or, (iii) a state may choose a certain minimum as soon as it appears (saturation convention), etc.

When the internal dynamics of a system is not of gradient type, the theory of bifurcation becomes very difficult, and is virtually unexplore But one may expect that the gradient-like situation keeps some validity though generalized catastrophes may spring up. For example, in a Morse-Smale system, made up of finite sets of fixed points and closed orbits (limit cycles) -- as, e.g. in the van der Pol case (Sec. 2) -- bifurcation is governed by the elementary catastrophes. However, extending the theory to cope with the creation, coalescence and bifurcation of limit cycles and high-dimensional attractors, passing to infinite-dimensional spaces and providing a dynamical setting for catastrophes in physical terms remain major challenges for further research. In pursuing this program one may then associate with each $\mu \in C$ a vector field $X_\mu(x)$ on a state space M and defined by $\dot{x} = X_\mu(x)$, $x \in M$. A vector field on M×C (and tangent to the fiber M) defines a metabolic field G on C, G generating a map of C into the space $\mathbb{D}$ of vector fields. If μ is on $\Sigma \subset C$, G_μ is a field of Σ (the complement of $\mathcal{S}$ in Fig. 6) and the local catastrophe set Σ in C appears as the counter-image, $\Sigma = G^{-1}(J)$, of a universal catastrophe set J in D if the map is transversal to J. A domain in C, e.g. space-time, with a metabolic field on it defines a morphogenetic field. When G is not transversal to Σ, the induced morphology G^{-1} becomes quite complicated and little can be said about the ensuing generalized catastrophes. On this background Thom has discussed a wealth of structures in the animate and inanimate world from the qualitative point of view, covering an enormous spectrum of material from developmental biology to linguistics to the structure of galaxies [2].

There remains a gap between the general theory of structurally stable models Thom puts forth and practical schemes for applying the theory of classification and deformation of singularities in the sciences. It is for this reason that the theory often looks more like an art than a science.

5. Catastrophes in Physics and Biology

When one accepts the idea that, because of their universality, critical phenomena in nature have a common topological basis, they must be governed by unfolding singularities in a structurally stable way, i.e. by the irreducible elementary catastrophes. Their physical origin may, roughly speaking, be traced to nonlinearities due to heterogeneities in otherwise homogeneous media (semiconductors, crystals, membranes, etc.). And it is in fact nonlinear dynamics where catastrophe geometry promises to provide both a general framework and a method. In a physical system with order parameters or behavior variables $x \in M$, driven by competing controls $\mu \in C$, one assumes tentatively a local gradient dynamics and chooses a "potential function" $V = V_o(x)$ having the origin (representing, in general, highest disorder or symmetry) as a degenerate singularity. This function acts as an "organizing center" and may represent, when unfolded into $V_\mu(x)$, a free energy , Lyapunov functions such as excess entropy or information gain etc. In biological systems, M may, for example, represent the state space of cells and C the space of morphogenes, with a subsequent map onto space-time revealing morphogenesis as images of catastrophes caused, e.g. by competing concentration gradients.

Having found the elementary geometric laws governing a structurally stable catastrophic world, let us try to understand them in physical terms. To begin with, we start from the linear oscillator's next best structurally stable version, the forced pendulum's anharmonic Duffing approximation, whose displacement $z(t)$ satisfies the equation $\ddot{z}+\gamma\dot{z}+\omega_o^2 z+\alpha z^3=F\sin\omega t$. To lowest order, neglecting γ for the moment, its iterative solution is $z(t)=A\sin\omega t+S(A,F)\sin 3\omega t$, where $3\alpha\Lambda^3/4-(\omega^2-\omega_o^2)\Lambda-F=0$ and $S=\text{const.}\Lambda^3/[\Lambda^3+4\Lambda\omega_o^2/3\alpha-4F/3\alpha]$. Identifying $\Lambda\to x$, $(\omega^2-\omega_o^2)\to u$, $F\to -v$, the oscillator's nonlinear resonance response x to the drives u,v is seen to be described by the overhanging cliff of the cusp catastrophe (Figs. 1b,8). A plane u=const through the cliff's surface shows the familiar resonance curve and hysteresis cycle, and the domain inside Σ corresponds to in/out-of-phase behavior. The poles of S determine the same surface, which is indicative of the role played by catastrophes in S-matrix theory [19]. Next to the shadow of our dog's mind (the area inside the cusp of Figs. 1a,1b where $-u-v=$rage and $-u+v=$fear) comes, of course, the transition of a ferroelectric near the Curie point from the non-pyroelectric to the pyroelectric phase. We describe it by the unfolding of the thermodynamic potential $V=\phi(D)$ around its singularity $D=0$ (D=x=polarization). With the simplest crystalline symmetry assumption and confining ourselves to a 2nd order transition (k=2), i.e., $V\to V_{uv}$, one finds Equ. (6) with the identification $u=T-T_c$, $v=E$ (E=electric field), and the behavior near the critical temperature T_c is therefore governed by the cusp catastrophe in accordance with the Ginzburg-Landau theory [8,11].

Proceeding with oscillator models, we observe that subharmonic oscillations start to bifurcate from harmonic ones as we go on to higher approximations, or to higher-order nonlinearities. They gain particular importance in entrainment and synchronization phenomena that abound in biological systems [24]. A relatively simple example is obtained by combining the van der Pol oscillator (Sec. 2) with the above anharmonic one into the equation

$$\ddot{z} + \gamma(z^2-\beta)\dot{z} + z + \alpha z^3 = f(t) \qquad (9)$$

where f is the external stimulus. Assuming $f=F\sin\omega t$, determining the periodic solution z(t) and examining its stability in the ($F=w$, $\omega=v$)-plane exhibits a swallowtail-like situation (Fig. 10b) with the region below the cusps' overlaps corresponding to harmonic entrainment, while beats occur beyond that domain. We hope to classify the--admittedly rather messy--entrainment phenomena in terms of catastrophes in a forthcoming paper [20]. They are difficult to analyze even topologically because of the occurrence of Smale's "horseshoes".

