A SPECIAL METHOD OF SUCCESSIVE APPROXIMATIONS FOR FREDHOLM INTEGRAL EQUATIONS

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Introduction. This paper deals with the integral equation

$$y(s) = f(s) + \lambda \int_a^b K(s, t)y(t)\, dt \tag{1}$$

with the continuous kernel $K(s, t)$ and the continuous function $f(s)$. The convenient method of successive approximations for this equation consists in computing a sequence

$$u_0(s),\ u_1(s),\ u_2(s),\ \cdots \tag{2}$$

of functions by means of the formula

$$u_{n+1}(s) = f(s) + \lambda \int_a^b K(s, t)u_n(t)\, dt. \tag{3}$$

Here u_0 is a given continuous function. It is well known that the convergence of sequence (2) depends both on u_0 and on λ. If convergence takes place for a given λ and for *any* continuous function u_0, we shall say that the convergence is *total* for λ.

If the kernel has no eigenvalues, total convergence takes place for any λ. If eigenvalues exist, they may be denoted by

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \cdots$$

(only different values appear in this sequence), where

$$|\lambda_i| \le |\lambda_k| \qquad (i < k)$$

is true. Then total convergence is restricted to $|\lambda| \le |\lambda_1|$, and it surely takes place for $|\lambda| < |\lambda_1|$.

Now we replace (3) by

$$u_{n+1}(s) = \theta u_n(s) + (1 - \theta)\lambda \int_a^b K(s, t)u_n(t)\, dt + (1 - \theta)f(s). \tag{3'}$$

Here θ is a fixed number. Generally the new sequence u_0, u_1, $\cdots$ based on (3′) will differ from the sequence (2), but they are identical for $\theta = 0$. If the new sequence converges for a given pair of numbers λ, θ and for any continuous function u_0, we shall say that the convergence is total for that pair.

I shall show in this paper that the generalized approximation method ac-

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This is possible because $|\theta| < 1$ (see (12)). For $n \geq N(\zeta)$ we can write for (9)

$$v_n(s) = \theta^n v_0(s) + \sum_{\nu=1}^{N-1} \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b K^{(\nu)}(s, t) v_0(t)\, dt$$

$$+ \sum_{\nu=N}^{n} \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b K^{(\nu)}(s, t) v_0(t)\, dt.$$

Furthermore

$$(18) \qquad \begin{aligned} |v_n(s)| \leq |\theta^n v_0(s)| &+ \left| \sum_{\nu=1}^{N-1} \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b K^{(\nu)}(s, t) v_0(t)\, dt \right| \\ &+ \eta^n \int_a^b |v_0(t)|\, dt. \end{aligned}$$

This means $\lim_{n\to\infty} v_n(s) = 0$ in view of $|\theta| < 1$ and $\eta < 1$.

Next we consider a kernel with at least one eigenvalue. We can divide the kernel into two parts

$$K(s, t) = A(s, t) + B(s, t),$$

where A and B are orthogonal, A being degenerate with the only eigenvalue λ_1, while B has the same eigenvalues as K with the exception of λ_1. We substitute $A + B$ for K. In view of (7) we can write for (9)

$$(19) \qquad \begin{aligned} v_n(s) = \theta^n v_0(s) &+ \sum_{\nu=1}^{n} \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b A^{(\nu)}(s, t) v_0(t)\, dt \\ &+ \sum_{\nu=1}^{n} \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b B^{(\nu)}(s, t) v_0(t)\, dt. \end{aligned}$$

The pair of numbers λ, θ is assumed to satisfy the conditions (11) and (12) with regard to $K(s, t)$. These conditions are satisfied too with regard to A and B because the eigenvalues of these two kernels are eigenvalues of K. Now A is degenerate. Since the conditions have already been proved to be sufficient for total convergence as to degenerate kernels, we can derive from (19)

$$\lim_{n\to\infty} \left\{ \theta^{n-\nu}(1-\theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b B^{(\nu)}(s, t) v_0(t)\, dt - v_n(s) \right\} = 0.$$

Therefore (11) and (12) are sufficient for total convergence with regard to the kernel $K(s, t)$ if they are sufficient as to $B(s, t)$. An evident generalization of this result is:

The conditions (11) and (12) are sufficient for total convergence with regard to $K(s, t)$ if they are sufficient with regard to a kernel $C(s, t)$ whose eigenvalues are

$$\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots,$$

these values belonging to the sequence of eigenvalues of $K(s, t)$.

If there is a finite number of eigenvalues of K only, we can determine $C(s, t)$ so as to have no eigenvalues. For such a $C(s, t)$ conditions (11) and (12) have been proved to be sufficient for total convergence.

If there is an infinite number of eigenvalues, we determine a positive number ζ such that

$$|\theta + \lambda(1 - \theta)\zeta| = \eta < 1. \tag{17}$$

Furthermore we choose a positive integer m for which $|\lambda_m|^{-1} < \frac{1}{2}\zeta$. Then to the kernel $C(s, t)$ with the eigenvalues

$$\lambda_m, \lambda_{m+1}, \cdots,$$

we can apply inequality (4) in the following form:

$$|C^{(n)}(s, t)| \leq (\tfrac{1}{2}\zeta + \epsilon)^n \qquad (n \geq N(\epsilon)).$$

The number $\epsilon > 0$ may be restricted to $\epsilon \leq \frac{1}{2}\zeta$. Then

$$|C^{(n)}(s, t)| \leq \zeta^n \qquad (n \geq N(\epsilon))$$

holds. This makes it possible to apply inequality (18) to the kernel $C(s, t)$. We obtain

$$\begin{aligned} |v_n(s)| \leq |\theta^n v_0(s)| &+ \left| \sum_{\nu=1}^{N-1} \theta^{n-\nu}(1 - \theta)^\nu \binom{n}{\nu} \lambda^\nu \int_a^b C^{(\nu)}(s, t) v_0(t)\, dt \right| \\ &+ \eta^n \int_a^b |v_0(t)|\, dt, \end{aligned} \tag{18'}$$

and in consequence of this inequality $\lim_{n\to\infty} v_n(s) = 0$. This is the end of the proof. We have shown that the conditions (11) and (12) are sufficient for total convergence in any case.

4. Two particular cases. We now apply the result of §3. At first we regard a kernel with real eigenvalues only. Then (11) is equivalent to

$$-2 < (1 - \theta)(\lambda\lambda_k^{-1} - 1) < 0.$$

This includes

$$\lambda\lambda_k^{-1} < 1 \qquad (k = 1, 2, \cdots). \tag{20}$$

If (20) is satisfied by λ we can find a θ that satisfies the conditions (11) and (12). If all eigenvalues are positive, condition (20) means that application of the proposed successive approximation method is possible for $\lambda < \lambda_1$ as already mentioned in the introduction.

Another application is possible in the case

$$K(s, t) \geq 0, \qquad \int_a^b K(s, t)\, dt \equiv 1.$$