On the Algebraic Aspects of Spinor Theory

I. The Real and Complex Pauli Algebras and their Representations

F. A. M. Frescura and B. J. Hiley
Department of Physics, Birkbeck College (University of London)

ABSTRACT

In this paper we analyse the structure of both the real and complex Pauli algebras in some detail. We are motivated partly by a recent suggestion of Bohm's, namely, that algebraic structures seem to offer a natural way of exploring certain new orders in physics and partly by a desire to obtain a better understanding of the spinor.

In the real case, it is shown that the algebra contains exactly one type of left-invariant subspace while in the complex case it is shown to contain two. The latter support inequivalent irreducible representations of the complex algebra, which in turn induce the irreducible representations $D_{\frac{1}{2},0}$ and $D_{0,\frac{1}{2}}$ of SL(2,C). They also support the representation $D_{\frac{1}{2}}$ of SU(2), indicating the relation of the minimal left ideals of this algebra to the Pauli spinor.

We also point out that the Pauli algebra contains a multivalued $(\infty\to 1)$ representation of O(3) which includes the usual 2-1 representation provided by SU(2) as a special case.

1. INTRODUCTION

The interpretation of quantum mechanics and its relativistic generalisation still presents fundamental difficulties. While it is generally recognised that these theories call for profound changes in outlook, the exact nature of the changes is still far from clear. In recent years there has developed a tendency to minimise the radical nature of quantum phenomena and tacitly to ignore features already implicit in the theory because one's thinking tends to be dominated by older classical orders which use primitive terms, like particle, field, locality, interaction, etc. Of course, we do not question the validity of these notions in their own domain, but their necessity at a more primitive level is far from obvious.

(1970, 1971 and 1973)

In a series of papers, Bohm and his co-workers/have suggested that the possibility of using certain new concepts and orders in physics should be explored. By taking cues from various fields of experience it was suggested that, as an alternative approach, primary relevance should be given to activity, through movement and/or process, where this activity is not to be thought of as things-in-movement, but rather as a structured movement which itself is taken as primary. This view questions the relevance of the notion of dynamics described in terms of particles and their trajectories in space-time. These classical forms will then appear only as abstractions at some higher level: for example, a particle would be some form of quasi-local stability which exhibits some degree of autonomy in the general field of activity.

In such a scheme, the evolution of a structured movement will be in terms of a restructuring rather than a linear mapping on some given vector space of the type commonly used. This approach would appear more appropriate to the discussion of the 'ding an sich', the operators being regarded directly as changing one such object into another similar object. (For example, for the retention of the identity of the moving particle.) Consequently, the use of differential equations seems inappropriate. Bohm has suggested that a more relevant mathematisation should involve algebras and their automorphisms, both inner and outer.

The use of algebras does not exclude discussion of the 'ding an sich'. In fact, certain features of the algebraic approach can be considered in a certain sense as being equivalent to these identity retaining transformations. But we find in addition that the underlying algebraic structure offers the possibility a more general form of change. We suggest that it is the implication contained in this type of change that may be of considerable importance in understanding those features of quantum mechanics that are more appropriately described in terms of Heisenberg's matrix mechanics rather than through the more usual Hilbert space formalism. (This suggestion is contained implicitly in Dirac (1965) and more overtly in Finkelstein (1969).)

We propose therefore that a detailed investigation of those algebras known to underlie quantum theory might prove fruitful in the understanding of the role of movement and process as fundamental forms in physics. This work forms part of such an investigation. We shall concentrate at first particularly on the theory of the Clifford algebras, which structures are known ultimately to be associated with the theory of spinors (Cartan 1966). More especially, we will focus attention on the Clifford algebra for the 3-d Euclidian space, which we call the Pauli algebra.

A clear understanding of the algebraic structure underlying the spinor becomes all the more important when it is realised that many of the notions discussed in this paper can be used in an analysis of a symplectic algebra, which can be shown to be an extension of the Heisenberg algebra. This work will be reported in a later paper.

Although the mathematical structures underlying the spinor have been extensively studied, the implications for the general point of view we are adopting have been overlooked. Indeed, the relation of the algebraic spinor structure to physical theories still requires clarification.

For example, in relativity, where differential geometry appears to offer a natural description of the associated physical phenomena, the spinor appears as no more than an interloper. Yet it is known that there exists a correspondence between spinors and tensors, with the 2-1 isomorphism between, say, the spinor and the vector, suggesting that the spinor in some way is more primitive to the concept of geometry than is the vector. Consequently, it would appear that the corresponding matrix description of a vector should be taken as an entity more basic than the vector itself. In fact, this is the approach suggested by Penrose (1968) who shows that in Minkowski space a van der Waerden spinor, ω^A, can be pictured as a kind of null-flag in terms of which a non-local geometrical distinction between ω^A and $-\omega^A$ is possible. Thus while mathematically an account in terms of differential geometry can be developed elegantly in terms of the spin-bundle formalism, nevertheless, as we see it, this account by itself provides no compelling a priori reason for adopting the spinor as a basic form. In the context of the differential geometry, it could equally well be regarded as a convenient calculating device.

Then again, in the non-relativistic limit, i.e. in E_3, a 2-1 isomorphism between the spinor and the vector is still possible, as was first pointed out by Cartan. However, the physical meaning in such a space of the null-vectors and the imaginary angles used to explain this feature is far from clear. In turn, this lack of clarity has been partially responsible for the belief that spin is purely a relativistic phenomenon. However, in section 2, we show that this is not true and, in fact, as Bohm, Schiller and Tiomno (1955) have already indicated, the global properties in Minkowski space to which Penrose refers occur also in E_3 and are describable in terms of the Pauli spinor. In fact, it is not difficult to build a mechanical model to demonstrate these properties directly (Misner, Thorne & Wheeler, 1973).

Again, in the group theoretic approach (Cartan 1966), it is found that the spinor is introduced more as a convenient mathematical construct than as a fundamental necessity. The representation space carrying a group representation is viewed as no more than a convenient adjunct to the group space which facilitates the calculation of the invariants of the group. In other words, the spinor introduced in this way plays no more than a secondary and utilitarian role whereas in physics and, in particular, in quantum mechanics it assumes no such subordinate position. Its properties are essential for the description of fermions.

In an attempt to clarify the role of the spinor, we have been led to reconsider a suggestion by Riesz (1946) that the spinor might be better understood as an algebraic object rather than directly as a geometric or group theoretic one. Hestenes (1966, 1967 & 1971) has attempted already to explore some of these possibilities. It is our intention to investigate further the algebraic approach, though from a somewhat different standpoint based on certain of Bohm's proposals.

Riesz (1946) suggests that the spinor could be treated as an element of a minimal left ideal of a Clifford algebra. In this algebra the proper orthogonal transformations are expressible in terms of inner automorphisms. Thus it is possible to regard both the group of orthogonal rotations of the space generating the algebra and the spinor space on which the representation of the group is effected as substructures of one single mathematical system, namely, the Clifford algebra. Although these algebras have been extensively studied by Riesz (1958) and Chevalley (1946 & 1954), the algebraic properties of the spinor, the new role it assumes within the theory, its relation to the more familiar geometric objects and its novel possibilities seem never to have been fully explored, except perhaps for the single instance when the covariants of the orthogonal group are derived from the multiplicative properties of the elements of the left and right ideals. It does not seem to have been realised that the algebraic definition yields an object of greater generality than the group theoretic spinor in the sense that, while it possesses all the properties characteristic of the group concept, it also possesses certain features that are not present in the older theory. It is to these aspects that we wish to draw attention.

So as to manifestly display these features we have found it necessary to reexamine some of the details of the underlying algebraic structure. This will be done using well established techniques (van der Waerden 1970). The Pauli algebra has the merit of being relatively straightforward and yet containing sufficient structure to illustrate the points we have in mind. Unfortunately there exists a considerable degree of confusion in the literature regarding the relations between the real Pauli algebra, the complex Pauli algebra and their representations using complex 2 x 2 matrices. We feel it necessary therefore to clarify these issues before exploring the new implications in detail. The extension of these ideas to the relativistic case in terms of the Dirac algrebra of gamma matrices is comparatively straightforward. Our aim will be primarily to show the relation of the present vector space approach to the algebraic. We will leave the discussion of the relation of the algebraic view of the spinor and its significance to the notion of movement and process for a later paper.

2. THE PAULI EQUATION

Before proceeding with a detailed analysis of the structure of the Pauli algebra we want to provide a physical motivation for our interest in this particular structure. Consider the Pauli equation and its relevance to the question of spin.

Originally it was deduced on purely phenomenological grounds (Pauli 1927), the main feature of the derivation being the representation of the angular momentum operators by means of 2 x 2 complex matrices having eigenvalues ± ½. Following the standard Hilbert space procedures, it was assumed that the operand wavefunction would be necessarily a two dimensional complex vector but, of course, this is not the only possibility. In fact any 2 x n matrix would fulfil all the mathematical requirements (though perhaps not all the physical ones) demanded by the form of the equation.

From the mathematical point of view we can distinguish two possibilities; either the operator and the operand are objects of a different mathematical character or they are to be of a similar kind. The phenomenological derivation favours neither case and it is only the existence of the Hilbert space formalism that forced Pauli's original choice. In this paper we explore the alternatives. For reasons to be stated in a later paragraph, we shall prefer to adopt the second possibility rather than the first, regarding both operator and operand as elements of a common structure.

The Pauli equation can be written as

$$\left[\frac{1}{2m}\left(\underline{p}+\frac{e}{c}\underline{A}\right)^2+\frac{e}{2mc}\underline{B}.\underline{\sigma}-E\right]\psi=0 \tag{1}$$

where $\underline{B}.\underline{\sigma}=B^1\sigma_1+B^2\sigma_2+B^3\sigma_3$, $\underline{B}$ being the magnetic field and $\underline{\sigma}$ the Pauli spin matrices. Since the matrices can be shown to be representations of the generating elements of the Pauli algebra, $\mathcal{P}$, (see section 3), the term $\underline{B}.\underline{\sigma}$ can be regarded as a vector in E_3 with components B^i. The other operators in the Pauli equation are scalars and so the operator itself can be regarded as an element of the algebra. Then we can re-write the equation as

$$\pi\psi = 0$$

with $\pi \in \wp$. In this form, the unnaturalness of Pauli's assumption is made manifest, the more natural assumption being that both π and ψ should be regarded as objects in the Pauli algebra.

It has been claimed that spin is essentially a relativistic phenomena so that little understanding of the phenomenon of spin will be gained except through the Dirac algebra. But this is not true, as we show in the following manner. Let us regard the momentum as a vector in E_3. In the presence of a magnetic field B we may write

$$p = (p^j - \frac{e}{c}A^j)\,e_j \qquad \text{where } p \in \wp$$

e_j being the generators of $\wp$.

If we now form the kinetic energy $\frac{p^2}{2m}$ by regarding the square as a Clifford product, viz. $\frac{p^2}{2m} = \frac{1}{2m}(p^j - \frac{e}{c}A^j)\,e_j\,(p^k - \frac{e}{c}A^k)\,e_k$, we find $\frac{p^2}{2m} = \frac{1}{2m}(p - \frac{e}{c}A)^2 - \frac{e}{2mc}B^j e_j$ which shows that it is the Clifford product that produces the required gyromagnetic factor for the electron. Thus the appearance of spin together with the appropriate magnetic moment is intimately associated with the use of the Clifford algebra rather than with the properties of the Lorentz group.

The above derivation of the Pauli equation due to Feynmann (1962) clearly demonstrates that the dynamical operators of the equation are specifically elements of a Clifford algebra. More precisely, they are elements of the Pauli algebra. By implication then, if we require consistency of form in the Pauli equation by demanding that the operators and operands be of like character, then the operand wavefunctions themselves should also be elements of the Pauli algebra. Consequently, we anticipate a coherent account of the Pauli spinor to be possible in terms of the Pauli algebra.

While this unification is of some significance in its own right,there is another imporant aspect of an algebraic approach which is of considerable relevance to the outlook that we wish to explore. Both Hamilton and Clifford were aware of the possibilities that the algebra provided, but their views have been neglected in favour of the present emphasis on the individuality of tensorial structures in differential geometry.

Hamilton always stressed the interplay between the more intuitive and philosophical ideas on the one hand, and the mathematical symbolism on the other (see for example Hankins(1976)). He was unhappy with the logical foundations of algebra and was exploring the possibility of finding an intuitive substructure. In developing this theme Hamilton writes "In algebra the relations which we first consider and compare are relations between successive states on some changing things or thought ... Relations between successive thoughts thus viewed as successive states of one more general and changing thing, are the primary relations of algebra ...". This underlying recurrent theme of change reappears in the quaternion ~~where~~ where he recognised the quaternion as an operator transforming by multiplication a vector into some other vector.

