Relativistic Phase Space Arising out of the Dirac Algebra

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1. Introduction

The spinor plays a fundamental role in the quantum theory (especially in its relativistic form) but its over-all structure and physical significance is not clear.

Various attempts have indeed been made to interpret the meaning of the spinor. Of these, the most elementary is the use of stereographic projection[1]. From this, one may come to the interpretation through a vector plus a flag[2], and also through a set of triads[3]. Next, there is a different but closely related approach, in terms of the chords of circles[4].

What appears, at least at first sight, to be quite another idea is to start with a Clifford algebra, and to interpret the coefficients as antisymmetric tensors[5]

$$\phi = A_o + A_\mu \gamma^\mu + A_{[\mu\nu]}\gamma^\mu\gamma^\nu + A_{[\mu\nu\lambda]}\gamma^\mu\gamma^\nu\gamma^\lambda + A_{[\mu\nu\lambda\gamma]}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\gamma \qquad (1)$$

Still another way of dealing with such questions, but along the same general lines is Schönberg's connection of spinors with phase space[6].

In this paper, we shall start by combining the Clifford algebra with Schönberg's phase space interpretation, showing the inter-relationship of these two, and developing the whole theory in a systematic way. In doing this, we will use the algebraic approach in such a way as to bring out the connection between quantum theory and classical theory[7]. This will enable us directly to comprehend the meaning of Dirac's equation in the classical limit.

We begin by showing how the Dirac algebra can be expressed as the direct product of two dual Grassmann algebras, which can in turn be written as products of fermionic creation and anihilation operators
Each antisymmetric tensor of a given rank is then the coefficient of a term in the algebra corresponding to the excitation of a number of vector" fermions" equal in number to the rank of the tensor . From this, a connection between the fermionic operators and boundary operators, similar to that of Kähler[8], will be derived.

Finally, starting with the Dirac equation, containing electromagnetic potentials, we obtain a Liouville equation, in which spin is seen to correspond to a rotary transformation on a set of constants of the motion, thus showing how the notion of spin goes beyond ordinary classical concepts as expressed in terms of the Liouville equation.

2. Brief Summary of Connection Between Non-Relativistic Quantum Algebra and an Algebra on Phase Space

In a previous paper[7] we have developed in some detail the connection of the non-relativistic quantum algebra with an algebra on phase space. Briefly, we start by proposing that the density matrix, $\rho(x',x)$, or more generally a certain non-Hermitian extension of it, called the characteristic matrix, $\xi(x',x)$, be taken as a basic concept while the state vector is seen as an abstraction from this. We then make the Wigner-Moyal transformation[7]

$$F(X,P) = \frac{1}{2\pi}\int \xi(X-\tfrac{\eta}{2}, X+\tfrac{\eta}{2})e^{-iP\eta}d\eta \qquad (2)$$

where $X = \frac{x+x'}{2}$, $\eta = x - x'$

and we obtain the generalized Liouville equation

$$\frac{\partial F}{\partial t} + LF = 0 \qquad (3)$$

where L is the generalised Liouville operator, which reduces exactly to the well known classical form

$$LF = \frac{\partial H}{\partial P}\frac{\partial F}{\partial X} - \frac{\partial H}{\partial X}\frac{\partial F}{\partial P} \qquad (4)$$

for the special case of the harmonic oscillator and approximately for the general hamiltonian, in the classical limit (of high temperatures).

When $\xi(x',x)$ is a hermitian operator, F is real, but not in general non-negative. It is this which has prevented a direct interpretation of F as a probability density. Instead, we have proposed, for the general (non-Hermitian) $\xi(x',x)$, F = U + iV be regarded as a pair of related constants of the motion (which may be negative as well as positive).

