ALGEBRAIC QUANTUM MECHANICS AND PREGEOMETRY

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ABSTRACT

We discuss the relation between the q-number approach to quantum mechanics suggested by Dirac and the notion of 'pregeometry' introduced by Wheeler. By associating the q-numbers with the elements of an algebra and regarding the primitive idempotents as 'generalized points' we suggest an approach that may make it possible to dispense with an a priori given space manifold. In the approach the algebra itself would carry the symmetries of translation, rotation, etc. Our suggestion is illustrated in a preliminary way by using a particular generalized Clifford algebra proposed originally by Weyl, which approaches the ordinary Heisenberg algebra in a suitable limit. We thus obtain a certain insight into how quantum mechanics may be regarded as a purely algebraic theory, provided that we further introduce a new set of "neighbourhood operators", which remove an important kind of arbitrariness that has thus far been present in the attempt to treat quantum mechanics solely in terms of a Heisenberg algebra.

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1. INTRODUCTION

Quantum field theories are generally constructed by assuming a basic a priori given space-time structure where the coordinates x^μ are treated as parameters belonging to a pre-Hilbert space. The successes of this formalism, particularly for the electromagnetic and weak forces suggest that a similar method should be carried through for the quantization of gravity where the metric tensor now becomes the subject of quantization. Unfortunately, renormalisation presents serious difficulties to such a programme and one of the more recent suggestions is that the root of the problem may lie in the intimate relation between gravity and the space-time structure itself. There is now a growing realisation that perhaps the use of a differential manifold will have to be called into question and other possibilities explored (see, for example, Taylor, 1979, t'Hoof, 1978 and Wheeler, 1980).

Indeed, a decade ago Finkelstein (1972) had already expressed dissatisfaction with quantum field theory by pointing out that present theories are essentially hybrids in which classical space-time (c) is combined with quantum matter (q). What is required, he suggests, is not a cq-theory but a purely q-theory where no reference is made to an a priori given space-time. In such an approach, space and time would emerge from some deeper theory. Since the deeper theory can no longer use the properties of a differential manifold in a basic way we should follow Wheeler (1980) and regard such a theory as 'pregeometric'. But this immediately raises the question as to the nature of the elements of such a theory. The purpose of this paper is to examine this question in the context of the present quantum formalism which we will analyse in a manner that is basically different from the usual one. We will show that our approach leads to a possibility of realising such a theory although at this stage we will make no attempt to connect our structure with gravity.

Our ideas start by noting that a 'pregeometric' theory is already to some extent implicit in the original Heisenberg matrix approach to quantum theory when what is regarded as 'position' becomes part of a matrix algebra. Here the 'position' was represented as a matrix q(n,m) which, in the way it was developed, appeared to be more like a transition or 'two-point' object than a means of labelling the points of a continuum. It was only through the Schrödinger representation that $|x\rangle$ became associated with a way of labelling points of an underlying differential manifold. The operator X is taken as a description of the way we locate objects at some position in space using a suitable measuring instrument. The fact that the two approaches (Heisenberg and Schrödinger) have generally been regarded as equivalent has led to the main emphasis being placed on the Schrödinger representation which, as is well known, has what appears to be a particularly simple interpretation through the wavefunction. The success of the formalism has strongly reinforced the idea that the underlying differential manifold must play a fundamental role in the theory.

When we consider the case of a system with an infinite number of degrees of freedom we find that the above equivalence breaks down. Dirac (1963) has illustrated this point by means of a simple example in which he shows that Heisenberg's approach yields results that cannot be derived from a wave-function even though the energies are well defined. He also points out that some of the infinities of the conventional theories are no longer present in the q-number approach.

As a result of this work, Dirac suggests that perhaps the q-number approach should be taken more seriously, but one of the main problems in developing such a theory lies in the apparent lack of any compelling physical interpretation of the q-numbers. Recently, Bohm (1973) has developed some very general ideas involving what he calls the implicate order. This approach, we believe, begins to provide a more general framework in which to view a q-number formalism, and although we will occasionally be guided by general considerations appearing in the implicate order, much of our work stems directly from the mathematical structures used in this paper.

We will not discuss the case of a system with an infinite number of degrees of freedom. Rather our aim here will be to motivate an exploration of purely q-number theories and to do this, we will consider the finite case only[1]. Furthermore, we will restrict ourselves at this stage to a discussion of the non-relativistic theory in order to make clear the concepts used in these initial investigations.