Clifford (1882) made this aspect even clearer. In discussing the difference between the Grassmann outer product and his own form of product, Clifford contrasted the two attitudes that could be adopted towards any product of objects of a similar mathematical nature in the following way. He considered the simple product 2 x 3 = 6; here 6 may be regarded as a number derived from the numbers 2 and 3 by a process in which they play identical roles; or we may regard the number as derived from the number 3 by the operation of doubling. In the former view 2 and 3 are both numbers while, in the latter, 3 is a number, but 2, even though coincidentally it is also a number, is being used in an active sense and the two numbers play distinct roles. The Grassmann product is founded on the first view, while the Clifford product is founded on the notion of activity. Naturally, it is this aspect of the Clifford algebra which will ultimately connect with the representations of the orthogonal transformations but we will show in paper II that giving the algebra primary significance enables one to view the spinor in a very different light, thereby opening up the possibility of pursuing Bohm's suggestions concerning movement and process.

3. THE STRUCTURE OF THE PAULI ALGEBRA

The Pauli algebra $\mathfrak{P}$ over a field $\mathfrak{F}$ (assumed of characteristic $\neq 2$) is formally defined to be the algebra generated by the unity element 1 of the field $\mathfrak{F}$ together with three elements e_1, e_2, e_3 such that $e_i e_j + e_j e_i = 2\delta_{ij}$ This algebra has dimension 8 over $\mathfrak{F}$ and any element p of the algebra can be written in the form

$$p = a + \sum_i a_i e_i + \sum_{i<j} a_{ij} e_i e_j + \sum_{i<j<k} a_{ijk} e_i e_j e_k = \sum_A a_A e_A \qquad (2)$$

where $a, a_i, a_{ij}, a_{ijk} \in \mathfrak{F}$. The generating elements e_i can be regarded as an orthonormal basis of a three-dimensional Euclidian vector space E_3 over the field $\mathfrak{F}$ and the algebra can be regarded as an algebra of antisymmetric tensors $T = \mathfrak{F} + E_3 + B + P$.

Thus any element p is the direct sum of the space of scalars, vectors, E_3, bivectors, $\mathfrak{B}$, and pseudoscalars, $\mathfrak{P}$, for the three-dimensional geometry. Thus the vector and multivector calculi for E_3 are seen to be incorporated into a single closed system by $\mathfrak{P}$, and all the geometric forms which find description in terms of these calculi will find a description in the calculus provided by the algebra $\mathcal{P}$.

In the attempt to incorporate the spinor into the algebra we must ensure that those phenomena correctly described by the complex two-dimensional Hilbert space formalism are also accounted for by the algebra replacing the spin space. Now it is well known that within any algebra there can exist subsets of elements that transform into each other under the product action of any element of the algebra and, therefore, behave like invariant linear subspaces under the algebraic product. These ~~elements~~ subsets are the ideals. For reasons that will become apparent later, it is necessary to distinguish products from the left and from the right respectively. We define a left ideal $\mathcal{I}_L$ of the algebra $\mathcal{A}$ as a set of elements such that if $a \in \mathcal{I}_L$, $b \in \mathcal{I}_L$ then $(a \pm b) \in \mathcal{I}_L$ and if $a \in \mathcal{I}_L$ and $p \in \mathcal{A}$ then $pa \in \mathcal{I}_L$. In the case of the right ideal the product is $a'p \in \mathcal{I}_R$ for $a' \in \mathcal{I}_R$ $\forall$ $p \in \mathcal{A}$. A two-sided ideal is an ideal under both left and right multiplication. We will show in Section 9 that the spinors can be considered to be elements of the left and right ideals.

It can be shown that the Pauli algebra is a non-nilpotent algebra with a unity containing exactly one principal idempotent, namely, the unity itself. It also can be shown that if the field $\mathcal{F}$ does not contain an element i such that $i^2=-1$ then $\mathcal{P}$ is simple (for example the real Pauli algebra) but if it does contain such an element then $\mathcal{P}$ is semi-simple (e.g. the complex Pauli algebra) (Chevalley 1946). The real Pauli algebra contains only the trivial two-sided ideals $\{0\}$ and $\mathcal{P}$, while the complex algebra contains two non-trivial two-sided ideals $\mathcal{I}'$ and $\mathcal{I}''$ generated by the central idempotents

$$\varepsilon' = \tfrac{1}{2}(1+ie_{123}) \quad \text{and} \quad \varepsilon'' = \tfrac{1}{2}(1-ie_{123}).$$

Thus the complex Pauli algebra can be written as the direct sum

$$\mathcal{P} = \mathcal{P}\varepsilon' \oplus \mathcal{P}\varepsilon''$$

of mutually orthogonal sub-algebras $\mathcal{P}\varepsilon'$ and $\mathcal{P}\varepsilon''$. They are mutually orthogonal by virtue of the fact that $\varepsilon'\varepsilon''=0=\varepsilon''\varepsilon'$. Both $\mathcal{J}'$ and $\mathcal{J}''$ are simple.

4. THE CENTRE OF THE PAULI ALGEBRA

Let us now consider the centre, Z_1 of the algebra which consists of elements which commute with every element of the algebra. For the Pauli algebra, the centre is of dimension 2 and is spanned by 1 and the element e_{123}. Any non-zero element of Z_1 can be written as $z=\alpha+\beta e_{123}$. Let $\mathcal{J}_{Z_1}$ $(\neq\{0\})$ be an ideal of Z_1 the since $(\alpha-\beta e_{123})\in Z_1$ we have

$$(\alpha-\beta e_{123})(\alpha+\beta e_{123}) = (\alpha^2+\beta^2)1 \quad \in \mathcal{J}_{Z_1}$$

so that we must distinguish two cases.

Case (a):

If the field $\mathcal{F}$ does not contain a root of -1 then $(\alpha^2+\beta^2)\neq 0$ for all $\alpha,\beta\neq 0$ in $\mathcal{F}$. Hence for all non-zero z, $(\alpha^2+\beta^2)$ has an inverse in $\mathcal{F}$, $\Rightarrow 1\in\mathcal{J}_{Z_1}, \Rightarrow \mathcal{J}_{Z_1}=Z_1$ and the algebra Z_1 is therefore simple.

Now, by Wedderburn's theorem, every simple algebra can be written as the product of a full matrix algebra of order m with a division algebra of order t. The order of Z_1 over $\mathcal{F}$ is two, thus:

$$2 = m^2t$$

and the only solution to this equation is m=1, t=2. Hence Z_1 is expected to be a division algebra. This is verified by noting that for each non-zero z=a+be_{123} we have $(a^2+b^2)\neq 0$ so that

$$z' = (a^2+b^2)^{-1}(a-be_{123})$$

is a well-defined element of Z_1 satisfying :

$$zz' = z'z = 1$$

showing that every non-zero element of Z_1 is invertible.

If Z_1 is regarded as a field in its own right, then:

(i) Z_1 is of order two over $\mathcal{F}$ and

(ii) Z_1 contains a root of -1, namely, the element e_{123}.

Z_1 can therefore be regarded as the 'complexification' of the field $\mathcal{F}$.

Case (b):

If the field $\mathcal{F}$ contains a root i of -1, then either $(\alpha^2+\beta^2) \neq 0$ in which event $\mathcal{J}_{Z_1} = Z$ as before, or $(\alpha^2+\beta^2) = 0$ for $\alpha, \beta \neq 0$ in which event we conclude that $\mathcal{J}_{Z_1}$ must contain one or both of the idempotents $\frac{1}{2}(1+ie_{123})$, $\frac{1}{2}(1-ie_{123})$. Because these indempotent elements do not possess an inverse in Z_1, we conclude that Z_1 is not in this case a division algebra. Indeed, Z_1 is a semi-simple algebra consisting of the direct sum of two simple sub-algebras corresponding to the decomposition:

$$Z_1 = Z_1\tfrac{1}{2}(1+ie_{123}) \oplus Z_1\tfrac{1}{2}(1-ie_{123})$$

Thus we have established the result that the centre of the real Pauli algebra is a division algebra isomorphic to the complex numbers while the centre of the complex Pauli algebra is semi-simple having two simple components, each being an isomorphic copy of the complex numbers. We can illustrate the point more overtly by considering the representations of the centres.

4.1 REPRESENTATION OF THE CENTRE OF THE REAL PAULI ALGEBRA BY REAL MATRICES

Since the centre Z_1 is a simple division algebra, every non-zero element in Z_1 is invertible. Consequently, Z_1 contains no idempotents other than 0 and 1. The regular representation is thus irreducible and is itself the required representation of Z_1

Put $\varepsilon_1 = 1$ and $\varepsilon_2 = e_{123}$ then

and
$$1 : \varepsilon_i \to \varepsilon_i \qquad \text{for } i = 1,2.$$
$$e_{123} : \varepsilon_1 \to \varepsilon_2$$
$$: \varepsilon_2 \to -\varepsilon_1$$

The regular representation is therefore:

$$1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$e_{123} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the general element of Z_1 is represented as:

$$z = a + b e_{123} \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with a and b real, which is recognised as the real representation of the complex numbers.

4.2 REPRESENTATION OF THE CENTRE OF THE COMPLEX PAULI ALGEBRA BY COMPLEX MATRICES

The field of complex numbers contains two roots of -1, namely, i and -i. Thus, in this case, the centre is semi-simple consisting of the direct sum of two simple algebras of dimension one, that is to say, it is the direct sum of two fields, according to:

$$Z_1 = Z_1 \tfrac{1}{2}(1 + i e_{123}) \oplus Z_1 \tfrac{1}{2}(1 - i e_{123})$$

Thus put $\varepsilon_1 = \frac{1}{2}(1 + i e_{123})$ and $\varepsilon_2 = \frac{1}{2}(1 - i e_{123})$. The vectors ε_1 and ε_2 clearly constitute a basis for the algebra Z_1. Then:

$$1 : \varepsilon_i \longmapsto \varepsilon_i \qquad \text{for } i = 1,2$$
and
$$e_{123} : \varepsilon_1 \longmapsto -i\varepsilon_1$$
$$\varepsilon_2 \longmapsto i\varepsilon_2$$

The regular representation is then:

$$z = \alpha + \beta e_{123} \longmapsto \begin{pmatrix} \alpha - i\beta & 0 \\ 0 & \alpha + i\beta \end{pmatrix}$$

with α and β complex. The two irreducible representations of $\mathcal{Z}_1$ contained in the regular representation are not equivalent, as can be seen from:

$$x^{-1}(\alpha+i\beta)x = (\alpha+i\beta)x^{-1}x$$
$$\neq (\alpha-i\beta)$$

for every $x \in \mathbb{C}_1$.

4.3 REPRESENTATION OF THE CENTRE OF THE REAL PAULI ALGEBRA BY COMPLEX MATRICES (NON-STANDARD REPRESENTATIONS)

The field of complex numbers can be considered as a scalar extension of the field of real numbers. In turn, the complex Pauli algebra will be related to the real algebra through this scalar extension. For the case in question, we can also regard the complex field as a division algebra of order two over the real field. Then the complex algebra $\mathcal{Z}(\mathbb{C})$ will be the product of the real algebra $\mathcal{Z}(\mathbb{R})$ with the algebra $\mathbb{C}$ of complex numbers. We can therefore regard $\mathcal{Z}(\mathbb{R})$ as that sub-algebra of $\mathcal{Z}(\mathbb{C})$ obtained by restricting the scalar coefficients to the field of real numbers. The representation of the complex algebra $\mathcal{Z}(\mathbb{C})$ will thus induce a representation of the real algebra through a corresponding restriction.

In particular, the regular representation of the centre $\mathcal{Z}_1$ of the complex Pauli algebra is:

$$z = \alpha + \beta e_{123} \longmapsto \begin{pmatrix} \alpha - i\beta & 0 \\ 0 & \alpha + i\beta \end{pmatrix}$$

with α and β in $\mathbb{C}_1$. By restricting the coefficients α and β to real values, we obtain the representation:

$$z = a + b e_{123} \longmapsto \begin{pmatrix} a - ib & 0 \\ 0 & a + ib \end{pmatrix}$$

with a and b real, which yields two non-equivalent representations of $\mathcal{Z}_1(\mathbb{R})$ by complex matrices, the first representation being the complex conjugate of the second.