Not satisfied to learn catastrophe physics in terms of a single oscillator, let us consider many, and indeed a continuum of oscillators. That's where field theory starts, classical or quantized, and the trouble begins--also with the catastrophes, because their geometry becomes now a part of a much more comprehensive theory which is as yet almost unexplored. If we consider a partial differential equation, whose characteristics satisfy Equ. (4), it is clear that the solutions of the latter, and therefore the catastrophes, characterize the propagation of singularities of the former [17]. When these principles are suitably extended and bifurcation theory [16] finally joins catastrophe theory [2], a practicable scheme for handling singularities and modeling continuum phenomena in the sciences will ultimately emerge. But even at the present stage, recognizing the cuspoids of the corang-1-singularities as loci of caustics and projections of ruled surfaces leaves no one surprised at finding magnetohydrodynamic [21] or even chemical shock waves to be the result of cusp geometries [20]--fusion, and fission for that matter, being commonly accepted catastrophes anyway [25]. The single-mode laser [22] is, perhaps, the best known example of a continuous set of nonlinear oscillators of the above type, exhibiting sudden transitions in power output if the input exceeds threshold. From the microscopic theory with its enormous number of degrees of freedom, Haken derived for the electric field strength E of the laser light the equation $\dot{E} = a\nabla^2 E + bE + cE^3 + F$ which, when the space-dependent part $\nabla^2 E$ and the fluctuation force F are neglected, reduces (via $\dot{E} = -\text{grad}V(E)$) to the (properly amputated) cusp catastrophe for the onset of coherent radiation. This is no longer surprising when one views the onset of laser radiation as a phase transition. Nor should, then, the chemists among you be surprised on discovering that the very same laser equation--with E replaced by chemical concentration variables Fe^{++}/Fe^{--}, etc.--is at the root of Winfree's scroll wave models [23].

Phase transitions--in ferroelectrics, antiferromagnets (metamagents), ^3He-^4He solutions, multicomponent-fluid mixtures, chemical reactions, particle reactions, and so on--are, as you will believe me by now, a conceptually simple theoretical (though not simple experimental) laboratory for catastrophe geometry, and, in fact, provide its most direct physical realization. It is, of course not so that one should look at Fig. 1b as representing the liquid-gas transition, which rather is given by the amputated swallowtail (Fig. 10b) of the Gibbs free energy G(P,T) [11]. Instead, we have to start from the microscopic dynamics and--this is the clue offered by our theorem in Sec. 4--single out (via the grand canonical partition function, by averaging, by the Bogoliubov-Uhlenbeck contraction, etc.) of the underlying R^n those few (n' = corang R^n) macroscopic order parameters that are ultimately the most relevant observable physical ones [11]. This results in the following <u>classification hypothesis</u>: One-order-parameter transitions are dominated by the cuspoid geometry of corang 1, two-order-parameter transitions by the umbilic geometry of corang 2. Then, under the assumption of structurally stable scaling, a Landau-type mean-field theory (zeroth approximation in the system dimension (d)) will describe the critical behavior qualitatively correctly. After all, although its quantitative predictions are not too good, they are not that bad, either: you cannot expect more from topology alone. Confining ourselves for the moment to the cuspoid geometry with one order parameter $\eta = x$, we describe a multicomponent system by the unfolding $V_\mu = \phi := \sum_{r=0}^{2N} a_\nu x^\nu$ $(\mu = \{a_\nu\} \in C = R^k)$ of a free energy V_o with 2N-th order singularity at x = 0, where the a_ν are related to the physical "field"variables (P,T, chemical potentials, etc.) by a diffeomorphism [11]. With the "coexistence curves" (conflict strata or coexisting minima of equal depth) indicated by dotted lines in Figs. 1c, 10, 11, 13b, one immediately discovers the tricritical and higher order critical points in fluid mixtures, etc. where three or more phases become simultaneously identical. I don't know if they have all been detected experimentally by now. At least the critical point of the star pocket (Fig. 13b) seems not yet to be on the experimentalist's desk--but it must be there, sooner or later, otherwise the Gods must have already made a serious mistake when they created Pythagoras.

Things become even more exciting when we look for "corang-2-singularities", i.e. for systems with two coupled order parameters x,y. Ferro-antiferromagnetic, ferro-piezoelectric and crystalline-superfluid systems provide familiar examples, along with two-mode lasers and BCS pairings [26, 11]. In these cases we start with the partition function Z = $Tr\{\exp[-\beta(H_o + V_\mu(x,y))]\}$ and describe the system as before by the free energy ϕ given by the (averaged) universal unfolding $V_\mu(x,y)$ corresponding to the umbilics or double cusps (with k=3,4 or k=7 type critical points, respectively) given in Sec. 4. Deferring a detailed discussion, we show in Figs. 12, 14 and 15 a few topological situations which also are typical of binary mixtures, MHD and biological systems as well. A consideration of bifurcation phenomena in dynamical equations and of the role caustics play in critical phenomena may be found in [11].

Let me finally touch on a field I have already been bothering with for two decades and which I see now sailing rapidly toward an edge of the overhanging cliff. I mean particle physics and quantum field theory in terms of structural stability concepts. Because of the close analogy between critical behavior in phase transitions (Widom scaling) and deep inelastic particle scattering (Bjorken scaling), critical point dominance in particle physics may be expected to follow rules similar to those developed above [19,27]. After all, scaling is but a special sort of stable mapping, and symmetry breaking is an obvious part of it.