An essential step in our work has been to treat $\xi(x',x)$ in two ways. First of all, it is a matrix operator, in a vector space, indexed by x. To simplify the discussion, let us approximate the range, x as discrete and finite, so that it has n "states". Then there will be n^2 independent elements in $\xi(x',x)$. What we now do is to regard these as components of a higher vector space, having n^2 elements. (As n approaches infinity, this becomes effectively the phase space). We call these vectors, ξ_α. We then introduce new matrix operators, $0_{\alpha',\alpha}$ having n^4 elements, which evidently are the basis of a matrix algebra on the phase space. When we make the Wigner-Moyal transformation we are regarding $\xi(x',x)$ as a vector in this higher dimensional space, on which the Liouville equation acts as an operator. The interchange between operator and vector interpretation for $\xi(x',x)$, is essential for expressing clearly the relationship between classical and quantum laws of motion.

In our paper, we have developed this interpretation in some detail, proposing a new kind of non-negative probability function and also a way of extending the theory, to cover certain new domains. All of this may ultimately be significant in the relativistic theory, but for the purposes of the present paper, what is important will be just the connection between quantum mechanics and Liouville's equation, whose solutions are to be considered as constants of the motion, with the aid of the double interpretation of $\xi(x', x)$ as an operator in the configuration space and as a vector in the phase space.

3. Connection Between Clifford Algebras and Grassmann Algebras

Before going in detail into the discussion of relativistic phase space, it is necessary to express in a simple and systematic way the connection between Clifford algebras and Grassmann algebras.

As indicated in Sec. 2, we begin with the density matrix, which must however now be enriched with spinor indices, so that we write

$$\rho = \rho_{ij}(x'^{\mu}, x^{\mu}) \tag{5}$$

where i and j are the usual set of four Dirac indices. We then extend the above to the non-Hermitian matrix

$$\xi = \xi_{ij}(x'^{\mu}, x^{\mu}) \tag{6}$$

and we note that ξ, like ρ satisfies the two Dirac equations

$$\gamma^{\mu} \frac{\partial \xi}{\partial x'^{\mu}} = 0, \quad \frac{\partial \xi}{\partial x^{\mu}} \gamma^{\mu} = 0 \qquad (7)$$

ξ can be expressed as a Clifford algebra, essentially the same as (1) except that the coefficients are, in general complex. That is to say we are treating ξ as a Dirac matrix. But, as suggested in Sec. 2, we are also going to regard ξ as a vector V in a higher dimensional vector space.

We now note that when γ^{μ} operates on ξ from the left, this will commute with γ^{μ} operating on ξ from the right. To distinguish these two operators when they act on the higher vector space, we shall designate them respectively as

$$\vec{\gamma}^{\mu} \text{ and } \overleftarrow{\gamma}^{\mu}, \text{ with}$$

$$\left[\vec{\gamma}^{\mu}, \overleftarrow{\gamma}^{\nu} \right] = 0 \qquad (8)$$

Of course, we also have the usual anticommution rules

$$\left\{ \vec{\gamma}^{\mu}, \vec{\gamma}^{\nu} \right\} = \left\{ \overleftarrow{\gamma}^{\mu}, \overleftarrow{\gamma}^{\nu} \right\} = 2g^{\mu\nu}$$

What we now aim to do is to arrive at Schönberg's representation of the Clifford algebra in terms of fermionic creation and anihilation operators, (but from an opposite point of departure). The first step is to obtain a set of eight operators, similar to the $\vec{\gamma}^{\mu}$ and $\overleftarrow{\gamma}^{\mu}$ (which however all anticommute in spite of eqn. (8) which defines two commuting sets of four operators). To obtain this, we consider

$\vec{\gamma}^5 = i\vec{\gamma}^1\vec{\gamma}^2\vec{\gamma}^4\vec{\gamma}^4$ and $\overleftarrow{\gamma}^5 = i\overleftarrow{\gamma}^4\overleftarrow{\gamma}^3\overleftarrow{\gamma}^2\overleftarrow{\gamma}^1$

the product, $\vec{\gamma}_5\overleftarrow{\gamma}_5$ anticommutes with all the $\vec{\gamma}^\mu$ and the $\overleftarrow{\gamma}^\mu$.