5. REPRESENTATIONS OF THE REAL PAULI ALGEBRA

Traditionally Physics has been preoccupied with the representation of algebras rather than the abstract structures themselves. Of particular interest is the regular representation which, for the finite dimensional case, contains every irreducible representation of the algebra. The irreducible representations are effected on left invariant subspaces , that is to say, on the minimal left ideals of the algebra. (By a minimal left ideal we mean a left ideal which contains no left ideal other than itself and $\{0\}$). These ideals are generated by primitive idempotents, that is, idempotents which cannot be decomposed in to the sum of mutually orthogonal idempotents i.e. $\nexists$ idempotents u and u' in $\wp$ such that

$$\varepsilon = u + u'$$

with $uu' = 0 = u'u$; $u \neq 0$, $u' \neq 0$

It can be shown that ε is primitive iff

$$\varepsilon x \varepsilon = \lambda \varepsilon$$

with λ an element of the centre of the algebra (see Boerner 1963).

As in the case of group theory, it is of interest to distinguish those irreducible subspaces which are equivalent from those which are not. This entails establishing whether or not it is possible to find a similarity transformation internal to the algebra, mapping one invariant subspace onto another. It can be shown that two irreducible subspaces are equivalent iff there exist elements of the form $u x u'$ with $x \in \mathcal{J}_L$ and u' is a generating idempotent of $\mathcal{J}'_L$ (Boerner 1963).

Consider the two idempotents of the real Pauli algebra

$$u = \tfrac{1}{2}(1 + e_3) \quad \text{and} \quad u' = \tfrac{1}{2}(1 - e_3)$$

such that $uu' = 0 = u'u$. The unity of the algebra can be written as $1 = u + u'$

Let us form the subalgebra $u\wp u$ of $\wp$ which will be spanned by the elements ue_Au where $\{e_A\}$ is a basis of $\wp$

Now

$$\begin{array}{ll} u1u = u & ue_{123}u = e_{123}u \\ ue_1u = 0 & ue_{12}u = e_{12}u = e_{123}u \\ ue_2u = 0 & ue_{13}u = 0 \\ ue_3u = u & ue_{23}u = 0 \end{array}$$

showing that $u\wp u = Z u$

which is a division algebra having only the idempotents u and 0 within its structure. Thus u is primitive in $\wp$. Similarly it can be shown that

$$u'\wp u' = {}'Z u'$$

which is also a division algebra containing only the idempotents u' and 0 , so that u' also is primitive in $\wp$. We have thus proved that in the real Pauli algebra the decomposition $1 = u + u'$ is a complete decomposition of the unity element into a sum of mutually orthogonal primitive idempotent elements. Furthermore this means that we can decompose $\wp$ into the direct sum of two minimal left ideals according to

$$\wp = I_{Lm} + I'_{Lm} = \wp\tfrac{1}{2}(1+e_3) + \wp\tfrac{1}{2}(1-e_3) \qquad (3)$$

These ideals can be shown to be equivalent.

It should be noted that choice of primitive idempotents giving rise to decomposition (3) is unique only up to an equivalence transformation. For if $1 = u + u'$ is some decomposition of the unity into the sum of mutually orthogonal primitive idempotents $1 = g^{-1}ug + g^{-1}u'g$ is another such decomposition for every invertible element $g \in \wp$. This implies that there is a non-denumerable set of equivalent minimal left ideals in the Pauli algebra and we have chosen one particular pair.

The real Pauli algebra is simple and, as we have already remarked, every simple algebra is expressible as a direct product of a full matric algebra of order m^2 with a division algebra of order t. Since $\wp$ is of order eight, we must have

$$8 = m^2t$$

which has just two solutions, namely,

$$m = 1; \quad t = 8 \text{ and } m = 2; \quad t = 2.$$

The first solution can be discounted since it would imply that $\wp$ is a division algebra, which it cannot be since it contains idempotent elements other than the unity. This leaves the second solution which proves that the real Pauli algebra can be represented as a direct product of a full 2 x 2 matric algebra over the *real* numbers with a division algebra of order two over the *reals*. (Recall that the Pauli spin matrices are a 2 x 2 matrix algebra over the complex numbers.)

Now let us turn to consider the two-sided Pierce decomposition (See Albert 1961) of the real Pauli algebra in the form

$$\wp = u\wp u \oplus u\wp u' \oplus u'\wp u \oplus u'\wp u'.$$

where each term in the direct sum is an algebra in its own right, the second and third terms being nilpotent algebras. We note the following property:

$$(u\wp u')(u'\wp u) = u(\wp u'\wp)u$$

But $\wp u'\wp$ is an ideal in $\wp$ which is not the zero ideal. Thus, since $\wp$ is simple, we must have:

$$\wp u'\wp = \wp$$

Hence:

$$(u\wp u')(u'\wp u) = u\wp u \supset \{0\}$$

which has been shown to be a division algebra. Similarly:

$$(u'\wp u)(u\wp u') = u'\wp u' \supset \{0\}$$

is a division algebra.

The vector space $u\mathfrak{p}u'$ is spanned by the elements ue_Au'. But:

$$\begin{aligned} ulu' &= 0 & ue_{123}u' &= 0 \\ ue_1u' &= e_1u' = ue_1 & ue_{13}u' &= -e_1u' = -ue_1 \\ ue_2u' &= e_2u' = ue_2 & ue_{23}u' &= -e_2\,u' = -ue_2 \\ ue_3u' &= 0 & ue_{12}u' &= 0 \end{aligned}$$

showing this space to be of dimension two. Noting that:

$$ue_2 = -e_{123}ue_1$$

it is seen that every element x of this space can be written as:

$$x = zue_1$$

with $z \in \mathbb{Z}$. In similar fashion, it can be shown that the vector space $u'\mathfrak{p}u$ is spanned by the two independent vectors $e_1u(=u'e_1)$ and $e_2u(=u'e_2)$ so that every $y \in u'\mathfrak{p}u$ can be written as:

$$y = ze_1u$$

with $z \in \mathbb{Z}$.

Consider the element $E_{21} = e_1u$ of $u'\mathfrak{p}u$. Then:

$$(u\mathfrak{p}u')\,E_{21} \neq \{0\}$$

so there is at least one element, say a_{12}, of $u\mathfrak{p}u'$ such that:

$$a_{12}E_{21} \neq 0$$

In particular, the element $a_{12} = ue_1$ satisfies this condition, with:

$$a_{12}E_{21} = u$$

which is an element of the division algebra $u\mathfrak{p}u$ and which therefore has an inverse, namely:

$$(a_{12}E_{21})^{-1} = u$$

Now, put:

$$\begin{aligned} E_{12} &= (a_{12}E_{21})^{-1}a_{12} \\ E_{11} &= E_{12}E_{21} = u \\ E_{22} &= E_{21}E_{12} = e_1ue_1 = u' \end{aligned}$$

Then the four elements E_{ij} constitute a basis for a 2x2 matric algebra since: $E_{ij}E_{mn} = \delta_{jm}E_{in}$

We have thus distinguished a matric subalgebra of $\mathfrak{p}$, namely the algebra built on the subspace spanned by the elements $1, e_1, e_3, e_{13}$, that is to say the space:

$$M = \langle 1\rangle \oplus \langle e_1\rangle \oplus \langle e_3\rangle \oplus \langle e_{13}\rangle$$

where clearly: $MM \subseteq M$

in accordance with the property that M is a subalgebra of $\mathfrak{p}$.

Consider now the set $\mathfrak{P}^M$ of all elements of $\mathfrak{P}$ which commute with all the elements of M. Since $e_{13} = -e_{123}e_2$, an element x of $\mathfrak{P}$ will commute with the elements of M if and only if $x \in \mathfrak{Z}$. Also, from the relations:

$$e_{123}e_1 = e_{23} \quad ; \quad e_{123}e_3 = e_{12} \quad ; \quad e_{123}e_{13} = e_2$$

it is evident that $M\mathfrak{Z} \supseteq \mathfrak{P}$ and we have therefore

proved that the real Pauli algebra can be written as the direct product:

$$\mathfrak{P} = M_2(\mathbb{R}) \times \mathfrak{Z}$$

where M is the matric subalgebra of $\mathfrak{P}$ spanned by the set $\{1, e_1, e_3, e_{13}\}$ and where $\mathfrak{Z}$ is the centre.

Consequently, every element of $\mathfrak{P}$ can be written uniquely either as:

$$p = m_o 1 + m_1 e_{123}$$

where m_o and m_1 are uniquely determined elements of M_2, or as:

$$p = \sum_{ij} z_{ij} E_{ij}$$

where z_{ij} are uniquely determined elements of $\mathfrak{Z}$. Each of these expressions is itself derived from the complete expression:

$$p = \sum_{ijk} \alpha_{ijk} E_{ij} \varepsilon_k$$

where the α_{ijk} are uniquely determined elements in the field of real numbers, $\{E_{ij}\}$ is a basis for the subalgebra M_2 and $\{\varepsilon_k\}$ is a basis for the subalgebra $\mathfrak{Z}$.

Consider now the uniqueness of the above decomposition. From the theory of simple algebras, we have the result that if:

$$\mathfrak{P} = M'' \times D''$$

is another decomposition of the same kind, then there is some invertible element g in $\mathfrak{P}$ such that:

$$g^{-1} M g = M'' \qquad \text{and} \qquad g^{-1} \mathfrak{Z} g = D''$$

But $\mathfrak{Z}g = g\mathfrak{Z}$ for all $g \in \mathfrak{P}$, thus $D'' = \mathfrak{Z}$ and the decomposition is unique up to an equivalence transformation on M by some invertible element $g \in \mathfrak{P}$. Note however that even though M and M" are equivalent algebras, yet they are distinct, that is to say, $M \cong M''$ but in general $M \neq M''$.

6. THE PAULI ALGEBRA OVER THE FIELD OF COMPLEX NUMBERS

The field of complex numbers contains roots of -1. Hence the complex Pauli algebra can be shown to be semi-simple, consisting of two simple component subalgebras according to the decomposition:

$$\mathfrak{P} = \mathfrak{P}\tfrac{1}{2}(1+ie_{123}) \oplus \mathfrak{P}\tfrac{1}{2}(1-ie_{123})$$

This decomposition is unique up to the order of the terms. We examine each of the two component subalgebras individually. Consider first the simple algebra $\mathfrak{P}\frac{1}{2}(1+ie_{123})$. Denote the idempotents $\frac{1}{2}(1+ie_{123})$ and $\frac{1}{2}(1-ie_{123})$ by ε and ε' respectively. Then noting the relations:

$$e_{12}\varepsilon = -ie_3\varepsilon \;;\; e_{13}\varepsilon = ie_2\varepsilon \;;\; e_{23}\varepsilon = -ie_1\varepsilon \;;\; e_{123}\varepsilon = -i\varepsilon$$

we see that the set $\{\varepsilon, e_A\varepsilon \mid A=1,3,13\}$ is a basis for $\mathfrak{P}\varepsilon'$. The centre of $\mathfrak{P}\varepsilon$ is the set $\mathfrak{P}\varepsilon\cap\mathfrak{Z}$ consisting of all $z\in\mathfrak{Z}$ such that $z\varepsilon = z$, that is to say:

$$\mathfrak{P}\varepsilon\cap\mathfrak{Z} = \mathfrak{Z}\varepsilon \cong \mathbb{C}$$

showing that the algebra is normal simple. Considering the decomposition of $\mathfrak{P}\varepsilon$ into the direct product of a matric algebra with a division algebra, we obtain the equation:

$$4 = m^2t$$

having the two solutions $m=1;t=4$ and $m=2;t=1$. Since $\mathfrak{P}\varepsilon$ is not a division algebra, the first solution can be discarded. Thus $\mathfrak{P}\varepsilon$ is a full matric algebra over $\mathbb{C}$. Now, the unity of $\mathfrak{P}\varepsilon$ is ε. One possible idempotent decomposition of ε is:

$$\varepsilon = \tfrac{1}{2}(1+e_3)\varepsilon + \tfrac{1}{2}(1-e_3)\varepsilon$$

and since:

$$\tfrac{1}{2}(1\pm e_3)\varepsilon\mathfrak{P}\tfrac{1}{2}(1\pm e_3)\varepsilon = \langle\tfrac{1}{2}(1\pm e_3)\varepsilon\rangle$$

it is seen that the idempotents $\frac{1}{2}(1+e_3)\varepsilon$ and $\frac{1}{2}(1-e_3)\varepsilon$ are both primitive. In terms of this idempotent decomposition then, we obtain a matric basis for $\mathfrak{P}\varepsilon$ by putting:

$$E_{11} = \tfrac{1}{2}(1+e_3)\varepsilon \qquad E_{12} = \tfrac{1}{2}e_1(1-e_3)\varepsilon$$

$$E_{21} = \tfrac{1}{2}e_1(1+e_3)\varepsilon \qquad E_{22} = \tfrac{1}{2}(1-e_3)\varepsilon$$

Entirely analogous results hold for the algebra $\wp\varepsilon'$ a matric basis for this algebra being given by the elements:

$$E'_{11} = \tfrac{1}{2}(1+e_3)\varepsilon'$$
$$E'_{22} = \tfrac{1}{2}(1-e_3)\varepsilon'$$
$$E'_{12} = \tfrac{1}{2}e_1(1-e_3)\varepsilon'$$
$$E'_{21} = \tfrac{1}{2}e_1(1+e_3)\varepsilon'$$

The expression of these algebras as full matric algebras is unique up to similarity transformation.