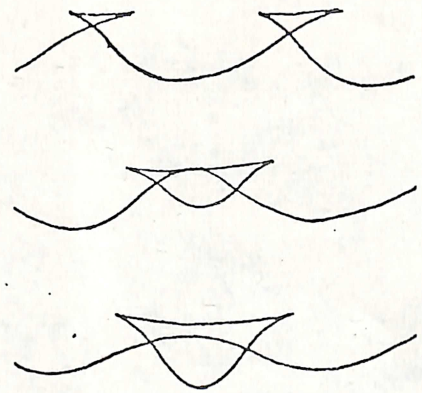


Fig. 15. Higher-order phase transition

Leaving the story of particle catastrophes for a more physically-minded audience, let me just add the following remarks. What we urgently have to do is to build fluctuations into the above frame and get at the thermodynamic roots of catastrophe geometry by considering the unfoldings $V_\mu(x)$ as an excess entropy or information gain $\delta^2 s$ [3] and, as the following discussion will show, extend the considerations to spaces with infinitely many dimensions.

While addressing the physicists, I feel that I may be boring the majority of the biologists among you. By the first day of this conference I already realized that some biologists--especially molecular ones--are nowadays a bit unsure of Thom's ideas, probably because he asked them recently [28] to "finally start thinking". Of course, but just extracting that little piece of theorem (Sec. 4) from Thom's writings and recognizing it as a first small step into an enormously wide field, is a task for an expert in topology, not molecular biology. On the other hand, biologists (like other scientists) are quick to recognize in the newcomers a lack of familiarity with the subtle complexities of their field and to dismiss them as naive, but often fail to understand the necessity of mathematical models because their own category of thought is basically unmathematical and, indeed, qualitative. Thus we are back to what I said at the beginning of this lecture--and what is my adaptation of Thom's philosophy--namely, that the qualitative laws of nature are written in the language of geometry. Biologists cannot be blamed for the sins of mathematicians who have entirely forgotten this fact, but they can help to revive the old geometric virtues. Their immediate task should be to look at biological phenomena the same global way Landau looked at phase transitions when creating his mean field theory of complicated dynamical situations, and adding geometric concepts the way I have just described. Let me, then, sketch in terms of a few examples how the beginning of such a program--anticipated by Thom with fascinating imagination--could be realized.

Modeling the evolution of biological macromolecules in terms of catastrophes is still in its infancy although, according to Eigen and others [29], the temporal evolution of such systems may be described in terms of the kinetic equation (1) with concentrations $x \in M$ of chemicals, and a set C of competing parameters μ representing monomers, etc. Thus, evolution and selection may quite well be describable in terms of catastrophes, though it is certainly not an easy task, even on the basis of the gradient system (4), to single out the relevant macroscopic order parameters. What one has to do, of course, is to supplement Equs. (1) or (4) by differential equations for the parameters μ, thought of as functions of the x, as seen, e.g., in the neuron example described later. I have learned lately that J. Tyson [30] is pursuing this program and I would, after all, not be too much surprised if selection, hypercycles and all that turn out to be phase-transition-like catastrophes. Formally analogous considerations are the main features of recent studies of the reaction-diffusion equation following from (1) by making x space-dependent:

$$\dot{x} = \nabla^2 x + X(x), \tag{10}$$

where X is a nonlinear function of the variables $x = x(t,\vec{r})$. Bifurcations and catastrophes caused by nonlinearities in $X(x)$ are physically due to autocatalytic and crosscatalytic reactions. Equ. (10) is a generalization of Lienard-type oscillators like (9) with the z,f made $\vec{r}$-dependent. As we have already seen, it governs laser transitions as well as scroll waves, and you will also believe that Equ. (10) describes mitosis, etc. [31]. That same equation is at the root of Lefever's, Nicolis', and Howard and Kopell's work [31,14], to name a few, but as we have seen before, the propagation of catastrophe singularities through the solutions of partial differential equations is a delicate matter.

This manifests itself already in topological models of neural behavior relating to the production, control and propagation of nerve impulses. If we denote by $I(x,t)$ a neuron's input current (related to the PSP) and by $\psi(x,t)$ the membrane potential at space point x and time t, then the following partial differential equation describes the neuron's behavior fairly reasonably [32]:

$$a\psi_{xx} + b\psi_{tt} + N'(\psi)\psi_t + c\psi_{xxt} + dN(\psi) = I + I_t \qquad (11)$$

Here $N(\psi)$ is the neuron's nonlinear current-voltage characteristics, $N'(\psi)$ its derivative, and since it is a cubic function, Equ. (11) is indeed of the same type as the equations we have found above in quite different contexts (laser, chemistry, etc.) Thus it is hardly surprising that neurons exhibit catastrophic phenomena: jumps, bursts and all that. What one discovers, e.g., when analyzing solutions of (11), are resonance phenomena (catastrophe concatenations) giving rise to rhythmic discharges as shown in Fig. 16, where the energy E stored in a neuron is plotted against the PSP. Things become almost trivial when in (11) we pass to a neural element and arrive at an equation identical with Equ. (9) with $z = \psi(t)$ and $f(t) = I(t) + \dot{I}(t)$ (the space coordinate drops out when passing to that limit) modeled by the circuit of Fig. 17. Writing it as a system of the type

$$\begin{aligned} \dot{x} &= -x^3 + ax + bu + cv \\ \dot{u} &= dx + eu \end{aligned} \qquad (12)$$

where $x = \psi(t)$, and identifying x with the membrane potential (or Na activation), u with the recovery variable (or H concentration) and v with the stimulus (PSP), Equ. (12) is obviously represented by the cusp catastrophe of Figs. 1b,8. This model (similar to FitzHugh's BvP model [33]) is of course far simpler and, through Equ. (11) more comprehensive than the one E.C. Zeeman recently proposed [13], but I refer the reader to [32] for an elaboration of simple neuron models.