Some preliminary work has already been done by Frescura and Hiley (1980a). They indicate in some detail some general lines along which one can make an essentially algebraic approach to quantum mechanics. What they do is to represent <u>all</u> physical features by the elements (q-numbers) of some suitable algebra, the nature of which is determined by the physical context. In such an approach there is no need for the disjoint features of the present mathematical formalism, namely, the operators on the one hand and the state space vectors on the other. Rather, one uses only a single type of object, the algebraic element (q-number). What is now taken to be the state vector is simply a distinguished element in the algebra, namely, the minimal ideal.

This idea is most easily illustrated in the case of the Pauli and the Dirac-Clifford algebras and, indeed, such a possibility was already anticipated by Riesz (1946). The polynomial Heisenberg algebra studied by Born and Jordan (1925) appeared to have no minimal ideals (except the trivial ones) so that the generalization of Reisz's work to quantum mechanics seemed impossible. However, a new element can be introduced into the polynomial algebra from which the minimal ideals corresponding to the state vectors can be generated. This new element plays a role similar to the standard ket in Dirac's (1947) bra-ket notation (Frescura and Hiley, 1980b). It was this additional feature that allows for the possibility of a completely q-number approach. In this paper we will try to bring out more clearly how the q-number theory is related to the usual bra-ket notation and, as a consequence, we will show that the q-number theory is potentially richer even in the case of a finite number of degrees of freedom. Furthermore, we will bring out how the q-number approach does not require an a priori given space manifold. In this sense it can be regarded as 'pregeometric'.

In Section 2 we will show how such a 'pregeometric' structure can be given an algebraic meaning in terms of the primitive idempotents and their corresponding minimal left and right ideals. These features provide some of the necessary concepts that are needed to understand the q-number approach in which space is not taken as an a priori given structure. In Section 3, we discuss how these concepts can be illustrated through a particular algebraic structure, namely, the collection of algebras known as generalized Clifford algebras (Morris, 1967). These algebras have been studied in some detail by Ramakrishnan, Santhanam and their coworkers (see Santhanam, 1977, for an extensive list of references).

We will discuss a particular generalized Clifford algebra C_2^n which can be used to set up quantum mechanics in a discrete one-dimensional space. In the limit as $n \to \infty$, C_2^n approaches the Heisenberg algebra as was pointed out many years ago by Weyl (1932). Santhanam and Tekumella (1975) have already used this structure to suggest a discrete quantum mechanics using the vector space approach. We analyse this structure purely from the q-number point of view and show in Section 4 that contrary to the conclusions of Santhanam and Tekumella (1975) the discrete space does possess an uncertainty principle.

In the final section we discuss how the geometric order emerges from the pre-geometric algebraic structure. In particular, we find that the physical meaning of the Heisenberg algebra can be determined only by introducing a new set of neighbourhood operators, which are parts of this algebra. In this way, we remove a certain arbitrariness that is inherent in the very concept of a Heisenberg representation, and determine which algebraic elements are the ones that represent the actual physical space.

2. THE ROLE OF THE PRIMITIVE IDEMPOTENT

In order to motivate our approach, let us briefly recall some of the ideas that prompted Wheeler to suggest some form of pregeometric theory[2]. In the classical approach space-time is assumed to be a continuum. In order to give meaning to matter and gravity in this context, Sakharov (1967) has suggested the possibility of regarding the continuum as some form of 'elastic' medium out of which matter is formed. The concentrations of matter then leave something analogous to a stress in the medium. The average properties of these stresses can then be described by the curvature tensor which in turn can be associated with the average distribution of matter so that gravitation becomes the'metric elasticity' of space.

However, in the case or ordinary matter, elasticity has its origins in the atoms and the forces between them; atoms are not built from elasticity. Could not the metric geometry be hiding a deeper 'atomic' structure which is revealed in some way by the appearance of particle-anti-particle pairs which arise from fluctuations of the vacuum. By associating change with small-scale topological features of space-time, Wheeler argues that the quantum fluctuations with their implications for a continual change in topology suggest something more like a foam (Hawking 1976), or perhaps like some form of disordered lattice structure continually undergoing restructuring (Hiley 1980). For such structures the continuum with its well defined and fixed neighbourhood relations may not be an appropriate starting point. Rather we should begin with a set of basic elements, which, for convenience, we can call 'generalized points', and the relationships between them. This will give rise to a generalized structure in which neither a fixed neighbourhood relation nor a fixed dimensionality have any direct relevance in the small scale. Our conventional space-time is then an abstraction which emerges from this structure through some form of macroscopic averaging.