7. REPRESENTATION OF THE REAL PAULI ALGEBRA BY REAL MATRICES

Here the field used in the representation is the same as the base field of the algebra. Thus, since the algebra is simple, it will have exactly one irreducible representation by real matrices obtainable from a reduction of the regular representation.

As has been shown, a decomposition of the unity element of $\wp$ into the sum of mutually orthogonal primitive idempotent elements is given by:

$$1 = \tfrac{1}{2}(1+e_3) + \tfrac{1}{2}(1-e_3) \qquad (4)$$

Thus, under regular representation, $\wp$ will decompose into the direct sum of two minimal left ideals or, in other words, two irreducible invariant vector spaces such that

$$\wp = \wp\,\tfrac{1}{2}(1+e_3) \oplus \wp\,\tfrac{1}{2}(1-e_3)$$

showing that the regular representation consists of the direct sum of two irreducible components.. Since the algebra possesses only one distinct irreducible representation by real matrices, these two components must be equivalent.

A basis for the minimal left ideal $\wp \frac{1}{2}(1+e_3)$ is given by the four basic elements

$$\epsilon_1 = \tfrac{1}{2}(1+e_3)\ ;\ \epsilon_2 = e_1\epsilon_1\ ;\ \epsilon_3 = e_2\epsilon_1\ ;\ \epsilon_4 = e_{12}\epsilon_1$$

while the four elements:

$$\epsilon_5 = \tfrac{1}{2}(1-e_3)\ ;\ \epsilon_6 = e_1\epsilon_5\ ;\ \epsilon_7 = e_2\epsilon_5\ ;\ \epsilon_8 = e_{12}\epsilon_5 .$$

constitute a basis for the minimal left ideal $\wp \frac{1}{2}(1-e_3)$ Together, these eight vectors span the algebra $\wp$. In terms of this basis, the regular representation takes the form:

$$p = p^A e_A \longrightarrow \begin{bmatrix} A & C & & \\ D & B & & O \\ & & B & D \\ O & & C & A \end{bmatrix}$$

where:

$$A = \begin{bmatrix} p^o+p^3 & p^1-p^{13} \\ p^1+p^{13} & p^o-p^3 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} p^o-p^3 & p^1+p^{13} \\ p^1-p^{13} & p^o+p^3 \end{bmatrix}$$

$$C = \begin{bmatrix} p^2-p^{23} & -p^{12}-p^{123} \\ p^{12}-p^{123} & -p^2-p^{23} \end{bmatrix} \quad ; \quad D = \begin{bmatrix} p^2+p^{23} & -p^{12}+p^{123} \\ p^{12}+p^{123} & -p^2+p^{23} \end{bmatrix}$$

The equivalence of the two irreducible components of the representation is easily demonstrated by considering the similarity transformation generated by the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1}$$

We have:

$$S^{-1}\begin{pmatrix} A & C \\ D & B \end{pmatrix} S = \begin{pmatrix} B & D \\ C & A \end{pmatrix}$$

as required. Thus the correspondence:

$$p = p^A e_A \longrightarrow \begin{bmatrix} p^o+p^3 & p^1-p^{13} & p^2-p^{23} & -p^{12}-p^{123} \\ p^1+p^{13} & p^o-p^3 & p^{12}-p^{123} & -p^2-p^{23} \\ p^2+p^{23} & -p^{12}+p^{123} & p^o-p^3 & p^1+p^{13} \\ p^{12}+p^{123} & -p^2+p^{23} & p^1-p^{13} & p^o+p^3 \end{bmatrix} \qquad (5)$$

furnishes the only irreducible representation (unique up to equivalence) of the real Pauli algebra in terms of real matrices from which all other representations of $\mathfrak{p}$ by real matrices can be constructed. We thus have proved that: there is only one irreducible representation of the real Pauli algebra in terms of real matrices and this representation is faithful. Every other real representation of $\mathfrak{p}$ is fully reducible and faithful.

8. REPRESENTATION OF THE COMPLEX PAULI ALGEBRA BY COMPLEX MATRICES

We have shown that a complete decomposition of the unity element of $\mathfrak{p}$ into the sum of mutually orthogonal primitive idempotent elements is given by:

$$1 = u\varepsilon + u'\varepsilon + u\varepsilon' + u'\varepsilon'$$

where $u = \frac{1}{2}(1+e_3)$, $u' = \frac{1}{2}(1-e_3)$, $\varepsilon = \frac{1}{2}(1+ie_{123})$ and $\varepsilon' = \frac{1}{2}(1-ie_{123})$.
Thus under the regular representation $\mathfrak{p}$ decomposes into the sum of four minimal left ideals according to:

$$\mathfrak{p} = \mathfrak{p}u\varepsilon \oplus \mathfrak{p}u'\varepsilon \oplus \mathfrak{p}u\varepsilon' \oplus \mathfrak{p}u'\varepsilon'$$

We choose the following basis for the algebra:

(i) basis for $\mathfrak{p}u\varepsilon$: $\epsilon_1 = u\varepsilon$; $\epsilon_2 = e_1\epsilon_1$

(ii) basis for $\mathfrak{p}u'\varepsilon$: $\epsilon_3 = u'\varepsilon$; $\epsilon_4 = e_1\epsilon_3$

(iii) basis for $\mathfrak{p}u\varepsilon'$: $\epsilon_5 = u\varepsilon'$; $\epsilon_6 = e_1\epsilon_5$

(iv) basis for $\mathfrak{p}u'\varepsilon'$: $\epsilon_7 = u'\varepsilon'$; $\epsilon_8 = e_1\epsilon_7$

Then, relative to this basis, the regular representation takes the form:

$$p = p^{A}e_{A} \longrightarrow \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{bmatrix}$$

with:

$$A = \begin{bmatrix} (p^{o}+p^{3})-i(p^{12}+p^{123}) & (p^{1}-p^{13})+i(p^{2}-p^{23}) \\ (p^{1}+p^{13})-i(p^{2}+p^{23}) & (p^{o}-p^{3})+i(p^{12}-p^{123}) \end{bmatrix}$$

$$B = \begin{bmatrix} (p^{o}-p^{3})+i(p^{12}-p^{123}) & (p^{1}+p^{13})-i(p^{2}+p^{23}) \\ (p^{1}-p^{13})+i(p^{2}-p^{23}) & (p^{o}+p^{3})-i(p^{12}+p^{123}) \end{bmatrix}$$

$$C = \begin{bmatrix} (p^{o}+p^{3})+i(p^{12}+p^{123}) & (p^{1}-p^{13})-i(p^{2}-p^{23}) \\ (p^{1}+p^{13})+i(p^{2}+p^{23}) & (p^{o}-p^{3})-i(p^{12}-p^{123}) \end{bmatrix}$$

and
$$D = \begin{bmatrix} (p^{o}-p^{3})-i(p^{12}-p^{123}) & (p^{1}+p^{13})+i(p^{2}+p^{23}) \\ (p^{1}-p^{13})-i(p^{2}-p^{23}) & (p^{o}+p^{3})+i(p^{12}+p^{123}) \end{bmatrix}$$

where the coefficients p^{A} are drawn from the field of complex numbers.

The matrix element P of the regular representation corresponding to the algebraic element p of $\mathfrak{p}$ is thus of the form:

$$P = A \oplus B \oplus C \oplus D$$

with each of A,B,C and D irreducible. Now, if the first component be written as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

comparison of the entries of the matrix A with those of B will show that B is of the form:

$$B = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

Thus using the matrix:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1}$$

we have:

$$B = S^{-1}AS$$

showing that the irreducible representations carried by the ideals $\mathfrak{p}u\mathcal{E}$ and $\mathfrak{p}u'\mathcal{E}$ of $\mathfrak{p}$ are equivalent. Similarly, it can be shown that the representations carried by the ideals $\mathfrak{p}u\mathcal{E}'$ and $\mathfrak{p}u'\mathcal{E}'$ of $\mathfrak{p}$ are equivalent by the same matrix S. It is not difficult to show the inequivalence of the representations A and C.

Since the regular representation of $\mathcal{P}$ contains exactly two distinct irreducible components, we see that the algebra $\mathcal{P}$ has exactly two irreducible representations which we denote by $\Gamma(1)$ and $\Gamma(2)$. Also, since every finite dimensional representation of a semi-simple algebra is fully reducible, every representation of $\mathcal{P}$ is obtainable from these two irreducible components by direct summation. We note also the following results:

(a) The irreducible representations are not faithful; the homomorphism of $\mathcal{P}$ onto $\Gamma(j)$ is two to one.

(b) Every faithful representation of $\mathcal{P}$ must contain at least one of each of the irreducible types $\Gamma(1)$ and $\Gamma(2)$.

Hence also every representation of $\mathcal{P}$ for which each irreducible component is of a single type, $\Gamma(j)$ say, is not faithful and every representation containing at least one irreducible component of each type is faithful. We have thus proved:

> The complex Pauli algebra has exactly two distinct irreducible representations $\Gamma(1)$ and $\Gamma(2)$. Every representation of $\mathcal{P}$ is of one of two types, either:
>
> (a) $\Gamma=\Gamma(j)+\Gamma(j)+\ldots.+\Gamma(j)$ with n terms, for either j=1 or j=2, in which case the representation is a two to one homomorphism of $\mathcal{P}$ onto Γ, or
>
> (b) $\Gamma= \underbrace{\Gamma(1)+\ldots+\Gamma(1)}_{\text{n terms}}+\underbrace{\Gamma(2)+\ldots+\Gamma(2)}_{\text{m terms}}$ in which case the representation is faithful.

Note that since the coefficients p^A of p are drawn from the complex number field, we do not have $\Gamma(1)\cong\Gamma^*(2)$ in spite of the deceptive appearance of the corresponding matrices.

8. REPRESENTATION OF THE REAL PAULI ALGEBRA BY COMPLEX MATRICES

Here we are interested in establishing a clear connection between the real Pauli algebra and the Pauli spin matrices. Therefore it is of value to consider the representation of an algebra by an algebraic structure whose base field is not its base field but rather a scalar extension of it. In this case, the standard results of the theory of the representation of algebras are no longer valid and must be suitably modified to accommodate the situation at hand. More particularly, a simple algebra will be found to have more than one irreducible representation, the exact number of distinct irreducible representations existing being governed by the nature of the extension of the field $\mathcal{F}$ used.

The real Pauli algebra $\mathfrak{P}(\mathbb{R})$ can be considered as a subalgebra of the complex algebra in the following way. The field of complex numbers is a division algebra of order two over the field of real numbers. We can thus write:

$$\mathfrak{P}(\mathbb{C}) \cong \mathfrak{P}(\mathbb{R}) \text{ x } \mathbb{C}(\mathbb{R})$$

The real Pauli algebra is then a subalgebra of the complex algebra obtainable from it by restriction of the coefficients p^A of p to the real numbers. Applying this same restriction to a representation of the complex algebra, we will arrive at a representation of the real algebra by complex matrices. In particular, from the results of section 7, we will obtain exactly two irreducible representations of the real algebra by complex matrices as follows:

$$\Gamma(1): p^A e_A \longrightarrow A = \begin{bmatrix} (p^{o}+p^{3})-i(p^{12}+p^{123}) & (p^{1}-p^{13})-i(p^{23}-p^{2}) \\ (p^{1}+p^{13})-i(p^{23}+p^{2}) & (p^{o}-p^{3})-i(p^{123}-p^{12}) \end{bmatrix} \qquad (7)$$

and

$$\Gamma(2): p^A e_A \longrightarrow C = \begin{bmatrix} (p^{o}+p^{3})+i(p^{12}+p^{123}) & (p^{1}-p^{13})+i(p^{23}-p^{2}) \\ (p^{1}+p^{13})+i(p^{2}+p^{23}) & (p^{o}-p^{3})+i(p^{123}-p^{12}) \end{bmatrix} \qquad (8)$$

It is easily demonstrated that the representations $\Gamma(1)$ and $\Gamma(2)$ of $\mathfrak{P}(\mathbb{R})$ are not equivalent and that they are faithful. Also, since the coefficients p^A are real, it is evident that:

$$\Gamma(1) \cong \Gamma^*(2)$$

where Γ^* denotes the representation complex conjugate to Γ.