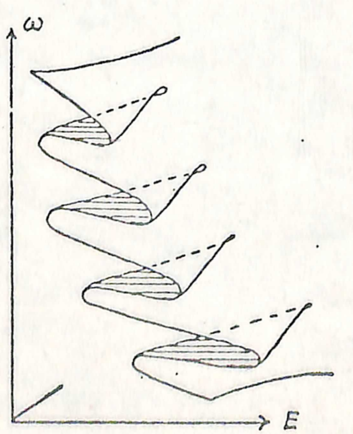


Figure 16

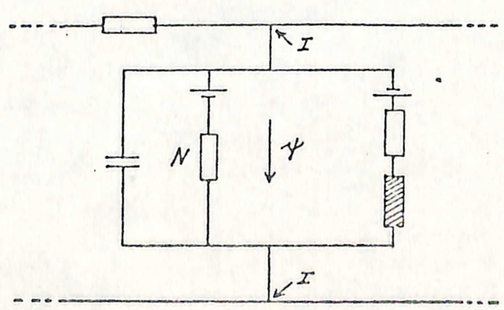
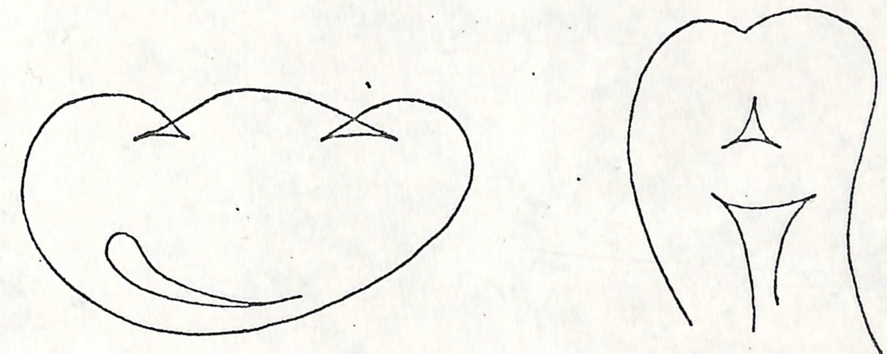


Figure 17

A set of diffusion-reaction equations of the type (10) are well-known from Prigogine's [3], Turing's, and Gierer-Meinhardt's [34] models of pattern formation due to competing concentration gradients μ. Describing slime molds and brains [35] may well follow similar lines. It is at this point where the map of M×C onto space-time I talked about at the beginning of this section comes into play, and the morphogenetic field of Sec. 4.

Identifying morphogenesis, and developmental biology in particular, with fundamental geometric structures is Thom's basic idea. One can indeed easily identify the phase transition cuspoids and umbilics in Figs. 10-15 with developing phases in the embryo (Fig. 18, [36]) and infer from the existence of an eventually unfolding organizing center (V_o) in the former case the existence of such a center in the latter one. This center may be the result of a local minimum of entropy production, but making such intuitions explicit is the great burden Thom has put upon us.



Amphibian gastrulation Neural tube

Figure 18

6. Conclusion

We started by saying that the qualitative laws of nature are written in the language of geometry. This provides us with something to hold on to in our search for a natural philosophy. Though all of Thom's colleagues admire his beautiful and comprehensive view of the universe, few, if any, embrace the totality of his ideas, and only very few, if any, claim to understand them. But he has undoubtedly opened the door to the next great era of awakening of human intellect--that of understanding the qualitative content of complex phenomena.

Acknowledgments

I thank René Thom for persuading me that his statements are too natural to need proving, Horst Eikemeier for convincing me that this is so, and Tom Güttinger for modeling and drawing the catastrophes.

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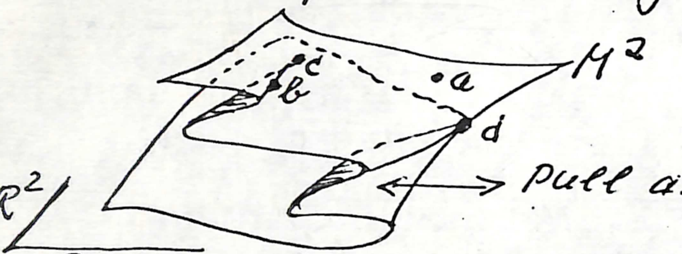
Diff. dyn. Systems : Supplement

W. Güttinger

Thom-Whitney cusp catastrophe : Examples

① Deform a piece of elastic fiber $\rightarrow M^2 \subset \mathbb{R}^3$. Map M^2 vertical onto $\mathbb{R}^2$. $\Rightarrow$:
3 Possible forms near any point : (a), (b), (c) as shown in Fig.



Deform fiber near a, b, c : Their location will be shifted a bit but their "quality" remains the same.
Do a lot of experimentation $\Rightarrow$

1) All points which are not of type a, b, c will be eliminated by small deformations of M^2.
2) But points of type a, b, c cannot be made to disappear under such perturbations of the material

Def : a : regular point (lies smooth over $\mathbb{R}^2$)
b : singular point (fold point : lies on edge of fold)
c) singular point (cusp point : marks the beginning of an "overhanging cliff").

Theorem : Transversal map $M^2 \rightarrow \mathbb{R}^2$. The only singular points which are stable under small deformations of M^2 are folds and cusps. (Thom - Whitney).

Problem : Generalize to $M^m \subset \mathbb{R}^n$: That's what Thom did! Further generalizations were made by Arnold.

② Duffing - Equation (Anharmonic Oscillator)

$$\ddot{x} + \omega_0^2 x + \varepsilon\mu x^3 = \varepsilon F \cos(\omega t) \quad (1), \quad x = x(t), \quad 0 < \varepsilon \text{ small}$$

(forget about damping term $\delta\dot{x}$ on l.h.s.)

