Lowest Order "Divergent" Graphs in ν-dimensional Space

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Abstract

The lowest order "divergent" graphs, for a scalar $\psi^2\varphi$-coupling theory, are computed as functions of the number of dimensions ν of the space. The result is seen to be finite for odd-dimensional spaces. For even-dimensional spaces it is shown that ν can be used as an analytic regularization parameter.

It is a known fact that the behaviour of the solutions of the wave equation depends on the number of dimensions and in particular upon the even or odd character of this number. For instance (Ref. 1), the Huyghen's principle is valid only when the number of dimensions is even. Similarly, when looking at the Fourier transform of $(R + i0)^{\lambda}$ (Ref. 2) - R being a quadratic form, one sees that it is an analytic function of λ, the positions of the poles depending on the number of dimensions. On the light of these facts, it seems interesting to look into the behaviour of Quantum field theory with respect to the number of dimensions of the space. We are particularly interested in the incidence of the number of dimensions on the divergences of Quantum field theory. With this aim we have computed the self-energy, vacuum polarization and vertex graphs in the lowest order for a massless scalar field φ interacting with a massive scalar one Ψ through the coupling: $g\Psi^2\varphi$.

The propagators satisfy the eq.

(1) $$L\Delta(x) = \delta \quad \text{with}$$

(2) $$L \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_{\nu-q}^2} - \frac{\partial^2}{\partial x_{\nu-q+1}^2} - \dots - \frac{\partial^2}{\partial x_\nu^2} - m^2 \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2$$

ν is the number of dimensions and q the number of negative terms in the metric.

We obtain then for the massive field

(3) $$\Delta(R) = -\mathcal{F}^{-1}\left\{(P+m^2-i0)^{-1}\right\} = -\frac{i^q m^{\frac{\nu}{2}-1} K_{\frac{\nu}{2}-1}(m\sqrt{R+i0})}{2^{\frac{\nu}{2}} \pi^{\frac{\nu}{2}} (\sqrt{R+i0})^{\frac{\nu}{2}-1}}$$

and for the massless field*

(4) $$D(R) = -\frac{i^q\,\Gamma(\frac{\nu}{2}-1)}{4\pi^{\frac{\nu}{2}}\, R^{\frac{\nu}{2}-1}}.$$

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* For $\nu = 2$ a special treatment is necessary. The finite part of (4) is the logarithmic potential

Where $R = \eta^{\alpha\beta} x_\alpha x_\beta$; $P = \eta^{\alpha\beta} p_\alpha p_\beta$

From now on $R \equiv R + i0$; $P \equiv P - i0$,

which leads to causal distributions. In what follows we make frequent use of the generalized Bochner Theorem (Ref. 3)

(5) $$\mathcal{F}\{f(R)\} = \frac{2^{\frac{\nu}{2}} \pi^{\frac{\nu}{2}}}{i^q \sqrt{P}^{\frac{\nu}{2}-1}} \int_0^\infty f(x^2)\, J_{\frac{\nu}{2}-1}(x\sqrt{P})\, x^{\frac{\nu}{2}}\, dx$$

Self-Energy.

(6) $$\Sigma(P) = \int \frac{d^\nu k}{k^2[(p-k)^2+m^2]} = \tilde{D} * \tilde{\Delta} = (2\pi)^\nu \mathcal{F}(D.\Delta)$$

where $d^\nu k$ is the volume element of the ν-dimensional space. $\sim$ means Fourier Transform and $*$ means convolution

$$\hat{\mathcal{F}}(D.\Delta) = \frac{i^q\, \Gamma(\frac{\nu}{2}-1)\, m^{\frac{\nu}{2}-1}}{4\pi^{\nu}\, \sqrt{P}^{\frac{\nu}{2}-1}} \int_0^\infty \frac{K_{\frac{\nu}{2}-1}(mx)}{x^{\frac{3}{2}\nu-3}} \cdot J_{\frac{\nu}{2}-1}(x\sqrt{P})\, x^{\frac{\nu}{2}}\, dx\, ,$$

which gives[4]

(7) $$\Sigma(P) = -i^q \pi^{\frac{\nu}{2}} m^{\nu-4}\, \Gamma(1-\tfrac{\nu}{2})\, F(1, 2-\tfrac{\nu}{2}; \tfrac{\nu}{2}; -\tfrac{P}{m^2})$$

Vacuum polarization.