The same results could have been derived by a different but equivalent and certainly more perspicuous method. We note first the following: if $\mathcal{A}$ and $\mathcal{B}$ are isomorphic algebras and if S and T are isomorphisms $: \mathcal{A} \longrightarrow \mathcal{B}$, then evidently:

$$\alpha = S^{-1}T : \mathcal{A} \longrightarrow \mathcal{A}$$

and

$$\beta = TS^{-1} : \mathcal{B} \longrightarrow \mathcal{B}$$

are automorphisms of the algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Indeed, every isomorphism $: \mathcal{A} \longrightarrow \mathcal{B}$ can then be written as $T = S\alpha$ or as $T = \beta S$. The problem of finding all isomorphisms between two algebras has thus been reduced to the problem of finding a single fixed isomorphism between them together with all the automorphisms of one of the algebras.

Now, it has been shown that:

$$\mathfrak{P}(\mathbb{R}) \cong M_2(\mathbb{R}) \times Z$$

However, since $\mathbb{Z} \cong \mathbb{C}$ we have:

$$\mathfrak{P}(\mathbb{R}) \cong M_2(\mathbb{R}) \times \mathbb{C}$$

showing that the real Pauli algebra is isomorphic to the full matrix algebra of two dimensional complex matrices. We can thus use the two dimensional complex matrices to give a faithful representation of $\mathfrak{P}(\mathbb{R})$, each possible isomorphism supplying a representation. Since $\mathfrak{P}(\mathbb{R})$ is simple and since the order of $\mathfrak{P}(\mathbb{R})$ is equal to the order of $M_2(\mathbb{C})$, $M_2(\mathbb{C})$ is simple. It will therefore supply an irreducible representation of $\mathfrak{P}(\mathbb{R})$. The problem of finding all irreducible complex representations of $\mathfrak{P}(\mathbb{R})$ is then equivalent to that of finding a single isomorphism between the two algebras together with all the automorphisms (over $\mathbb{R}$) of the complex matrix algebra.

An isomorphism $: \mathfrak{P}(\mathbb{R}) \longrightarrow M_2(\mathbb{R}) \times \mathbb{C}$, which we here use as the required fixed isomorphism, is given by:

$$S : \tfrac{1}{2}(1+e_3) \longmapsto E_{11}$$

$$: \tfrac{1}{2}(1-e_3) \longmapsto E_{22}$$

$$\begin{aligned} S &: \tfrac{1}{2}e_1(1+e_3) \longmapsto E_{21} \\ &: \tfrac{1}{2}e_1(1-e_3) \longmapsto E_{12} \\ &: e_{123} \longmapsto i(E_{11}+E_{22}) \end{aligned}$$

Now, every automorphism $f:\mathfrak{P}\rightarrow\mathfrak{P}$ when restricted to the centre $\mathfrak{Z}$ of $\mathfrak{P}$ induces an automorphism $f|_{\mathfrak{Z}}:\mathfrak{Z}\rightarrow\mathfrak{Z}$ on $\mathfrak{Z}$. But as $\mathfrak{Z}$ has exactly two automorphisms, namely the identity 1 and the conjugation $*$ defined by the requirement

$$* : (\alpha + \beta\, e_{123}) \longmapsto (\alpha - \beta\, e_{123})$$

we must have either $f|_{\mathfrak{Z}} = 1$ or $f|_{\mathfrak{Z}} = *$. If $f|_{\mathfrak{Z}} = 1$ we say that $f:\mathfrak{P}\rightarrow\mathfrak{P}$ is an automorphism of class A while if $f|_{\mathfrak{Z}} = *$ we say it is of class B. The automorphisms $F: M_2 x \mathbb{C} \rightarrow M_2 x \mathbb{C}$ of $M_2 x \mathbb{C}$, considered as automorphisms over the field of real numbers, similarly divide into exactly two classes according as they do or do not conjugate the subalgebra $\mathbb{C}$.

Consider first those irreducible representations of $\mathfrak{P}(\mathbb{R})$ arising from a combination FS of the fixed isomorphism S with a class A automorphism F of $M_2 x \mathbb{C}$. Then $F|_{\mathbb{C}} = 1$. Now, for each $p \in \mathfrak{P}(\mathbb{R})$

we have $\qquad S(p) = c_{ij}E_{ij} \quad$ with $c_{ij} \in \mathbb{C}$

so $\qquad FS(p) = F(c_{ij})F(E_{ij}) = c_{ij}F(E_{ij})$

But F is class A on $M_2 x \mathbb{C}$ and therefore inner, so $\exists$ some invertible $g \in M_2 x \mathbb{C}$ such that $F(m) = g^{-1}mg$ for all $m \in M_2 x \mathbb{C}$. Thus

$$FS(p) = g^{-1}c_{ij}E_{ij}g = g^{-1}S(p)\, g$$

showing that each class A automorphism F of $M_2 x \mathbb{C}$ will give rise to a representation of $\mathfrak{P}(\mathbb{R})$ equivalent to the representation S.

Let $*^M$ be the conjugation automorphism on $M_2 x \mathbb{C}$ defined by the requirement

$$*^M|_{M_2} = 1 \quad \text{and} \quad *^M|_{\mathbb{C}} = *$$

Then every class B automorphism F^* on $M_2 x \mathbb{C}$ can be written as

$$F^* = F \,.\, *^M$$

where F is class A. Thus for each class B automorphism F^* of $M_2 x \mathbb{C}$

we have $\qquad F^* \,.\, S(p) = F \,.\, *^M \,.\, S(p) = g^{-1}\left[*^M . S(p) \right] g$

for some invertible $g \in M_2 x \mathbb{C}$, showing that all representations arising from the class B automorphisms are equivalent to the representation $\left[*^M . S \right]$.

Since $*^M$ is not an inner automorphism, it follows that the representations arising from S and from $*^M.S$ are not equivalent. We have therefore proved:

> There are exactly two distinct irreducible representations of the real Pauli algebra by complex matrices.

The two distinct irreducible representations supplied by the isomorphisms S and $*^M.S$ can be shown to coincide with the representations $\Gamma(1)$ and $\Gamma(2)$. Note also that since the algebra $\mathfrak{P}(\mathbb{R})$ is simple, it is also semi-simple. Hence every representation of $\mathfrak{P}(\mathbb{R})$ is fully reducible and equivalent to a direct sum of irreducible components. All the complex representations of $\mathfrak{P}(\mathbb{R})$ are faithful.

Finally, to complete the discussion, we exhibit the relationship that the complex representations of $\mathfrak{P}(\mathbb{R})$ bear to the regular representation. It follows from the above that the two irreducible complex representations of $\mathfrak{P}(\mathbb{R})$ arise essentially from the possibility of representing inequivalently the centre $\mathfrak{Z}$ of the algebra by $\mathbb{C}$. We have:

$$\mathfrak{P}(\mathbb{R}) = M_2(\mathbb{R}) \times \mathfrak{Z}$$

where M_2 is the matric subalgebra spanned by the vectors $1, e_1, e_3, e_{13}$. Choosing the idempotent decomposition:

$$1 = \tfrac{1}{2}(1+e_3) + \tfrac{1}{2}(1-e_3)$$

and spanning the minimal invariant subspaces $M\frac{1}{2}(1+e_3)$ and $M\frac{1}{2}(1-e_3)$ of M_2 by the elements $\frac{1}{2}(1+e_3)$, $\frac{1}{2}e_1(1+e_3)$ and $\frac{1}{2}(1-e_3)$, $\frac{1}{2}e_1(1-e_3)$ respectively, we obtain a regular representation of M_2:

$$: e_1 \longrightarrow \left[\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right]$$

$$: e_3 \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & -1 & \multicolumn{2}{c}{0} \\ \hline & & -1 & 0 \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right]$$

$$: e_{13} \longmapsto \left[\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array}\right]$$

Then, representing $\mathcal{Z}$ according to:

$$:(a+be_{123}) \longmapsto (a+ib)$$

representations equivalent to $\Gamma(2)$ are obtained in both of the components $M\frac{1}{2}(1+e_3)$ and $M\frac{1}{2}(1-e_3)$ of the regular representation. Representing $\mathcal{Z}$ according to:

$$:(a+be_{123}) \longmapsto (a-ib)$$

representations equivalent to $\Gamma(1)$ are obtained in each component of the regular representation.

Thus when e_{123} is represented by the complex unit i, the representation carried by the space $M\frac{1}{2}(1+e_3)$ is readily recognised as the algebra of Pauli matrices used in the quantum theory, e_1, e_2 and e_3 corresponding to the Pauli matrices. The representation carried by the space $M\frac{1}{2}(1-e_3)$ is equivalent to it.

9. GEOMETRIC CONTENT OF THE PAULI ALGEBRA

As we have indicated in section 2 we can write

$$\mathcal{P} = \mathcal{J} \oplus E_3 \oplus \mathcal{B} \oplus \mathcal{P}. \tag{9}$$

The decomposition is manifest in the representation of the real Pauli algebra by complex matrices. For example, eqn. (8) for $\Gamma(2)$ can be written as

$$\begin{pmatrix} p^0 & 0 \\ 0 & p^0 \end{pmatrix} + \begin{pmatrix} p^3 & p^1-ip^2 \\ p^1+ip^2 & -p^3 \end{pmatrix} + \begin{pmatrix} ip^{12} & ip^{23}-p^{13} \\ ip^{23}-p^{13} & -ip^{12} \end{pmatrix} + \begin{pmatrix} ip^{123} & 0 \\ 0 & ip^{123} \end{pmatrix} \tag{10}$$

A comparison with Cartan's (1966) discussion shows explicitly that the expression is of the form (9) provided we identify p^i, p^{ij} and p^{123} with the components of a vector $\underline{x}$, bivector $\underline{x} \wedge \underline{y}$ and pseudoscalar $\underline{x} \wedge \underline{y} \wedge \underline{z}$ respectively.

The relation between the Pauli algebra and the rotation group in E_3 is known to arise through an inner automorphism on the algebra. Indeed, every otation can be written in the form $p\,\underline{x}\,p^{-1}$ where $\underline{x}$ E_3 and $p = \underline{a}\,\underline{b}$

is the product of two vectors in E_3. If we denote the unit vector in the direction of $\underline{a}$ as $\hat{e}_1$ then any real or complex vector $\underline{b}$ can be written as

$$\underline{b} = b_1\hat{e}_1 + b_2\hat{e}_2 = \mathrm{Cos}\,\theta\, b\,\hat{e}_1 + \mathrm{Sin}\,\theta\, b\,\hat{e}_2$$

where $\hat{e}_2$ is a unit vector orthogonal to $\hat{e}_1$, θ is the angle between $\underline{a}$ and $\underline{b}$, and $b^2 = (\underline{b}.\underline{b})$. Then

$$\underline{a}\underline{b} = ab\,(\mathrm{Cos}\,\theta + \mathrm{Sin}\,\theta\,\hat{e}_{12}) \text{ and } (\underline{a}\underline{b})^{-1} = (ab)^{-1}(\mathrm{Cos}\,\theta - \mathrm{Sin}\,\theta\,\hat{e}_{12})$$

So that

$$p\underline{x}p^{-1} = (x^1\mathrm{Cos}\,2\theta + x^2\mathrm{Sin}\,2\theta)\hat{e}_1 + (-x^1\mathrm{Sin}\,2\theta + x^2\mathrm{Cos}\,2\theta)\hat{e}_2 + x^3\hat{e}_3$$

where x^i are the coefficients of $\underline{x}$ in the orthonormal triad $\hat{e}_i$. Thus we produce a rotation of each vector $\underline{x}$ of E_3 through an angle 2θ in the plane defined by the unit bivector $\hat{e}_{12}$. In this way we find a group homomorphism from the group of elements of $\mathcal{P}$ of the form:

$$p(\tfrac{1}{2}\theta, \hat{I}) = (\mathrm{Cos}\,\tfrac{1}{2}\theta + \mathrm{Sin}\,\tfrac{1}{2}\theta\,\hat{I}) \tag{11}$$

onto the group of rotations of E_3 according to

$$\rho : p(\tfrac{1}{2}\theta, \hat{I}) \longrightarrow \mathcal{R}(\theta, \hat{I})$$

where $\hat{I}$ is a unit bivector.