Try "ansatz" $x(t) = A\cos(\omega t) + \varepsilon\xi(t) + O(\varepsilon^2)$ (2)
Differentiate (2), insert into (1), use $\cos^3 \omega t = \frac{1}{4}\cos 3\omega t + \frac{3}{4}\cos \omega t$, and keep only terms of order $\leq \varepsilon^1 \Rightarrow$

$$-\omega^2 A\cos\omega t + \varepsilon\ddot{\xi} + \omega_0^2 A\cos\omega t + \omega_0^2\varepsilon\xi + \varepsilon\mu\frac{A^3}{4}\cos 3\omega t$$
$$+ \varepsilon\mu\frac{3}{4}A^3\cos\omega t + O(\varepsilon^2) = \varepsilon F\cos\omega t \quad (3)$$

Put $\omega^2 = \omega_0^2 - \varepsilon\beta$ (4) (or, e.g., $\omega = \omega_0 + \varepsilon\tilde{\omega}$ or so)

(3) $\Rightarrow$

$$\varepsilon[\tfrac{3}{4}\mu A^3 + \beta A - F]\cos\omega t + \varepsilon\{\ddot{\xi} + \omega_0^2\xi + \tfrac{\mu}{4}A^3\cos 3\omega t\} + O(\varepsilon^2) = 0 \quad (5)$$

(5) is satisfied if

(a) $[---] = 0$ $:\Rightarrow$ $\boxed{\tfrac{3}{4}\mu A^3 + \beta A - F = 0}$ (6)

(b) $\{---\} = 0$ $:\Rightarrow$ $\xi(t) = \text{const.}\cos 3\omega t + a\cos\omega_0 t + b\sin\omega_0 t$ (7)

[Alternatively: solve (5) for $\xi(t)$:

$$\ddot{\xi}(t) + \omega_0^2\xi = -[\;]\cos\omega t - \frac{\mu A^3}{4}\cos 3\omega t \quad (5a)$$

Require periodic solution for ξ $\rightarrow$ a term of the type $[--]t\sin\omega t$ must be made to vanish $\Rightarrow$ (6)

Equivalently: Write (1) as

$$\ddot{x} + \omega^2 x = (\omega^2 - \omega_0^2)x - \varepsilon\mu x^3 + \varepsilon F\cos\omega t \quad (1a)$$

Use $x_0(t) = A\cos\omega t$ [= solution of the homogeneous equation pertaining to (1a): $\ddot{x} + \omega^2 x = 0$] as a first approximation to (1a). Insert x_0 into the right hand side $\Rightarrow$

$$\ddot{x}_1 + \omega^2 x_1 = [--]\cos\omega t - \frac{\varepsilon\mu}{4}A^3\cos 3\omega t.$$

$\Rightarrow x_1 = B\cos\omega t + [--]t\sin\omega t + D\cos 3\omega t$ as a 2nd approximation

Require x_1 to be periodic $\Rightarrow [--] = 0 \Rightarrow$ (6)]

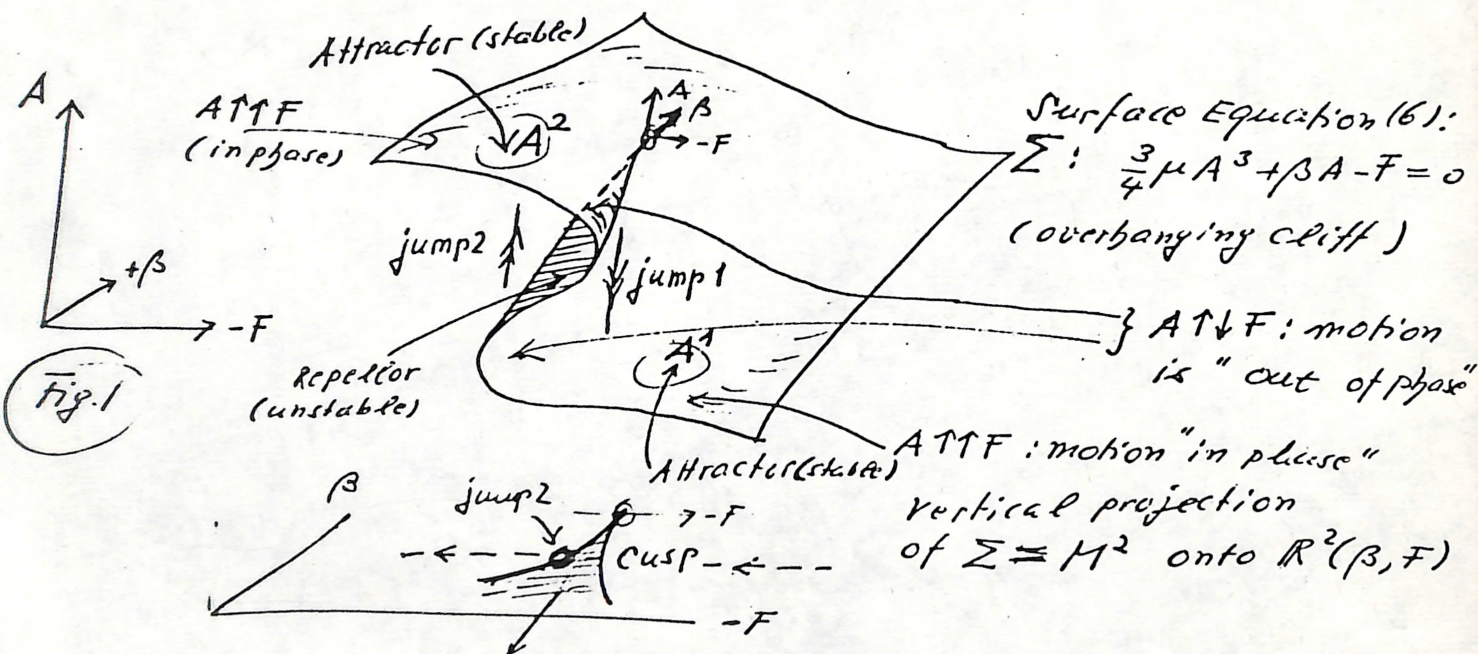
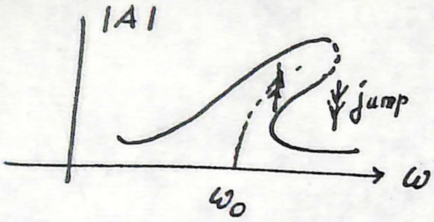


Fig. 1

Fix β and change F slowly from F negative to F positive ($\leftarrow$----) $\Rightarrow$ A jumps rapidly from the the lower sheet ("stable") to the upper one when $\leftarrow$--- crosses 2nd branch of cusp. Similarly A jumps over the edge of the cliff downward ($A^2 \rightarrow A^1$) if F changes slowly into the other direction: Cusp-"Catastrophe"!