We then define

$$\overset{+}{\gamma}{}^\mu = \vec{\gamma}^\mu$$
$$\overset{-}{\gamma}{}^\mu = \vec{\gamma}^5\overleftarrow{\gamma}^5\overleftarrow{\gamma}^\mu \qquad (7)$$

and obtain

$$\{ \overset{+}{\gamma}{}^\mu, \overset{-}{\gamma}{}^\nu \} = 0 \qquad \{ \overset{+}{\gamma}{}^\mu, \overset{+}{\gamma}{}^\nu \} = 2g^{\mu\nu}$$
$$\{ \overset{-}{\gamma}{}^\mu, \overset{-}{\gamma}{}^\nu \} = -2g^{\mu\nu} \qquad (8)$$

We now go on to define the operators

$$a^{+\mu} = \frac{\overset{+}{\gamma}{}^\mu + \overset{-}{\gamma}{}^\mu}{2}$$
$$a^{\mu} = \frac{\overset{+}{\gamma}{}^\mu - \overset{-}{\gamma}{}^\mu}{2} \qquad (9)$$

These satisfy the anticommutation relationships

$$\{ a^{+\mu}, a^{+\nu} \} = \{a^\mu, a^\nu\} = 0$$
$$\{a^{+\mu}, a^\nu\} = g^{\mu\nu} \qquad (10)$$

We have thus arrived at Schönberg's set of creation and anihilation operators, which we have, however, derived from the two supplementary Clifford algebras, generated respectively from $\overleftarrow{\gamma}^\mu$ & $\vec{\gamma}^\mu$ (which are however to be distinguished from the original Clifford algebra given by eqn. (1). Clearly the a^μ

define a Grassmann algebra with 16 elements, as do the $a^{+\mu}$, so that together, they cover the same 256 elements that are covered by the product of the two supplementary Clifford algebras.

The infinitesimal element of the Lorentz group is given by

$$L_g = \frac{\varepsilon^{\mu\nu}}{4}(\vec{\gamma}^{\mu}\vec{\gamma}^{\nu} + \overleftarrow{\gamma}^{\mu}\overleftarrow{\gamma}^{\nu}) = \varepsilon_{\mu\nu}\, a^{+\mu} a^{\nu}. \qquad (11)$$

From the above, we readily verify that $a^{+\mu}$ and a^{ν} do indeed transform as vectors.

When we use the "fermionic" operators $a^{+\mu}$ and a^{ν}, it is convenient to define an "empty" state vector, V_e, satisfying

$$a^{\mu} V_e = 0 \text{ for all } \mu \qquad (12)$$

Let us now go back to the representations (1) in terms of the original Clifford algebra, and consider the unit scalar

$$S = (1)_{ij}$$

We have

$$a^{\mu} S = \left(\frac{\overset{+}{\gamma}{}^{\mu} - \overset{-}{\gamma}{}^{\mu}}{2}\right) S = (\gamma^{\mu} 1 - \gamma^{5} 1\, \gamma^{5}\, \gamma^{\mu}) = 0$$

where we have written $(\vec{\gamma}^{\mu})1 = \gamma^{\mu}$ and $\vec{\gamma}^{5} 1 = \gamma^{5}$. $1.\overleftarrow{\gamma}^{5} = \gamma^{5}$, and $1.\overleftarrow{\gamma}^{\mu} = \gamma^{\mu}$. This means that the scalar S is equivalent to V_e, the "empty" state of the "fermionic" representation.

We have also

$$a^{+\mu} S = \frac{\gamma^{\mu} 1 + \gamma^{5} 1 \gamma^{5} \gamma^{\mu}}{2} \doteq \gamma^{\mu}$$

and similarly, we can easily show that products, such as $a^{+\mu} a^{+\nu} S$ with $\mu \neq \nu$, also reduce to the corresponding products, $\gamma^{\mu}\gamma^{\nu}.1 = \gamma^{\mu}\gamma^{\nu}$. This means that we can write for the general state vector

$$V = \left[A_0 + a^{+\mu} A_{\mu} + a^{+\mu} a^{+\nu} A_{[\mu\nu]} + a^{+\mu} a^{+\nu} a^{+\alpha} A_{[\mu\nu\alpha]} + a^{+\mu} a^{+\nu} a^{+\alpha} a^{+\beta} A_{[\mu\nu\alpha\beta]}\right] V_e \tag{13}$$

This expresses the connection between the Clifford algebra and Schönberg's use of a Grassmann algebra.