In view of eqn (9) we can write $\mathcal{P} = \mathcal{P}^+ \oplus \mathcal{P}^-$ where $\mathcal{P}^+ = \mathcal{J} \oplus \mathcal{B}$ is the even component of $\mathcal{P}$. As $\mathcal{P}^+\mathcal{P}^+ \subseteq \mathcal{P}^+$ and $\mathcal{P}^-\mathcal{P}^- \subseteq \mathcal{P}^+$, we see that $\mathcal{P}^+$ is a subalgebra of $\mathcal{P}$ whereas $\mathcal{P}^-$ is not. Since any element of $\mathcal{P}^+$ can be written as $p^+ = a + b\hat{I}$ which contains eqn. (11), ~~so that~~ in the real Pauli algebra every rotation can be generated by elements of p^+ with $a = \lambda\,\mathrm{Cos}\,\tfrac{1}{2}\theta$ and $b = \lambda\,\mathrm{Sin}\,\tfrac{1}{2}\theta$ with $\lambda \in \mathbb{R}$. In this way we obtain a group homomorphism

$$\rho : G_p(\mathcal{P}^+) \longrightarrow O^+(3)$$

which has as kernel $\mathcal{P}^+ \cap \mathcal{Z}_1 = \mathbb{R}$, $'\mathcal{Z}_1$ being the centre of the Pauli algebra. Thus $\mathcal{P}^+$ provides a multivalued (infinity to one) representation of $O^+(3)$. That is to say

$$G_p(\mathcal{P}^+)/\mathbb{R} \cong O^+(3).$$

This analysis shows that $G_p(\mathcal{P}^+)$ contains more structure than had hitherto been realised and we have the possibility of new multivalued representations of $O(3)$. It is evident that the common assertion that "all the representations of the rotation group are contained in the series $\mathcal{D}_{p/2}$" (Wigner 1959) is incorrect.

Considering the subset $U = \{p^+ \mid \lambda = \pm 1\}$ of $Gp(\mathcal{P}^+)$ as a group, we have the 2:1 group homorphism $\rho|_U : U \longrightarrow O^+(3)$ with kernel $\{1, -1\}$. Thus U provides a two valued representation of the rotation group, so that

$$U / \{1, -1\} \cong O^+(3) \tag{12}$$

We can show U to be the group SU(2) most easily by looking at the representations of $\Gamma(1)$ and $\Gamma(2)$ given by eqn (7) and (8). Writing $p^0 = \cos\frac{1}{2}\theta$, $p^{ij} = b^{ij}\sin\frac{1}{2}\theta$, $p^i = p^{123} = 0$

we have

$$\Gamma(1): p \longmapsto \pm \begin{pmatrix} \cos\frac{1}{2}\theta - i b^{12}\sin\frac{1}{2}\theta & -(b^{13} - i b^{23})\sin\frac{1}{2}\theta \\ (b^{13} + i b^{23})\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta - i b^{12}\sin\frac{1}{2}\theta \end{pmatrix}$$

and

$$\Gamma(2): p \longmapsto \pm \begin{pmatrix} \cos\frac{1}{2}\theta + i b^{12}\sin\frac{1}{2}\theta & -(b^{13} - i b^{23})\sin\frac{1}{2}\theta \\ (b^{13} + i b^{23})\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta - i b^{12}\sin\frac{1}{2}\theta \end{pmatrix}$$

These are unitary unimodular matrices, showing that SU(2) is isomorphic to the subgroup U contained in the Pauli algebra. One can also establish that every irreducible representation of the Pauli algebra (real or complex) induces the irreducible representation $\mathcal{D}_{1/2}$ of the group SU(2) by the restriction of the representation of $\mathcal{P}$ to the subgroup U of $\mathcal{P}$.

We have noted already that not every representation of the rotation group is contained in the series $\mathcal{D}_{p/2}$, the above method providing a large number of multivalued representations not found among the $\mathcal{D}_{p/2}$ (Frescura 1977). What is proven by the method of characteristics as applied to the representations $\mathcal{D}_{p/2}$ (for example, Wigner 1959) is that in the series $\mathcal{D}_{p/2}$ is to be found every single valued representation of the group SU(2) and consequently also every multivalued representation of $O^+(3)$ arising from the homomorphism SU(2) $\longrightarrow O^+(3)$, but not every multivalued representation of $O^+(3)$.

10. THE ALGEBRAIC SPINOR.

The possibility of regarding SU(2) as a subspace of the Pauli algebra $\mathfrak{P}$ paves the way for an approach to the theory of spinors different to that of group theory.

Traditionally, the spinor has been regarded as an element of a vector space supporting a representation $\mathcal{D}_{1/2}$ of SU(2). Now, consider the left regular representation of $\mathfrak{P}$. In this representation, the algebra $\mathfrak{P}$ simultaneously assumes two distinct roles, namely that of operator and also that of operand, so that the representation space in the regular representation is the algebra $\mathfrak{P}$ itself. As a representation space, $\mathfrak{P}$ will divide into the direct sum of invariant subspaces, each of which carries an irreducible representation of the algebra $\mathfrak{P}$ (and so also of SU(2)). These invariant subspaces are easily shown to be minimal left ideals of the algebra, the decomposition of $\mathfrak{P}$ into invariant subspaces corresponding to some decomposition of the unity element 1 into the sum of mutually orthogonal primitive idempotents (Weyl 1950). Hence, when SU(2) is regarded as a subspace of the Pauli algebra, an irreducible representation of $\mathfrak{P}$ will induce an irreducible representation $\mathcal{D}_{1/2}$ of SU(2) whose representation space will be an isomorphic copy of a minimal left ideal $\mathscr{I}_{Lm}$ of $\mathscr{P}$. In this way, the spin space of group theory can be identified with a vector space representation of a minimal left ideal of the Pauli algebra, each spinor representing some element of $\mathscr{I}_{Lm}$.

According to the approach advocated in this paper, the algebraic structure is to be regarded as of primary relevance. It seems desirable therefore to attempt a redefinition of the spinor in such a way as to shift the focus of attention away from representations of the algebra while giving due prominence rather to purely algebraic features. The above analysis suggests that we define a Pauli spinor as an element of a minimal left ideal of the Pauli algebra. The consequences of this definition are far reaching both from the point of view of formalism and also of interpretation, for while the algebraic spinor can be shown to incorporate all the features associated with its group theoretic counterpart, it also possesses certain characteristic properties absent from the older notion.

The relation of the algebraic approach to the conventional group theoretic one and some of the consequences of the algebraic view of the theory of spinors will be discussed in our next paper.

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On the Algebraic Aspects of Spinor Theory

II. The Relation between the Algebraic Spinor and the Group Theoretic Spinor

F.A.M. Frescura and B.J. Hiley

Department of Physics, Birkbeck College (University of London)

ABSTRACT

We examine in some detail the consequences of the algebraic characterisation of the spinor discussed in a previous paper. The algebraic approach is compared and contrasted with the usual group theoretic approach for the particular case of the real and complex Pauli algebras. We show that the cogredient and the contragredient spinors introduced as two separate ad hoc spaces in the group theoretic approach appear together naturally in the algebra as left and right ideals respectively. Furthermore, we indicate how the algebraic approach contains features that are reminiscent of the dotted and undotted van der Waerden spinors but argue that the algebraic characterisation of the notion of a 'conjugate spinor' is more general than that originally proposed. We conclude that the algebraic definition of a spinor yields an object of greater generality than the corresponding group theoretic object.

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1. Introduction

In the previous paper (Frescura and Hiley 1977) we were able to justify Riesz's proposal (Riesz 1946, 1953) that a spinor can be regarded as an element of a minimal left ideal of a Clifford algebra. We considered in particular the spinor of the three-dimensional Euclidian geometry, showing its relation to the Pauli algebra. The extension of these ideas to include also the Dirac spinor is straightforward and requires little elaboration, this spinor being identifiable as an element of a minimal left ideal of the Clifford algebra for the Minkowskian geometry (the algebra of Dirac γ- matrices).

The previous discussion however in no way demonstrates the equivalence of this algebraic definition with the group theoretic one. But, as we will show, the algebraic definition yields rather an object of character more general than that of group theory in the sense that while possessing all the properties characteristic of the group theoretic spinor, it possesses also certain properties which are not present in the older theory.

Consider more closely some elementary similarities and differences of the two approaches. It was shown in paper I that both in the real and in the complex case, every minimal left ideal of the Pauli algebra is a vector space of two complex dimensions. Thus the spin space of the group theoretic approach is a vector space isomorphic to the minimal left ideals of the Pauli algebra, that is

$$S \underset{v}{\cong} \mathcal{I} \tag{1}$$

We have also shown that

$$\mathcal{P} \supseteq U \underset{g}{\cong} SU(2) \longrightarrow O^{+}(3) \tag{2}$$

where $\underset{g}{\cong}$ denotes a group isomorphism, so that the subgroup U of $\mathcal{P}$ yields a two valued representation of the rotation group $O^{+}(3)$. But $\mathcal{I}$ is a left ideal of $\mathcal{P}$ so that

$$U\mathcal{I} \subseteq \mathcal{I}$$

and we can thus consider the group U as a set of operators on the vector space $\mathcal{I}$. This is equivalent to saying that any basis of the space $\mathcal{I}$ constitutes a tensor for the group U. Hence, by virtue of the isomorphism (2), any basis of $\mathcal{I}$ will also constitute a tensor for the group SU(2). It is easily demonstrated that, as this tensor is irreducible, and since $\mathcal{I}$ is of two complex dimensions, it must be equivalent to the tensor $\mathcal{D}_{1/2}$.

Insomuch as these two properties constitute a full characterisation of the group theoretic spinor, it is evident that the algebraic definition of the spinor fully incorporates within itself all the essential features of the older definition. More precisely, the traditional theory of the spinor can be regarded as just the theory of the representation of the algebraic spinor, the spin space S being identifiable with a vector representation of a left ideal $\mathcal{I}$ of $\mathfrak{P}$ and the group theoretic spinor with an element of this representation space S. The group theoretic spinor is thus an isomorphic image of the algebraic spinor having all the vector properties of its algebraic counterpart but itself having none of the properties arising from the algebraic multiplication operation by virtue of the necessary loss of 'definition' resulting from the representation of an algebraic subspace by means of a vector space.

In the sense of the above, the algebraic approach might be said to give rise to an object more general in character than that of the older definition. By regarding the algebraic object as being in some way the more primitive, new avenues for the understanding of the role of the spinor are opened up. Thus, for example, the spinor as an algebraic element can be regarded both as an operator and as an operand, leading to the possibility of it playing an active role in the theory, as was suggested in Paper 1. Indeed, it is just the elimination of this operator-operand distinction that opens the way for a novel approach to quantum phenomena such as that proposed by Bohm (1973). It also helps to fulfil the original aim of Heisenberg, which was to express the quantum theory solely in terms of matrix operators, without the need for a state space. In formal terms, this coequality gives rise to the possibility of equations containing not only terms of the form $\Lambda\psi$ where ψ is a spinor and $\Lambda \in \mathfrak{P}$, but also of the form $\psi\Lambda$. The equations currently in use in physics, containing solely the former variety, only partially define the ideals from which the solutions are to be drawn. They might in this sense be considered to be incomplete, a complete specification of the ideal requiring terms of the second type also.

2. THE MULTIPLICITY OF SPIN SPACES

According to the group theoretic definition of the spinor, it is proper to regard the spin space as a uniquely defined unit which remains invariant under the group of transformations.
It is thus customary to speak of 'the spin space' without fear if ambiguity.

However, the algebraic definition of the spinor introduces an entirely new feature, namely, that of the existence of a multiplicity of spin spaces each possessing indentical characteristic properties but each existing as a distinct unit in its own right. This is seen from the following considerations.