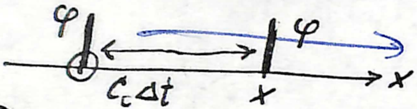
Relation to nonlinear resonance: From (6):



F fixed.

Solution to 1st order $x = A\cos\omega t + \text{const}\,\frac{A^3\cos 3\omega t}{[\dots(6)]}$

③ Nonlinear Wave Propagation



Disturbance φ of something at $x=0$'s $t=0$. Assume this disturbance to propagate with constant speed c_0 in positive x-direction. (without distortion or damping etc). Then at point x at time t you have the same disturbance as the one you had at $x=0$ at time $t-\Delta t$ where $\Delta t = x/c_0$: $\varphi(x,t) = \varphi(0, t-x/c_0) =: f(x-c_0t)$. Such a propagating disturbance you call a wave ($x, t \in \mathbb{R}^1$).

$$\varphi(x,t) = f(x-c_0t) \qquad (1)$$

satisfies the p.d.e. ($\varphi_t = \partial\varphi/\partial t$, $\varphi_x = \partial\varphi/\partial x$)

$$\varphi_t + c_0\varphi_x = 0 \qquad (2)$$

but also the p.d.e. (standard linear wave equation)

$$\varphi_{tt} - c_0^2\varphi_{xx} = 0 \qquad (3)$$

General solution of (3) [since $\varphi_{tt} - c_0^2\varphi_{xx} = (\partial_t + c_0\partial_x)(\partial_t - c_0\partial_x)$] is

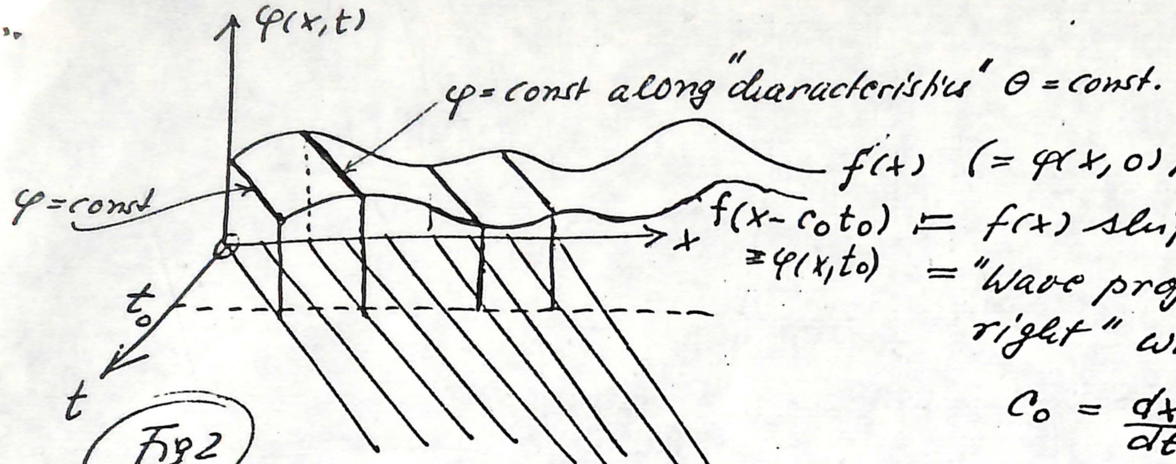
$$\varphi = f(x-c_0t) + g(x+c_0t) \qquad (4)$$

where f, g: arbitrary functions (to be fixed by initial conditions).

Consider again (2): General solution of (2) is (1) with f arbitrary, i.e. $\varphi(x,0) = f(x)$ [initial condition] may be arbitrarily prescribed [Cauchy problem for the p.d.e. (2)] $\Rightarrow$

$$\begin{cases} \varphi_t + c_0\varphi_x = 0 \\ \varphi(x,0) = f(x) \end{cases} \Rightarrow \varphi(x,t) = f(x-c_0t) \qquad (5)$$

$\Theta := x - c_0t$ = "phase" of the wave φ. φ = const for all x, t satisfying Θ = const, i.e. φ = const on all straight lines Θ = const as shown in Fig. 2 [these lines are called "characteristics" of the p.d.e.] $\varphi(x,t)$ = surface over (x,t)-plane:



$c_0 = \frac{dx}{dt}$

Fig 2

$\theta := x - c_0 t = \text{const}$

Nonlinear Waves: Assume c_0 to be no longer constant but that it depends on the magnitude of φ : $c_0 \to c(\varphi)$. Then (5) $\Rightarrow$

$$\begin{cases} \varphi_t + c(\varphi)\varphi_x = 0 \\ \varphi(x,0) = f(x) \text{ arbitrarily given} \end{cases} \qquad (6)$$