In the Clifford algebra, the operator $i\gamma^5 = -\gamma^1\gamma^2\gamma^3\gamma^4$ produces what has been called a duality transformation That is

$$\gamma^5 \phi = \tilde{A}_0 + \tilde{A}_{\mu}\gamma^{\mu} + \tilde{A}_{[\mu\nu]}\gamma^{\mu}\gamma^{\nu} + \tilde{A}_{[\mu\nu\lambda]}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda} + \tilde{A}_{[\mu\nu\lambda\alpha]}\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\alpha} \tag{14}$$

where the $\tilde{A}$ are the duals, as usually defined. For example,

$$\tilde{A}_{\mu} = A_{[\nu\alpha\beta]} \frac{\varepsilon^{\nu\alpha\beta\gamma}}{3!} g_{\gamma\mu} .$$

Noting that γ^5 is operating from the right, we obtain, in the "fermionic" representation

$$\gamma^5 = \vec{\gamma}^5 = i\vec{\gamma}^1\vec{\gamma}^2\vec{\gamma}^3\vec{\gamma}^4$$

Let us choose any of the $\vec{\gamma}^\mu$, and express it as

$$\vec{\gamma}^\mu = \overset{+}{\gamma}^\mu = a^{+\mu} + a^\mu$$

We then obtain

$$\overset{+}{\gamma}^\mu \; a^\alpha \; \overset{+}{\gamma}^\mu = (a^{+\mu} + a^\mu) \; a^\alpha \; (a^{+\mu} + a^\mu)$$

when $\alpha = \mu$, we obtain $a^{+\mu}$, and when $\alpha \neq \mu$, we obtain $-a^\alpha$. So, the operator γ^μ transforms a^μ into $a^{+\mu}$, and (as can easily be shown), $a^{+\mu}$ into a^μ. The product, $i\vec{\gamma}^5$, thus interchanges all creation and anihilation operators. So the duality operation transforms the "empty" state vector, Ve into the "full" state V_f, and as can easily be verified, it changes an arbitrary state vector, V, into $\tilde{V}$.

Let us now take as an example the Dirac equation operating from the left on the original Clifford algebra

$$(\gamma^\mu \frac{\partial}{\partial x^\mu})\phi = (\vec{\gamma}^\mu \frac{\partial}{\partial x^\mu})\phi = 0$$

Writing once again

$$\vec{\gamma}^\mu = \overset{+}{\gamma}^\mu = a^{+\mu} + a^\mu$$

we obtain

$$(a^{+\mu} + a^{\mu}) \frac{\partial}{\partial x^{\mu}} V = 0$$

where V is the state vector corresponding to ϕ. Consider a typical term of V, such as $(a^{+\alpha} A_{\alpha})Ve$

Then $a^{+\mu} a^{+\alpha} \frac{\partial A_{\alpha}}{\partial x^{\mu}} Ve$ corresponds to the exterior derivative of A_{α}. This is the boundary operator [9] (which evidently satisfies $(a^{+\mu} \frac{\partial}{\partial x^{\mu}})(a^{+\lambda} \frac{\partial}{\partial x^{\lambda}}) = 0$, because of antisymmetry of all products of $a^{+\mu}$). Similarly $a^{\mu} \frac{\partial}{\partial x^{\mu}} \cdot A_{\alpha} a^{+\alpha} Ve = g^{\mu\alpha} \frac{\partial A_{\alpha}}{\partial x^{\mu}} Ve$, which corresponds to the co-boundary operator.

If D represents the duality operation (given by $i\vec{\gamma}^{5}$) we can write $a^{\mu} = D a^{+\mu} D^{-1}$, and obtain

$$(\vec{\gamma}^{\mu} \frac{\partial}{\partial x^{\mu}})V = a^{\mu} \frac{\partial}{\partial x^{\mu}} + D a^{\mu} \frac{\partial}{\partial x^{\mu}} D^{-1} \qquad (15)$$

and this is just the expression for the Dirac equation that Kähler[8] has derived, in another way.