Wheeler's specific suggestion was to build the pregeometry from a set of Boolean elements giving a dichotomic choice such as yes or no, true or false, on or off, etc. Although his original hope was to use some of the notions of formal logic to lay the foundations of pregeometry, he has since concluded that, in fact, formal logic leads in another direction away from quantum theory rather than towards it. However, his conclusion should not be taken to imply that a simple dichotomic element cannot be used as the basic descriptive form in pregeometry. What we will do here is to draw attention to the fact that such elements play an important role in the analysis of algebras in general and, furthermore, the dichotomy of yes/no arises in an essentially quantum mechanical way, namely, through the eigenvalues of the set basic elements. Here we are referring to the primitive idempotents.

This idea has already been discussed briefly in Frescura and Hiley (1980a). They argued that the link between the usual geometric entities, vectors, bivectors, etc., appeared as ordered products of elements taken from minimal left and right ideals. But these minimal ideals are generated from a set of primitive idempotents ε_i which satisfy the relations

$$\varepsilon_i^2 = \varepsilon_i \qquad (1)$$

$$\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i = 0$$

and for an algebra with a unity,

$$\sum_i \varepsilon_i = 1$$

These idempotents have eigenvalues 1 and 0 so that they become the algebraic equivalents to the yes-no logic.

It is clear that idempotents can be constructed in quantum mechanics where they are, of course, called projection operators. These projection operators are used to form the basis of the propositional calculus first introduced by Birkoff and von Neumann (1936) and which has since been developed into a formal structure called quantum logic (see, for example, Jauch 1968). We do not want to follow this approach for reasons similar to those given by Wheeler (1980) and, more importantly, because here we want to open up new possibilities, the relevance of which cannot be seen if the idempotents are treated as propositions. Indeed, there is no notion of projection in our algebraic structure.

Instead we will follow Eddington (1946) who argued that within a purely algebraic approach, which he regarded as providing a structural description of physics, there are elements of existence defined, not in terms of some hazy metaphysical concept of existence, but in the sense that existence is represented by a symbol which contains only two possibilities - existence or non-existence. Thus we will assume that the structural concept of existence is represented by an idempotent of some appropriate algebra. But recalling that any idempotent can always be decomposed into primitive idempotents satisfying the relations (1), we will take the primitive idempotents as our basic descriptive forms. They will be considered as the generalized points of our structure.

It should be noted that these primitive idempotents, although being used as basic elements of the description, should not be considered as some form of absolute element of reality. The symbols do not have any direct meaning in isolation. They are not dinge-an-sich but take their meaning from within the overall context of the given algebra, which, in turn, is determined from a particular physical context. In this way, we incorporate Bohr's notion of 'wholeness' and d'Espagnat's view of 'non'separability' in a very basic way. In this sense our generalized points could appropriately be called 'holons'.

The set of primitive idempotents defined by the relations (1) have been indexed by using a single symbol and thus we can represent the idempotent by a set of points labelled by a set of integers i, j ... But we also require a way of relating generalized points, i.e. how do we relate $i \rightarrow j$, etc.? To meet this requirement we will introduce a set of elements ε_{ij} to denote the relation between i and j. If we impose the multiplication rule

$$\varepsilon_{ij}\varepsilon_{kl} = \delta_{jk}\varepsilon_{il} \tag{2}$$

then the elements ε_{ii} will satisfy the relations

$$\varepsilon_{ii}\varepsilon_{ii} = \varepsilon_{ii} \tag{3}$$

$$\varepsilon_{ii}\varepsilon_{jj} = 0 \qquad \text{if } i \neq j$$

That is, they are primitive idempotents. These elements are then our basic q-numbers.

The discussion so far has been quite general and although we have been using the idea of 'generalized point' there is nothing yet in the structure to indicate that these 'points' can be related to the points of space. In fact, the primitive idempotents are not even unique since it is generally possible to find another set by using some inner automorphism of the algebra. In order to illustrate how the connection with space is made, it is necessary to turn to consider a particular algebraic structure and analyse it in the way we have suggested in this section.