In both the real and the complex Pauli algebras, every minimal ideal is generated by some primitive idempotent. Conversely, each primitive idempotent in $\mathfrak{P}$ generates a minimal ideal of $\mathfrak{P}$. Now suppose u and u' to be distinct primitive idempotent elements so that $\mathfrak{P}u$ and $\mathfrak{P}u'$ are minimal left ideals in $\mathfrak{P}$. Consider the intersection $\mathfrak{P}u \cap \mathfrak{P}u'$ of these two ideals. Suppose:

$$\mathfrak{P}u \cap \mathfrak{P}u' \neq \{0\}$$

Then if ψ is in this intersection, we must have:

$$\mathfrak{P}\psi \subseteq \mathfrak{P}u \quad \text{and} \quad \mathfrak{P}\psi \subseteq \mathfrak{P}u'$$

by the ideal property. Now, in the Pauli algebra, it is always possible to find some $\mathfrak{p} \in \mathfrak{P}$ such that for given ψ in $\mathfrak{P}u$ we have $\mathfrak{p}\psi = u$ whence:

$$\mathfrak{P}u \subseteq \mathfrak{P}\psi$$

so that:
$$\mathfrak{P}\psi = \mathfrak{P}u$$
and
$$\mathfrak{P}u \subseteq \mathfrak{P}u'$$
But $\mathfrak{P}u'$ is a minimal left ideal. Thus:

$$\mathfrak{P}u = \mathfrak{P}u'$$

and we see that two minimal left ideals of the Pauli algebra are either distinct in the sense of their intersection being zero, or equal. We have thus shown that two primitive idempotents u and u' of the Pauli algebra (real or complex) generate identical minimal ideals if and only if there is some $\mathfrak{p} \in \mathfrak{P}$ such that $\mathfrak{p}u = u'$. Otherwise, the ideals they generate are distinct, having no element in common except the zero of the algebra.

It follows that if there is no $p \in \mathfrak{P}$ such that $pu = u'$, the left ideals generated by u and u' will be distinct vector spaces contained in $\mathfrak{P}$. From the theory of the Pauli algebra it is then evident that the algebra will contain a non-denumerably infinite number of distinct minimal left ideals within its structure. The algebraic definition of the spinor therefore gives rise to the existence of a non denumerably infinite number of distinct (though isomorphic) spin spaces in $\mathfrak{P}$, so that it is no longer meaningful to speak of 'the spin space', as if it were an uniquely defined, immutable unit.

3. SPINOR TYPES.

In the group theoretic approach, the Pauli spinor appears with both upper and lower indices while the van der Waerden spinor contains in addition both dotted and undotted indices. Let us consider how these features are contained in the real and the complex Pauli algebras.

At this stage, it is important to emphasise the distinction between the real and the complex Pauli algebras. The real algebra is simple and hence contains only trivial two sided ideals while the complex algebra is semi-simple containing two inequivalent ones *. In fact, we have shown that $\mathfrak{P}(\mathbb{C}) = \mathfrak{P}\varepsilon \oplus \mathfrak{P}\varepsilon'$ with $\varepsilon = \frac{1}{2}(1 + i e_{123})$ and $\varepsilon' = \frac{1}{2}(1 - i e_{123})$.

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* By inequivalent we mean that there exists no $g \in \mathfrak{P}$ such that $\varepsilon' = g^{-1} \varepsilon g$.

Now each two-sided ideal is decomposable into primitive idempotents which generate the minimal left ideals. We have already seen that for

$$\wp(\mathbb{R}): \quad 1 = u + u'$$

and

$$\wp(\mathbb{C}): \quad \varepsilon = u\varepsilon + u'\varepsilon \quad \text{and} \quad \varepsilon' = u\varepsilon' + u'\varepsilon'$$

One can show that $u \sim u'$ so that the complex algebra contains two inequivalent minimal left ideals while the real algebra contains one only. In this way the complex algebra can accommodate both the dotted and undotted index of the van der Waerden type in contrast to the real algebra with only a single type of index.

In the algebraic approach we can show that one type of spinor (dotted say) is transformed into another type (undotted) by the CPT type of transformation. Here the CPT transformation results from those algebraic automorphisms corresponding to the improper transformations of the metric space. Thus both the use of two spinor spaces of different type and the use of CPT-like transformations find explanation in algebraic terms and, insofar as the Pauli algebra is a geometric algebra, they find explanation also in geometric terms. In the vector space approach, such transformations are introduced for a posteriori reasons only.

4. COGREDIENT AND CONTRAGREDIENT SPINORS

Every primitive idempotent of $\wp$ generates simultaneously a unique minimal left ideal $\wp u$ and a unique minimal right ideal $u\wp$. Therefore we have the possibility of a right spinor as well as a left spinor and it is these spinors which correspond to cogredient and contragredient spinors represented by row or column vectors, as we shall now show. Suppose we choose a basis $\{e_j\}$ for the algebra A. Then for the left regular representation we obtain

$$ax = ax^j e_j = x^j a e_j = x^j L^k{}_j(a)\, e_k$$

so that if the representation space A is considered as a column vector space, then the left regular representation in the basis $\{e_j\}$ gives

$$\rho_L : a \longrightarrow L^k{}_j(a).$$

On the other hand, the right regular representation gives

$$ax = x^j e_j a = x^j \hat{R}_j{}^k(a) e_k \quad \text{or} \quad x^j R^k{}_j(a)\, e_k$$

according as the representation space $\mathcal{A}$ is considered either as a space of column vectors or as a space of row vectors. If $\mathcal{A}$ be considered as a space of column vectors, then we obtain an algebra $\{\tilde{R}_j{}^k\}$ of matrices which is anti-isomorphic to the original algebra $\mathcal{A}$ and which therefore does not constitute a representation of $\mathcal{A}$ since, by definition, a representation of an algebra must be by homomorphic structures. On the other hand, if $\mathcal{A}$ be considered as a space of row vectors, then a true representation of $\mathcal{A}$ is obtained. It should be noted that these two anti-isomorphic matric algebras $\{R^k{}_j\}$ and $\{\tilde{R}_j{}^k\}$ obtained from the right regular representation are related to one another through the operation of transposition:

$$\tilde{R}_j{}^k = (R^k{}_j)^T$$

Inasmuch as we are interested in the representations of the algebra $\mathcal{A}$ we must prefer to adopt the latter approach to the right regular representation in which the space $\mathcal{A}$ of the representation is regarded as a space of row vectors.

If our sole interest were in obtaining the irreducible representations of the algebra & since all the irreducible representations must occur amongst the irreducible components of the regular representation (left or right), no advantage would be gained by considering simultaneously both the left and the right methods of regular representation. However, since the algebraic definition of the spinor elevates the role played by the representation space from a purely utilitarian one to one of fundamental importance, it follows that in spite of producing equivalent representations of the algebra, the decomposition of the algebra into left or into right ideals gives rise to two significantly different (one-sided) ideal structures, each of which in some sense plays the role of the 'spin space' to the algebra.

In order to exploit this distinction, let us examine the following structure. Suppose that u is a primitive idempotent of the Pauli algebra. We have shown that for $\wp$

$$u\wp u = Z_1 u \cong Z_1$$

where Z_1 is the centre of $\wp$. Any right spinor up defines a map

$$up : \wp u \rightarrow Z_1$$

$$: qu \rightarrow z$$

where the element z is uniquely defined by the relation

$$upqu = zu$$

Thus each right spinor up in $u\mathcal{P}$ can be regarded as a functional on the space $\mathcal{P}u$ of left spinors. In this sense we can regard the two spaces $u\mathcal{P}$ of right spinors and $\mathcal{P}u$ of left spinors as functional duals of one another, the 'scalars' of the duality being the centre Z_1 of the algebra.

We have already pointed out that there exists a non-denumerable infinity of minimal right ideals in $\mathcal{P}$. The above duality establishes a unique correspondence between the minimal left ideals and the minimal right ideals.. Furthermore it is not difficult to show that under an inner automorphism of the algebra (these automorphisms including in particular the rotations in E_3), the 'scalars' of the duality are unaltered showing that the left and right spinors of the Pauli algebra are directly analogues of the group theoretic ideas of contragredient and cogredient spinors respectively and that the left and right minimal ideals generated by the same idempotent element u transform covariantly under similarity transformations.

In this way we see the relationship between covariance and contravariance of spinors in a new light. In the group theoretic approach they are introduced through the existence of two separate spaces whose existence and relation is entirely ad hoc. Indeed, the notion that the objects in these spaces are intrinsically different is emphasised by the use of upper and lower indices, their relation being established through a special rank two spinor 'metric', ε^{AB} (see Penrose 1968).

In the algebraic approach, on the other hand, the spinor spaces and their 'duality' are directly a consequence of the intrinsic properties of the Pauli algebra itself and arise naturally within that structure. The use of the spinor indices thus also must be seen in a new light. For, while the use of these indices proved useful in the group theoretic approach, where no other formal characterisation of the properties of covariance or contravariance was available, in the algebraic theory these properties are fully characterised by the intrinsic algebraic features of the elements concerned so that the use of spinor indices is no longer needed*.

It should be noted that similar comments do not apply to the distinction between cogredient and contragredient vectors and tensors which, in this theory, arise from the properties of the metric space assumed to characterise the generators of the Pauli algebra. However this distinction can also be removed by means of another algebraisation which will be reported elsewhere.

5. THE SPINORS OF THE REAL PAULI ALGEBRA:

Consider the matrix representations of the real Pauli algebra and examine the explicit structure of its ideals. The complete reduction of the identity of the real Pauli algebra into a sum of orthogonal primitive idempotents is

$$1 = u + (1-u)$$

for a given primitive idempotent u of $\wp$. It was shown in Paper I that a matric basis E_{ij} can be found for $\wp(\mathbb{R})$ such that

$$E_{11} = u \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_{12} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$E_{21} \Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad E_{22} = (1-u) \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then any element of the left ideal takes the form

$$\psi = \psi^0 E_{11} + \psi^1 E_{21} \qquad \text{where } \begin{matrix} \psi^0 = z^{11} \\ \psi^1 = z^{12} \end{matrix} \ ; z \in Z_1$$

The centre Z_1 has two irreducible representations in the complex field C and C^* viz:

$$C : (a1 + b e_{123}) \longrightarrow a + ib$$
$$C^* : (a1 + b e_{123}) \longrightarrow a - ib$$

So that contragredient spinors can be represented by

$$\Gamma(1) : \psi \rightarrow \begin{pmatrix} \psi^0 & 0 \\ \psi^1 & 0 \end{pmatrix} \quad \text{or} \quad \Gamma(2) : \psi \rightarrow \begin{pmatrix} \psi^{0*} & 0 \\ \psi^{1*} & 0 \end{pmatrix}$$

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* In fact, there is no way in which the spinor indices can be introduced consistently into the algebraic formalism (Frescura 1977). The implications of this to our understanding of spinor geometry are far reaching, but will not be discussed here.

The spinor space covariant with fu is the right ideal uf. A typical element of this space is

$$\phi = \phi_0 E_{11} + \phi_1 E_{12} \qquad \phi_A \in Z_1$$

and the matrix representation is

$$\Gamma(1) : \phi \rightarrow \begin{pmatrix} \phi_0 & \phi_1 \\ 0 & 0 \end{pmatrix}$$

$$\Gamma(2) : \phi \rightarrow \begin{pmatrix} \phi_0^* & \phi_1^* \\ 0 & 0 \end{pmatrix}$$

It will be noted that in representation two forms of spinor appear, owing to the two representations of the centre Z_1 of the algebra. At first sight, it might appear that these correspond to the 'undotted' and the 'dotted' spinor as defined by van der Waerden (1929), for under given orthogonal transformation, they transform inequivalently as complex conjugates of one another. This correspondence however is purely accidental, the correct generalisation into the algebraic context of van der Waerden's classification of spinors being through the dissimilarity of their respective spinor spaces. Now, in the real Pauli algebra, all primitive idempotents are similar, and hence only one type of spinor will arise. The apparent existence of two spinor types in representation in no way contradicts this. These two spinor 'types' are to be found in two distinct representative algebraic structures which mathematically cannot be mixed. (Contrast this with the case of the complex algebra where the two spinor types will be found to occur simultaneously within the same algebraic structure.) This feature is due entirely to the unorthodox method of representation used in which an algebra over the real numbers ~~of~~ is represented by an algebra of matrices over the complex numbers. Had the more usual method of representation been used, in which an algebra over some given base field (in our case, representation by real matrices), then only one irreducible representation for the real algebra would have been found.

6. THE SPINOR OF THE COMPLEX ALGEBRA

The complex Pauli algebra has been shown to be semi-simple, consisting of the direct sum of two normal simple components. The primitive idempotent elements of the algebra therefore can be divided into two classes, all the elements of each being similar to one another by some invertible element of $\wp$ but not to the elements of the other. The complex Pauli algebra thus contains exactly two spinor types together with their contravariants.