The "wave surface" $\varphi(x,t)$ can still be built by rising straight line "characteristics" to an appropriate hight [Problem for you : show this !]. The solution φ of (6) can be directly guessed from (5) to be given implicitly by

$$\varphi = f(x - c(\varphi)t) \qquad (7)$$

[Check that φ defined by (7) satisfies (6) : just differentia

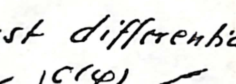
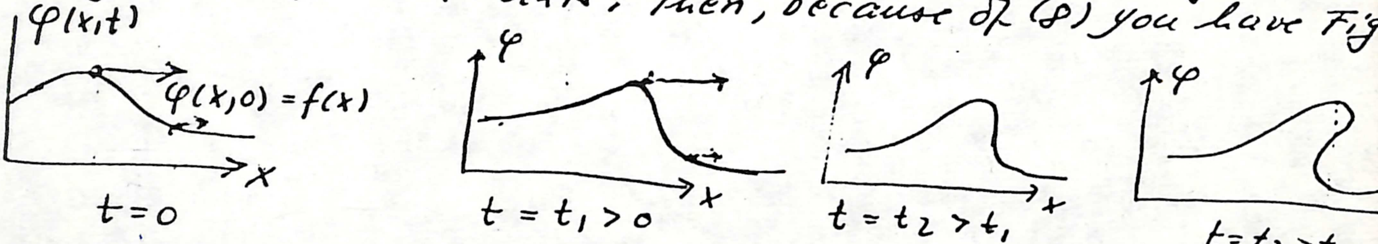
Assume that $c'(\varphi) > 0$ (8): This means that ( larger values of φ propagate faster than smaller ones. For example, let $\varphi(x,0) = f(x)$ have a decreasing branch along part of the x-axis. Then, because of (8) you have Fig

Fig 3



because larger values of φ propagate faster than smaller ones. The wave starts to break at time $t = t_2$ (vertical tangent. To see this mathematically, write (7) as

$$G(\varphi) := \varphi - f(x - c(\varphi)t) = 0 \qquad (9)$$

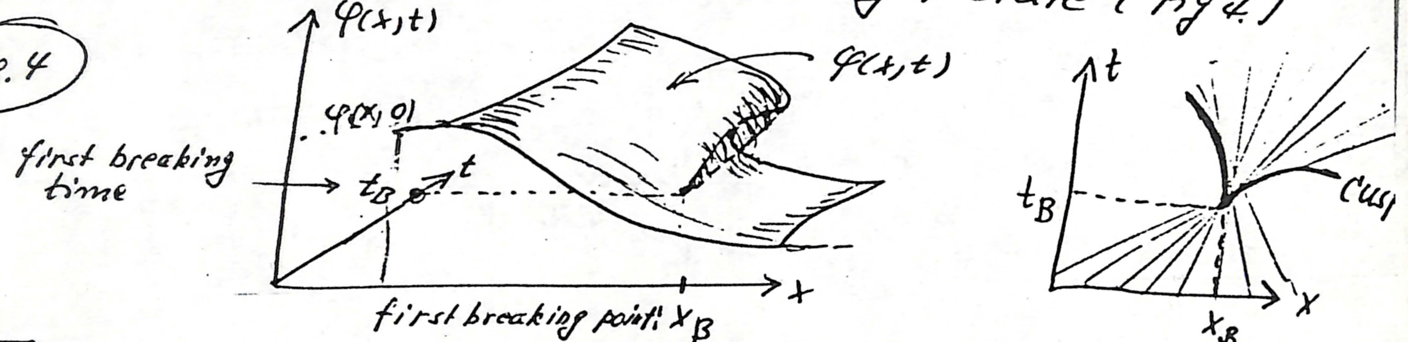
The implicit function theorem tells you at once that φ is a regular (single-valued) function of x, t with finite φ_x, φ

as long as ($'$ = diff. with respect to argument of quantity)

$$D \equiv \frac{dG}{d\varphi} = 1 + t f' c' \neq 0 \qquad (10)$$

Suppose $c' > 0$, but $f' < 0$ on part of x-axis (i.e. f has a decreasing branch) and that $t \geq 0$. Then $tf'c' \leq 0$ and (10) is obviously satisfied for t small enough. But if t becomes larger, (10) will be violated at, say, $t = t_B$: $D|_{t_B} = 1 + t_B f'c' = 0$. Then the implicit function theorem implies that $\varphi_t = \frac{a}{D} = \infty$, $\varphi_x = \frac{b}{D} = \infty$ at $t = t_B$. You get obviously the following picture (Fig 4.)

Fig. 4



This is nothing but a cusp catastrophe. Note [show this!] that $\varphi(x,t)$ can be built up by rising characteristics (which are straight lines in this case) to an appropriate (φ-dependent) height and that the cusp is nothing but an envelope – a caustics, in physical terms.

Now, what to do if φ represents the density, e.g., of water, which is always single valued? The answer in this case is that you should supplement (6) by some condition to avoid multivaluedness (e.g. by adding a dissipative term φ_{xx} to the r.h.s. of (6)). That's a physical assumption you have to make in that case and the result is that instead of the breaking you will get a <u>shock wave</u> which is single-valued. But mathematically no such assumption needs to be made. If φ does not represent a mass or density, you will have the cusp catastrophe and you have to accept it.

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ttinger

Summary of Catastrophe Theory

1. Catastrophe Theory is, first of all, a topological affair. It classifies the singularities which surfaces in R^m can possess if these surfaces are qualitatively insensitive to small deformations. The singular elements that can occur are Thom's famous "elementary catastrophes" (folds, cusps, swallowtail etc) and shown in the paper accompanying these lectures.

2. In its simplest form, the theory relates to any system governed by a "potential function" V, $\dot{x} = -\nabla V$,

$$V \equiv V_\mu(\underline{x}) = V(x_1, \dots, x_n; \mu) = V(x_1, \dots x_n; u, v, w, \dots) \quad (1)$$

where $\underline{x} := (x_1 \dots x_n) \in R^n$ is a set of "state variables" and $\mu := (u, v, w, \dots) \in R^k$ a set of "control variables." Assume that the equilibria of the system are given by the "stationary points" of $V_\mu(\underline{x})$, i.e., by the Equs.

$$\frac{\partial V_\mu(\underline{x})}{\partial x_i} = 0 \quad , i = 1, 2, \dots, n. \quad (2)$$

These n equations define "surfaces" in the $(k+n)$-dimensional $(\underline{x}, \mu)$-space $R^n \times R^k$. Stable stationary points correspond to minima of $V_\mu(\underline{x})$, unstable equilibria to maxima of V. Critical equilibrium states occur when the determinant $\det(\partial V_\mu(\underline{x}) / \partial x_i \partial x_k)$ vanishes, i.e., if

$$H := \left| \frac{\partial^2 V_\mu(x)}{\partial x_i \partial x_k} \right| = 0 \quad ; \quad i, k = 1, 2, \dots, n. \quad (3)$$

Then a loss of stability can be expected. If $\underline{x}$ is a scalar, i.e., $n = 1$, (3) means that $d^2 V_\mu(x)/dx^2 = 0$ so that the equilibrium given by (2), i.e. by $d V_\mu(x)/dx = 0$, is a saddle or inflexion point.