3. DEFINITION AND STRUCTURE OF THE FINITE WEYL ALGEBRA OF ORDER n^2

We begin by defining the finite Weyl algebra C_2^n of order n^2 as the polynomial algebra generated over the complex field by the set of generating elements $\{e_0^1, e_1^0\}$[3] subject to the relations:

$$(e_0^1)^n = e_0^n = 1 \qquad (4)$$

$$(e_1^0)^n = e_n^0 = 1 \qquad (5)$$

$$e_0^1 e_1^0 = \omega e_1^0 e_0^1 \qquad (6)$$

where $\omega = \exp\left(\frac{2\pi i}{n}\right)$

These relations define the C_2^n algebra completely.

By adopting an implied ordering of the generating elements, a general element has the form:[4]

$$e_b^a \triangleq e_0^a e_b^0 = \omega^{+ab} e_b^0 e_0^a \qquad a, b, = 0, 1 \ldots n-1$$

We may obtain the general rule of combination:

$$e_b^a e_d^c = \omega^{-bc} e_{b+d}^{a+c} \qquad (7)$$

with this single rule we may accomplish all the manipulations of algebra.

It is easy to demonstrate that the n^2 elements e_b^a form a basis for the C_2^n algebra. Thus every element of the algebra may be written as

$$A = \sum_{a,b=0}^{n-1} A_{ab} e_b^a \tag{8}$$

In terms of this basis, it is possible to obtain a complete set of pairwise orthogonal primitive idempotents ε_{ii}, one such set that we will use has the form:

$$\varepsilon_{ii} = \frac{1}{n}\sum_k \omega^{-ik} e_k^0 \tag{9}$$

and satisfies

$$\sum_i \varepsilon_{ii} = 1 \tag{9a}$$

It is quite easy to see by direct algebraic multiplication, and the use of rule (7), that the ε_{ii} satisfy relations (2) and (3).

The ε_{ij} associated with the primitive idempotents given by (9) can be written as

$$\varepsilon_{ij} = \frac{1}{n}\sum_r \omega^{-jr} e_r^{j-i} \tag{10}$$

Direct multiplication shows that this expression satisfies the multiplication rule (2) viz:

$$\varepsilon_{ik}\varepsilon_{km} = \delta_{kj}\varepsilon_{im} \tag{2}$$

Each of the n idempotents defines an n-dimensional subspace in the n^2 dimensional space associated with the C_2^n algebra. To illustrate the method let us arbitratily choose right and left ideals associated with the idempotent given by the index i = 0

$$\varepsilon_{00} = \frac{1}{n}\sum_k e_k^0 \tag{11}$$

By employing standard algebraic techniques we can find the right and left ideals $I_R^{(0)}$ and $I_L^{(0)}$ associated with this fundamental idempotent. These ideals form n-dimensional vector subspaces with basis vectors given by

$$I_L^{(0)}(i) = \frac{1}{n}\sum_k e_k^{-i} \tag{12}$$

$$I_R^{(0)}(i) = \frac{1}{n}\sum_k \omega^{ik} e_{-k}^{i} \tag{13}$$

It is now straightforward to show that

$$I_L^{(0)}(i)I_R^{(0)}(j) = \varepsilon_{ij} \tag{14}$$

and

$$I_R^{(0)}(i)I_L^{(0)}(j) = \delta_{ij}\varepsilon_{00} \tag{15}$$

where (15) expresses the fact that the left ideal and its dual are orthogonal. Comparison of (12) and (13) with (10) shows us that

$$I_L^{(0)}(i) = \varepsilon_{i0} \quad \text{and} \quad I_R^{(0)}(j) = \varepsilon_{0j}$$

so that (14) and (15) are consequences of (2).

We may now introduce an operator which, for reasons that will become apparent later, we denote by X. This operator will label the generalized points if we define it as

$$X = \frac{1}{n}\sum_{jk} j\omega^{-jk} e_k^0 = j\varepsilon_{jj} \tag{16}$$

Then immediately we see

$$X\varepsilon_{jj} = j\varepsilon_{jj} \tag{17}$$

Notice also that

$$X\varepsilon_{jm} = j\varepsilon_{jm} \tag{18}$$

In this way we see that the very structure of the idempotent given in equation (9) provides a serial order in the set of primitive idempotents. But, of course, it is always possible to obtain a new set of primitive idempotents under the inner automorphism

$$\varepsilon'_{jj} = S\varepsilon_{jj}S^{-1} \tag{19}$$

where S is any element of C_2^n. The new primitive idempotents will also provide an order but, in general, this order will not be simply related to the original order given to the first set of primitive idempotents. Closer examination of the effects of equation (19) suggests a kind of 'exploding' transformation in which each generalized point of the old set is spread out into some or all of the points of the new set. Yet the implications of the notation used in equations (17) and (18) are that X will be a position operator that can be used to locate or label a particular generalized point through an eigenvalue j that reflects the position of the point. However,

this order seems to be arbitrary. In Section 6 we shall enquire into how one can obtain an invariant meaning to the order rather than just imposing it from the outside.