(8) $$\Pi(P) = \int \frac{d^\nu k}{(k^2+m^2)[(p-k)^2+m^2]} = \tilde{\Delta} * \tilde{\Delta} = (2\pi)^\nu \mathcal{F}(\Delta^2)$$

Using Ref 4, p.901 and 1071,

(9) $$\Pi(P) = i^q m^{\nu-4} \pi^{\frac{\nu}{2}+\frac{1}{2}}\, G^{12}_{22}\left(\frac{P}{4m^2}\middle| \begin{matrix} \frac{\nu}{2}-1, 0 \\ 0, -\frac{1}{2} \end{matrix}\right) ,$$

(10) $$\Pi(P) = i^q m^{\nu-4} \pi^{\frac{\nu}{2}}\, \Gamma(2-\tfrac{\nu}{2})\, F(2-\tfrac{\nu}{2}, 1; \tfrac{3}{2}; -\tfrac{P}{4m^2}) .$$

Vertex

(11) $$\Lambda(p,q) = -\int \frac{d^\nu k}{k^2[(p-k)^2+m^2][(q-k)^2+m^2]}$$

$$(12)\quad \Lambda(p,p) \equiv \Lambda(P) = -\int \frac{d^{\nu}k}{k^2\left[(p-k)^2+m^2\right]^2} = \tilde{D} * \tilde{\Delta}^2 = (2\pi)^{\nu} \mathcal{F}(D\Delta_2),$$

$$\Delta_2(R) = \mathcal{F}^{-1}\{(P+m^2)^{-2}\} = \frac{i^q\, m^{\frac{\nu}{2}-2}\, K_{\frac{\nu}{2}-2}(m\sqrt{R})}{2^{\frac{\nu}{2}+1}\,\pi^{\frac{\nu}{2}}\,\sqrt{R}^{\frac{\nu}{2}-2}}$$

$$(13)\quad \Lambda(P) = i^q\, m^{\nu-6}\,\pi^{\frac{\nu}{2}}\left(2-\tfrac{\nu}{2}\right)\Gamma\left(1-\tfrac{\nu}{2}\right) F\left(1,3-\tfrac{\nu}{2};\tfrac{\nu}{2};-\tfrac{P}{m^2}\right).$$

One can verify the following Ward-type identity

$$(14)\quad \frac{\partial \Sigma(P)}{\partial m^2} = \Lambda(P)$$

DISCUSSION

One can see from form. (7) (10) and (13) that the lowest order results are finite and well defined when the number of dimensions is odd, while for ν-even a singularity appears explicitly in the form of a pole in the variable ν.* (Observe that from a mere counting of powers in numerators and denominators, one should expect the degree of divergence to increase with the number of dimensions). This result suggests that the the study of quantum field theory in odd number of dimensions has a particular interest.

Besides, (7) (10) and (13) are explicit analytic functions of the parameter ν. This suggests the use of ν as ragularizing parameter in an analytic regularization method. Then for $\nu = 4$ (which might be connected with reality!) Guelfand's prescription of taking the finite part of the distribution at the pole, gives the physical amplitude.

We have computed the finite part of eq. (6), for $\nu = 4$ using the analytic regularization method. (Ref. 5)

In this method, eq. 6 is replaced by

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* Note however, that (13) is finite for $\nu = 4$, as it should be from the definition (11).

$$(15)\quad \sum_\lambda (P) = \int \frac{d^4k}{k^{2\lambda}\left[(p-k)^2+m^2\right]} = i\,\Gamma(2-\lambda)\,m^{2-2\lambda}\pi^2\,\Gamma(\lambda-1)\,F\left(\lambda,\lambda-1;2;\frac{-P}{m^2}\right)$$

The residue at the pole $\lambda = 1$ is a constant, and the finite part is:

$$(16)\quad P.f.\sum(P) = \frac{d}{d\lambda}(\lambda-1)\sum_\lambda(P)\Big|_{\lambda=1} = \frac{-i\pi^2 P}{2m^2}\,F\left(1,1;3;-\frac{P}{m^2}\right) + \text{cte.}$$

On the other hand, the suggested new method gives:

$P.f.\sum(P) = \frac{d}{d\nu}(\nu-4)\sum_\nu(P)\Big|_{\nu=4}$ which using (7), coincides with (16).

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