The Dirac operator can thus be regarded from the point of view of algebraic topology as the sum of a boundary operator and a co-boundary operator, while the terms of the Grassmann algebra correspond to a set of complexes. We have however been able to derive/the result fairly directly, because the relationship of antisymmetric tensors of different rank is more simply expressed in terms of the Grassmann algebra rather than in terms of the Clifford algebra.

The result relates to a preliminary connection between quantum theory and co-homology theory that we have made earlier[10]. But now, we can develop this connection further by relating the homology operations to the Clifford and Grassman algebras which are basic to relativistic quantum theory. We expect to go into this question in later papers.

4. The Dirac Equation and Relativistic Phase Space

We return to the two Dirac equations (7) which are now written in terms of the vector space, V, .

$$\vec{\gamma}^{\mu} \frac{\partial V}{\partial x'^{\mu}} = 0 , \quad \overleftarrow{\gamma}^{\mu} \frac{\partial V}{\partial x^{\mu}} = 0 \tag{16}$$

As in eqn (2), we introduce $X = \frac{x'^{\mu} + x^{\mu}}{2}$

$\eta^{\mu} = x^{\mu} - x'^{\mu}$ along with a relativistic generalization of the Wigner-Moyal transformation

$$\bar{V}\,(X^{\mu}, P^{\mu}) = \frac{1}{2\pi} \int V(x'^{\mu}, x^{\mu}) e^{-iP_{\mu}\eta^{\mu}} d^4\eta \tag{17}$$

We multiply the second of the equations (16) by $\vec{\gamma}_5 \overleftarrow{\gamma}_5$ and we get

$$\overset{+}{\gamma}^{\mu} \frac{\partial V}{\partial x'^{\mu}} = 0 , \quad \overset{-}{\gamma}^{\mu} \frac{\partial V}{\partial x^{\mu}} = 0$$

From (9) we write the above as

$$(a^{+\mu} + a^{\mu}) \frac{\partial V}{\partial x'^{\mu}} = 0 , \quad (a^{+\mu} - a^{\mu}) \frac{\partial V}{\partial x^{\mu}} = 0$$

By adding and subtracting the above equations we obtain

$$\frac{a}{2}^{+\mu} \frac{\partial V}{\partial x^{\mu}} - a^{\mu} \frac{\partial V}{\partial \eta^{\mu}} = 0 \tag{18-a}$$

$$\frac{a}{2}^{\mu} \frac{\partial V}{\partial x^{\mu}} - a^{+\mu} \frac{\partial V}{\partial \eta^{\mu}} = 0 \tag{18-b}$$

Under the Wigner-Moyal transformation (18-a) becomes

$$\left(\frac{\overset{+}{a}^{\mu}}{2} \frac{\partial}{\partial x^{\mu}} - i\, a^{\cdot\,\mu} P_{\mu}\right) \overline{V} = 0$$

By applying the above operators twice we are led to

$$g^{\mu\nu} P_{\nu} \frac{\partial \overline{V}}{\partial x^{\mu}} = 0 \tag{19}$$

This is the (classical) Liouville equation for a free particle.

We have thus seen that certain combinations of the Dirac operators appearing in eqn (18-a), when applied twice imply the Liouville equation. If however we had added (18-a) and (18-b), and then applied the resulting operator twice we would have obtained the Klein-Gordon equation which, of course, contains further quantum-mechanical implications, beyond those of the classical Liouville equation. Some of these implications have been discussed in reference (7), but for the purposes of this paper, such questions can be set aside.

As pointed out in Sec. (2) the coefficients of the $\overline{V}$, as obtained by the Wigner-Moyal transformation on (13), are now to be interpreted as constants of the motion. The Dirac equation applied to the original Clifford algebra thus implies a whole set of antisymmetric tensors that are related constants of the motion, beyond the single pair (real and imaginary parts of a complex function) which appears in the non-relativistic theory. The appearance of such a set implies some new physical concepts, which we shall discuss, at least in part, further on in this section.