4. THE GEOMETRIC INTERPRETATION OF THE ALGEBRA C_2^n

In the usual Cartesian view, we can order a discrete set of equally spaced points on the real line by choosing an origin and defining a unit displacement. Successive applications of this unit displacement will take us through the series of points in the right order. Similarly, in terms of our generalized points, we can choose a basic primitive idempotent ε_{00} to serve as an origin and select an element T of the algebra to define a unit displacement through the relation

$$\varepsilon_{j+1\ j+1} = T\varepsilon_{jj}T^{-1} \qquad (\forall_j) \qquad (20)$$

For the primitive idempotent defined in (11), T has a very simple form, namely, $T^{-1} = e_0^1$. In fact, our choice of the basic idempotent (11) was determined by noticing that in this case e_0^1 acted as a translation operator. Using this to define the canonical order, we find

$$\varepsilon_{jj} = e_0^{-j}\varepsilon_{00}e_0^{j} \qquad (21)$$

Since the ε_{jj} are the generalized points of our structure which are labelled through the eigenvalue equation (17), we can interpret $T = e_0^{-1}$ as a translation operator on our discrete space.

Since the algebra is symmetrical in e_0^1 and e_1^0 we could raise the question as to whether the other generator e_1^0 could be used as a translation operator based on another set of generalized points ε_{jj}' so that

$$\varepsilon'_{00} = \frac{1}{n}\sum_k \omega^{-ik} e_0^k \tag{22}$$

then

$$\varepsilon'_{jj} = e_j^0 \varepsilon'_{00} e_0^{-j} \tag{23}$$

The generalized points defined by ε'_{jj} can then be labelled through

$$X'\varepsilon'_{jj} = j\varepsilon'_{jj} \tag{24}$$

and

$$X'\varepsilon'_{jm} = j\varepsilon'_{jm} \tag{25}$$

with

$$X' = \frac{1}{n}\sum_{jk} j\omega^{-jk} e_0^k \tag{26}$$

In this discrete space the translation operator is $T' = e_1^0$. Thus we have distinguished two discrete spaces, each comprising a set of points for which the generators define a translation operator via (21) and (23). These spaces are related through the transformation

$$Z\varepsilon_{jj}Z^{-1} = \varepsilon'_{jj} \tag{27}$$

where

$$Z = \frac{1}{\sqrt{n^3}} \sum_{ijk} \omega^{j(i-k)} e_k^{j-i} \tag{28}$$

Thus the symmetry between e_0^1 and e_1^0 establishes a kind of duality between two discrete subspaces. Specifically, the points of each space are related via (27) and (28). As we have seen, under any inner-automorphism the new primitive idempotents have a far from simple relationship to the original idempotents, nevertheless there remains a natural duality between the two sets of new primitive idempotents and hence between the two discrete spaces constructed from these idempotents.

We are familiar with this kind of natural duality in standard quantum mechanics where translations in space are generated by the momentum operator and translations in momentum space are generated by the position operator. In fact, the operator corresponding to a translation through a distance a is given by

$$T_x(a) = e^{-iaP}$$

This suggests that for a discrete space there exists a momentum operator p such that

$$e_0^1 = e^{\frac{2\pi iP}{n}} \tag{29}$$

Again in standard quantum mechanics a translation operator in momentum space is

$$T_p(a) = e^{+iax}$$

which again suggests

$$e_1^0 = e^{-\frac{2\pi i}{n}X} \tag{30}$$

Thus the two discrete spaces generated above are the discrete position and momentum spaces which imply that in equation (26) we should write X' = P.