The centre of the algebra is the subalgebra generated by the unity element 1. The cogredient spinors therefore can be considered to define complex functionals on the contragredient spaces according to:

$$\begin{aligned} uq : \wp u &\longrightarrow \mathbb{C} \\ : pu &\longrightarrow (uq, pu) \end{aligned}$$

where (uq,pu) is defined by the relation:

$$(uq, pu)u = uqpu$$

As with the real algebra, in any given representation of the complex algebra, the algebraic space simultaneously plays two roles, namely that of the representation space as well as that of the operators on that space. Hence the spinors will be represented simultaneously in two forms, firstly as operators and then also as operands.

The decomposition of $\wp$ (which is unique up to the order of terms) into the sum of normal simple components is given by:

$$\begin{aligned} \wp &= \wp \tfrac{1}{2}(1+ie_{123}) \oplus \wp \tfrac{1}{2}(1-ie_{123}) \\ &= \mathfrak{J}(+) \oplus \mathfrak{J}(-) \end{aligned}$$

where, denoting the central idempotents $\frac{1}{2}(1+ie_{123})$ and $\frac{1}{2}(1-ie_{123})$ by $\varepsilon(+)$ and $\varepsilon(-)$ respectively, we have:

$$\mathfrak{J}(\pm) = \wp\, \varepsilon(\pm)$$

Now, because the subalgebra $\mathfrak{J}(+)$ is simple, all its primitive idempotents are similar, and so also are all the primitive idempotents of the simple subalgebra $\mathfrak{J}(-)$

But no idempotent of $\mathcal{J}(+)$ is similar to any primitive idempotent of $\mathcal{J}(-)$ and vice versa. We see then that the division of the primitive idempotents of $\wp$ into mutually exclusive similarity classes can be achieved according to their membership of the simple components of $\wp$. We shall refer to the primitive idempotents of $\wp$ in $\mathcal{J}(+)$ as 'class 1' idempotents, while those in $\mathcal{J}(-)$ will be said to be of 'class 2'.

Consider now the simple subalgebra $\mathcal{J}(+)$. The unity of this subalgebra is $\mathcal{E}(+)$ and we have shown in paper I that if $u\,\mathcal{E}(+)$ is a primitive idempotent of $\mathcal{J}(+)$, then:

$$\mathcal{E}(+) = u\,\mathcal{E}(+) + (1-u)\,\mathcal{E}(+)$$

is a complete decomposition of the unity into the sum of mutually orthogonal primitive idempotents. Corresponding to this decomposition of the unit, we can find a matric basis E^{+}_{ij} for $\mathcal{J}(+)$ such that:

$$E^{+}_{11} = u\,\mathcal{E}(+) \qquad \text{and} \qquad E^{+}_{22} = (1-u)\,\mathcal{E}(+)$$

Corresponding to the primitive idempotent $u\,\mathcal{E}(+)$ of $\mathcal{J}(+)$, we have the ideal decomposition of $\mathcal{J}(+)$ given by:

$$\mathcal{J}(+) = \mathcal{J}(+)u \oplus \mathcal{J}(+)(1-u)$$

We call the elements of the left ideals of $\mathcal{J}(+)$ contragredient spinors of the first type. These will be of the form:

$$\psi = \alpha^{11} E^{+}_{11} + \alpha^{21} E^{+}_{21}$$

with $\alpha^{ij} \in \mathbb{C}$. Adopting the more common spinor notation in which the basis of the spinor space is labelled by 0 and 1, we can write:

$$\psi = \psi^0 E^+_{11} + \psi^1 E^+_{21}$$

with $\psi^0 = \alpha^{11}$ and $\psi^1 = \alpha^{21}$. Irreducibly representing the matric basis of $\mathcal{J}(+)$ in the standard way, we have:

$$E^+_{11} \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad E^+_{12} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$E^+_{21} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad ; \quad E^+_{22} \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then the contragredient spinor of the first type irreducibly represented as an operator on the space $\mathcal{J}(+)u$ takes the form:

$$\psi \longmapsto \begin{bmatrix} \psi^0 & 0 \\ \psi^1 & 0 \end{bmatrix}$$

In the smallest faithful (though not irreducible) representation of $\mathfrak{P}$, ψ will be represented as:

$$\psi \longmapsto \left[\begin{array}{cc|cc} \psi^0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ \psi^1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 0 \end{array}\right]$$

The spinor space contravariant with $\mathcal{J}(+)u$ is $u\mathcal{J}(+)$ and its elements are said to be cogredient spinors of the first type. The general element of this space is of the form:

$$\varphi = \varphi_0 E^+_{11} + \varphi_1 E^+_{12}$$

with $\varphi_A \in \mathbb{C}$. Irreducibly represented as an operator on the space $u\mathcal{J}(+)$, it takes the form:

$$\varphi \longmapsto \begin{bmatrix} \varphi_0 & \varphi_1 \\ 0 & 0 \end{bmatrix}$$

In the smallest faithful representation of $\mathfrak{P}$, it takes the form:

$$\varphi \longmapsto \left[\begin{array}{cc|cc} \varphi_0 & \varphi_1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 0 \\ & & 0 & 0 \end{array}\right]$$

If the spinor be considered in the role of an operand, the contragredient spinor of the first type will be represented either as:

$$\psi \longmapsto \begin{bmatrix} \psi^0 \\ \psi^1 \end{bmatrix}$$

or as:

$$\psi \longmapsto \begin{bmatrix} \psi^0 \\ \psi^1 \\ 0 \\ 0 \end{bmatrix}$$

according as we are considering it as an element of $\mathfrak{J}(+)u$ or of $\mathfrak{p}u$. Similarly, the cogredient spinor will be represented either as:

$$\varphi \longmapsto (\varphi_0, \varphi_1)$$

or as:

$$\varphi \longmapsto (\varphi_0, \varphi_1, 0, 0)$$

according as we consider it as an element of $u\mathfrak{J}(+)$ or of $u\mathfrak{p}$. The scalar product of a contragredient spinor with a cogredient one is obtained in the usual way.

Completely analogous considerations can be applied to the simple subalgebra $\mathfrak{J}(-)$, the spinors of this subalgebra being called spinors of the second type. The unity element of this subalgebra is the element $\varepsilon(-)$. The decomposition of $\varepsilon(-)$ into the sum of mutually orthogonal primitive idempotent elements which is complementary to the decomposition of $\varepsilon(+)$ given above is:

$$\varepsilon(-) = u\,\varepsilon(-) + (1-u)\varepsilon(-)$$

giving the left ideal decomposition:

$$\mathfrak{J}(-) = \mathfrak{J}(-)u \oplus \mathfrak{J}(-)(1-u)$$

In order to distinguish in the notation spinors of the first type from those of the second type, we use a dotted spinor index. Thus, using a matric basis E^-_{ij} for $\mathfrak{J}(-)$ such that $E^-_{11} = u\,\varepsilon(-)$ and $E^-_{22} = (1-u)\varepsilon(-)$, a typical element of the ideal $\mathfrak{J}(-)u$ will be of the form:

$$\xi = \xi^{\dot{0}}E^-_{11} + \xi^{\dot{1}}E^-_{21}$$

Represented as an operator, it will be either:

$$\xi \longmapsto \begin{bmatrix} \xi^{\dot{0}} & 0 \\ \xi^{\dot{1}} & 0 \end{bmatrix}$$

or:

$$\xi \longmapsto \left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & \xi^{\dot{0}} & 0 \\ & & \xi^{\dot{1}} & 0 \end{array}\right]$$

according as we regard ξ as an operator on the space $\mathfrak{g}(-)u$ or on the space $\mathfrak{p}u$. Represented as a vector, ξ will be of the form:

$$\xi \longmapsto \begin{bmatrix} \xi^{\dot{0}} \\ \xi^{\dot{1}} \end{bmatrix}$$

or:

$$\xi \longmapsto \begin{bmatrix} 0 \\ 0 \\ \xi^{\dot{0}} \\ \xi^{\dot{1}} \end{bmatrix}$$

according as it is regarded as an element of the space $\mathfrak{g}(-)u$ or of $\mathfrak{p}u$.

The cogredient spinor of the second type is of the form:

$$\eta = \eta_{\dot{0}} E^{-}_{11} + \eta_{\dot{1}} E^{-}_{12}$$

which is represented as an operator in the form:

$$\eta \longmapsto \begin{bmatrix} \eta_{\dot{0}} & \eta_{\dot{1}} \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \eta \longmapsto \left[\begin{array}{cc|cc} 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{0} \\ \hline & & \eta_{\dot{0}} & \eta_{\dot{1}} \\ \multicolumn{2}{c|}{0} & 0 & 0 \end{array}\right]$$

and is represented in vector form as:

$$\eta \longmapsto (\eta_{\dot{0}}, \eta_{\dot{1}}) \quad \text{or} \quad \eta \longmapsto (0, 0, \eta_{\dot{0}}, \eta_{\dot{1}})$$

The occurrence of two types of spinor in the complex algebra, in contrast to the occurrence of a single type in the real algebra, is a consequence of the important difference in the structure of the two. The relation in which the two spinor types stand to one another is not immediately evident, though we have hinted at it in the notation, and must be explicitly deduced and demonstrated. Consider the transformation induced in the spin spaces of the first kind and of the second kind by a single element p of $\mathfrak{p}$. That is, consider the transformation defined by:

$$p : a \longmapsto pa$$

for all $a \in \mathfrak{P}$. Suppose first of all that p is a real element, that is, that all the coefficients of p are drawn from the real field. Then comparison with the representation of $\mathfrak{P}$ given in Paper I, section 8, shows that the spinors of first and second type undergo simultaneous transformations which are complex conjugate to one another:

$$\begin{bmatrix}\psi^0\\ \psi^1\end{bmatrix} \longmapsto \begin{bmatrix}(p^o+p^3)-i(p^{12}+p^{123}) & (p^1-p^{13})+i(p^2-p^{23})\\ (p^1+p^{13})-i(p^2+p^{23}) & (p^o-p^3)+i(p^{12}-p^{123})\end{bmatrix}\begin{bmatrix}\psi^0\\ \psi^1\end{bmatrix}$$

and

$$\begin{bmatrix}\psi^{\dot{0}}\\ \psi^{\dot{1}}\end{bmatrix} \longmapsto \begin{bmatrix}(p^0+p^3)+i(p^{12}+p^{123}) & (p^1-p^{13})-i(p^2-p^{23})\\ (p^1+p^{13})+i(p^2+p^{23}) & (p^o-p^3)-i(p^{12}-p^{123})\end{bmatrix}\begin{bmatrix}\psi^{\dot{0}}\\ \psi^{\dot{1}}\end{bmatrix}$$

so that under the special restriction of $p \in \mathfrak{P}(\mathbb{R})$, the spinor spaces of the first and the second type are seen to correspond to the well known notions of undotted and dotted spinors respectively. Removing the restriction of reality on p, it is evident that we can no longer speak of the above matrices as complex conjugates of one another, since the coefficients of p appearing in their entries are themselves now complex numbers. But the transformations induced in the two spinor spaces nevertheless can be said to be 'conjugate' in the more general sense that they transform according to the two different irreducible representations of the algebra. Thus we have arrived at an algebraic characterisation of the notion of 'conjugate spinors' which is more general than that originally proposed by van der Waerden.

One very important difference between the real and complex algebras must be emphasised here. In the real algebra, as with the complex algebra, we obtained a set of objects which were in some ways analogous to the dotted and undotted spinors of van der Waerden. However, in the case of the real algebra, these arose artificially from the use of different representational techniques. The real algebra itself in fact contained only one spinor type. But in the case of the complex algebra, the two spinor types arise in a relevant way from the nature of the actual algebraic structure and not accidentally from the properties of the representation. In a very approximate sense, the complex algebra may be regarded as the juxtaposition of two real algebras, one being the 'conjugate' of the other. The existence of two spinor types is a consequence of this structural property.

Comments on the multiplicity of spin spaces and on the role of spinor indices in the theory similar to the ones made in the case of the real algebra are here applicable.

7. CONCLUSION

In this paper we have argued that the algebraic spinor produces an object of greater generality than its group theoretic counterpart. The cogredient and contragredient spinors are expressed through the right and left ideals respectively, these ideals being generated simultaneously by a single idempotent u of $\mathfrak{P}$. In this way, we see that the distinction between the cogredient and contragredient spinor spaces becomes subordinate to the primary idea of a 'pre - spinor' structure, namely the Pauli algebra itself. We can regard the present spinor geometry as being an abstraction from the 'pre - spinor' geometry determined by the particular choice of left and right ideals to serve in the role of the 'spin space' and its dual. The implications of this algebraic view of the spinor and its significance to movement and process will be discussed in a later paper.

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