Let us now consider the effect of electromagnetic potentials, $A_\mu(x^\nu)$. The Dirac equations (16) become

$$\vec{\gamma}^\mu \left(\frac{\partial}{\partial x'^\mu} - i\frac{e}{c}A_\mu(x'^\nu)\right) V = 0$$

$$\overleftarrow{\gamma}^\mu \left(\frac{\partial}{\partial x^\mu} + i\frac{e}{c}A_\mu(x^\nu)\right) V = 0 \qquad (20)$$

Expanding

$$A_\mu(x^\nu) \cong A_\mu(X^\nu) + \frac{\eta^\alpha}{2}\frac{\partial A_\mu}{\partial X^\alpha}(X^\nu)$$

and

$$A_\mu(x'^\nu) \cong A_\mu(X^\nu) - \eta^\alpha/2\,\frac{\partial A_\mu}{\partial X^\alpha}(X^\nu)$$

and following the same substitutions that led to eqn. (19), we obtain for the generalized Liouville operator,

$$L_q = g^{\mu\nu}(P_\mu + \frac{e}{c}A_\mu)\frac{\partial}{\partial X^\mu} + g^{\mu\nu}(P_\mu + \frac{e}{c}A_\mu)\frac{e}{c}\frac{\partial A_\nu}{\partial X^\sigma}\frac{\partial}{\partial P_\sigma}$$

$$- \frac{e}{c}\,\overset{+}{a}^\mu a^\nu \left(\frac{\partial A_\mu}{\partial X^\nu} - \frac{\partial A_\nu}{\partial X^\mu}\right) - i\frac{\overset{+}{a}^\mu \overset{+}{a}^\nu}{4}\frac{e}{c}\frac{\partial}{\partial X^\sigma}\left(\frac{\partial A_\mu}{\partial X^\nu} - \frac{\partial A_\nu}{\partial X^\mu}\right)\frac{\partial}{\partial P_\sigma} \qquad (21)$$

The first term in the above represents the modified contribution to the Liouville equation coming from the motion of the trajectory through space. The second represents the contribution of the electromagnetic force.

The third corresponds to the effects of "spin" of the Dirac electron. That is to say, $\bar{V}$ changes, not only because of changes of position and momentum in a trajectory, but also because the electromagnetic fields, $F_{\mu\nu}$, generate what is equivalent to a rotation (Lorentz) transformation among the components of the V. These latter thus cease to be constants of the motion. Spin is therefore a new kind of movement, which involves a change of what were originally constants of the motion. This means that the solutions of Liouville's equation now undergo a transformation that does not arise from the "particle" motions, so that they have begun to take on an independent physical meaning. Spin is thus not a property of trajectories but of the field, $\bar{V}$ (X P). (which is constituted of antisymmetric tensors)

Finally, there is the fourth term in (21). This represents a kind of "force" on the trajectories due to the electromagnetic field (because it involves $\frac{\partial}{\partial p_\sigma}$) But it also involves fields, $\bar{V}$ through the operators $a^{+\mu}a^{+\nu}$. We may regard it as describing the effect of spin on the trajectories. (this term will be unimportant if the field functions, $\bar{V}$, spread out over a large region in phase space).

5. Conclusion

We have developed an interpretation of Dirac's equation in terms of an original Clifford algebra (given in eqn (1)), which sytematically incorporates all the antisymmetric tensors, the Grassmann algebra and its dual, and a pair of supplementary (right and left) Clifford algebras. From there, we have derived a Liouville equation. By thus putting Dirac's equation into a "language" broad enough to relate it to classical mechanics, we can see in some detail how spin goes beyond ordinary classical concepts. These latter attribute all the electronic motion to a particle trajectory, and imply that the solutions of Liouville's equation are a mere "shadow" of the trajectories, which are regarded as descriptions of the basic reality. We see that on the contrary, the solutions of Liouville's equation are fields having certain independence of motion, and that the trajectories depend on these fields in an irreducible way. So the two together are needed for a complete account, and this is beyond the classical way of thinking.

A related result has been obtained non-relativistically in our previous paper[4], where we show that the concept of a self-determined trajectory has a limited applicability, and that these limits are determined by the solutions of Liouville's equations, which indeed also have a meaning going beyond that of being mere descriptions of trajectories.

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