In order to confirm this identification let us investigate the commutation between X and P as we go to the limit of $n \to \infty$. To do this let us first form the commutator of X defined by (16) and P(=X') defined by (26).

$$[X,P] = \frac{1}{n}\sum_{jkrs} (s-j)r\omega^{r(s-j)}\omega^{-jk}e_k^{j-s} \tag{31}$$

Now let us see what happens. To do this, the discrete indices are replaced by continuous indices viz:

$$\frac{1}{n}\sum_{jkrs} \to \frac{1}{2\pi}\iiiint dj\,dk\,dr\,ds$$

and

$$\omega^{\alpha} \to \exp\{+2\pi i\alpha\}$$

we have from (31)

$$[X,P] = \frac{1}{2\pi}\iiiint r \exp\{-2\pi ir(s-j)\}(s-j)\omega^{-jk}e_k^{j-s}djdkdrds$$

$$= \frac{+1}{i(2\pi)^2}\iiiint 2\pi i\frac{d}{d(s-j)} \exp\{-2\pi ir(s-j)\}(s-j)e_k^{j-s}\exp\{+2\pi ijk\}djdkdrds$$

Integrating over dr.

$$= \frac{-i}{2\pi}\iiint\delta'(s-j).(s-j).\exp\{2\pi ijk\}e_k^{j-s}djdkds$$

Intergrating over dj.

$$[X,P] = i\iint \exp\{2\pi i.jk\}e_k^0dkdj \qquad (32)$$

Where the latter double integral is the algebraic expression of the unity element 1 since it is none other than the completeness relation for the infinite dimensional case.

$$[X,P] = i$$

Thus, under the appropriate limiting procedure, the commutator of the X and P elements assumes the quantum mechanical value, confirming our interpretation of both X and P.

This result is not new. It was obtained essentially by Weyl (1932) and more directly by Santhanam (1977), but the method we have used is new and our whole approach throws a different light on quantum mechanics. It is not necessary to assume an a priori externally imposed position order together with an independent and externally imposed momentum order. The appropriate algebra already carries the order of space implicitly, provided the momentum is also part of the same structure. Thus the correlation between X and P has little to do with a duality of 'waves' or 'particles' but has to do with the description of structure process that does not require the external imposition of independent space and momentum orders.

The commutator (32) leads directly to the uncertainty principle and it is natural to assume that the discrete algebra ($n \neq \infty$) should also contain some form of uncertainty. However, Santhanam and Tekumalla (1975) state that the discrete case does not have an uncertainty principle. We believe this conclusion to be incorrect, as we demonstrate in the following way. It is well known that if A and B are two hermitian operators then

$$\Delta A \Delta B \geqslant \tfrac{1}{2}\langle | [A,B] | \rangle .$$

In our case A = X and B = P. It is easy to demonstrate that X and P are both hermitian and since they have a non-zero commutator (32) there must exist an uncertainty relationship between the two observables.

5. THE CONNECTION WITH THE BRA-KET NOTATION

In order to bring the discussion onto even more familiar ground, let us connect our approach with the usual bra-ket notation. As pointed out by Frescura and Hiley (1980a) there is a very close correspondence between the ket (bra) and the minimal left (right) ideals. Indeed, we can write

$$I_L^{(0)}(i) = |i\rangle \; ; \quad I_R^{(0)}(i) = \langle i|$$

then (14) and (15) become

$$I_L^{(0)}(i) I_R^{(0)}(j) = \varepsilon_{ij} = |i\rangle\langle j| \tag{33}$$

and

$$I_R^{(0)}(i) I_L^{(0)}(j) = \delta_{ij}\varepsilon_{00} \Rightarrow \langle i|j\rangle \tag{34}$$

Here the label (0) indicates the choice of the basic primitive idempotent. The usual bra-ket notation however suppresses the dependence of the ket (bra) on the choice of basic primitive idempotent by writing in general

$$I_L^{(m)}(i)I_R^{(m)}(j) = |i\rangle\langle j| \text{ and}$$

$$I_R^{(m)}(i)I_L^{(m)}(j) = \delta_{ij}\varepsilon_{mm} \Rightarrow \langle i|j\rangle \qquad (\forall m)$$

m being an arbitrary choice of basic primitive idempotent. So the bra-ket notation does not exploit all the q-number structure, and works with only one ideal.