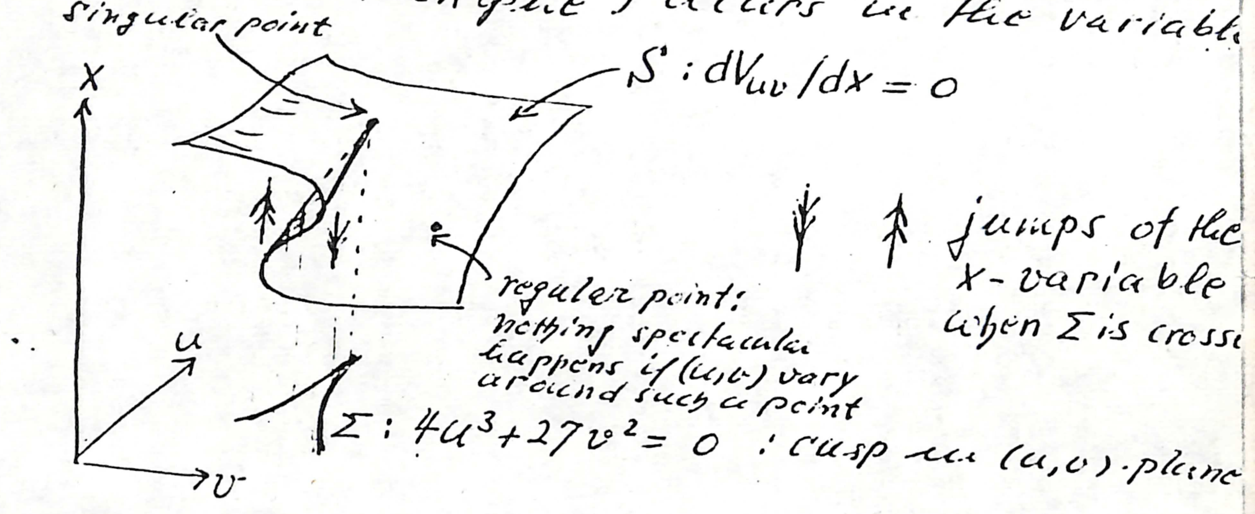
We ask which forms the equilibrium surfaces defined by the solutions $x_i = x_i(\mu)$ of (2) may take on, and what sort of instabilities, given by (3), may occur. Assume that the system described by (1) is structurally stable, i.e., that it preserves its quality under small perturbations (changing V by δV in $\dot{\underline{x}} = -\operatorname{grad} V_\mu(\underline{x})$, for example) so that a slightly deformed equilibrium surface (2) stays qualitatively the same. Then we have

3. <u>Thom's Theorem</u>:

Locally, near any point $(\underline{x}, \mu) \in \mathbb{R}^n \times \mathbb{R}^k$ the surface (2) is either smooth (i.e., $(\underline{x}, \mu)$ is a "regular" point: $H \neq 0$) or it has there a singularity ($H = 0$), out of a finite set called "elementary catastrophes" (if $k \leq 6$). That is to say, any singularity of the equilibrium surface, given by (2) and (3), is equivalent to an elementary catastrophe. One obtains them from the map $\mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ which is implied by the simultaneous solutions of (2) and (3), the latter determining the breakdown of stability. Furthermore, these elementary catastrophes (describing a sudden dynamical snap from one stationary state to another one) are the only naturally occuring forms of sudden change of behavior no matter how many (n) state variables there are in the system. For example, there are only 7 different catastrophes possible if $k = 4$ (there are 14 if $k = 6$) and only 2 of the n state variables are significant. The interpretation assumes that the k control variables $\mu = (u, v, w, \dots)$ are slowly varying. Then the changes in the remaining significant state variables, denoted by x, y, are fast if μ crosses the "bifurcation set" given by (2),(3).

4. Examples:

(a) Cusp catastrophe. If $k = 2$, i.e., $\mu = (u, v)$, then any $V_\mu(\underline{x}) = V(x_1, \dots, x_n; u, v)$ can be reduced to $V_{uv}(x) = x^4/4 + ux^2/2 + vx$ with one significant state variable $x \in R^1$. The equilibria are given by $dV_{uv}/dx = x^3 + ux + v = 0$ and form the surface S below. The singularity of V_{uv} is given by $d^2V_{uv}/dx^2 = 3x^2 + u = 0$ (vertical tangents to the equilibrium surface S), and eliminating x from both equations gives, as the projection of the verticals onto the (u, v) plane, the cusp Σ shown below. If (u, v) cross Σ, a sudden snap (catastrophe) occurs in the variable x.



(b) Swallow tail catastrophe: $k = 3$, i.e., $\mu = (u, v, w)$. Here one or two significant state variables (x or x, y occur). In case of one, V_μ reduces to $V = x^5/5 + ux^3/3 + vx^2/2 + wx$. Then from $dV/dx = 0$, $d^2V/dx^2 = 0$ Figure 10a of the manuscript follows as the basic catastrophe or bifurcation set (swallow tail). It plays in 3 dimensions (u, v, w) the same role as Σ in the (u, v)-plane shown above.