Thus we see that when quantum mechanics is viewed from the q-number theory, the ket notation hides the fact that each ket represents an object with two labels. Suppressing this dependence on two labels means that the eigenvalue equation (18) can replace equation (17) so that each point can now be labelled by a ket $|j\rangle$. Then by using (21b) we can write

$$|j + a\rangle = e_0^{-a}|j\rangle \qquad (35)$$

which shows that e_0^{-a} corresponds to a translation operator that takes you from the point labelled by $|j\rangle$ to the point labelled by $|j + a\rangle$. As we know that in the limit as $n \to \infty$ this particular discrete structure becomes continuous and C_2^n approaches the Heisenberg algebra for a one-dimensional continuum in this limit we may write $|j\rangle \to |x_j\rangle$ in which case equation (18) becomes

$$X|x_j\rangle = j|x_j\rangle \qquad (36)$$

In this equation X acts as a position operator on the points representing the individual elements of a given minimal left ideal. It is this structure that forms the starting point of Santhanam's work (1977).

To complete the picture we can introduce kets in the momentum space so that

$$|p_{j+b}\rangle = e_b^0 |p_j\rangle$$

Using the primitive idempotent defined in equation (22) it is now easy to construct explicitly the kets in momentum space, viz.

$$|p_j\rangle = \frac{1}{\sqrt{n^3}} \sum_{ik} \omega^{ij} e_k^{-i} \qquad (37)$$

with the relationship

$$|p_j\rangle = \frac{1}{\sqrt{n}} \sum_i \omega^{ij} |x_i\rangle \qquad (38)$$

which corresponds to the continuum limit

$$|p\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iPx} |x\rangle \, dx \qquad (39)$$

Thus (38) represents the finite dimensional version of the Fourier transform which forms the starting point of the recent work of Gudder and Naroditsky (1981), though here, we see that it arises directly from the q-number structure.

The use of equations (25) and (26) with X' = P gives

$$P|p_j\rangle = j|p_j\rangle$$

6. GEOMETRIC ORDER AS EMERGING FROM THE ALGEBRAIC STRUCTURE

Let us now return to the question of how the algebra may uniquely determine its own space structure. We have seen in Section 3 that the set of primitive idempotents can be used to order the "points" but that this order is arbitrary up to an inner automorphism given by equation (19). We shall now discuss the question of how the proper set of idempotents with its attendant order of "points" can be determined in an invariant way, rather than just be imposed from outside.

This order, which is intended to be _the_ geometric order as we observe and experience it, is actually determined by the basic equations for all the relevant underlying physical structures (e.g. fields and particles). Such basic equations are in fact always built out of forms, which are differential, in what is usually called the position representation. Under an inner automorphism, these equations will in general cease to be differential, because, as pointed out in the discussion around equation (19), this kind of transformation "explodes" each point into a distribution spreading throughout the original space.

The above means that in a certain sense, Heisenberg and Schrödinger pictures are not completely equivalent after such a transformation, because the basic physically significant operations have ceased to be "local" (in the sense of being constituted out of functions such as x and $\frac{\partial}{\partial x}$, which have zero matrix elements for points that have a finite separation). The significance of this kind of non-equivalence has generally not been properly appreciated. For it is the requirement of continuity and single valuedness of _all_ physically significant operators (including the "wave function" as a left ideal) which is needed to give rise to the correct energy levels and probabilities of transition. Thus far, it has been possible to write the requirement only in the "position" representations. But, as we have seen, this representation is arbitrary. Therefore, without a further specification of what the "position representation" is, the physical consequences of the theory are not defined. This means that additional new concepts are needed to complete the mathematical structure, so as to allow us to assert the complete equivalence of Heisenberg and Schrödinger pictures (i.e., in a way that determines all physical properties, independently of the representation).

In the discrete case, which we are treating in this paper, there are, of course, no differential operations. Instead, however, we may introduce neighbourhood operators, from which can be obtained operators that approach differential operators, in the limit as the density of points becomes infinite.

We can bring out what is meant here by replacing the differential operator as a typical physical equation (e.g., a wave equation) by difference operators, involving neighbouring points in a discrete array. Thus, consider a continuous field function $\psi(x)$, to be replaced by a corresponding set of discrete values, ψ_j, over our array. [In terms of our previous notation $\psi_j = \langle x|\psi\rangle$].
We then replace $\frac{\partial\psi}{\partial x}$ by $\psi_{j+1} - \psi_j$ and $\frac{\partial^2\psi}{\partial x^2}$ by $\psi_{j+1} - 2\psi_j + \psi_{j-1}$.
A typical wave equation would take the form

$$\frac{\partial^2\psi}{\partial t^2} = \alpha(\psi_{j+1} - 2\psi_j + \psi_{j-1})$$

where α is a suitable constant.

Under an inner automorphism, the point ψ, will go into a linear combination $\sum_k C_{jk}\psi_k$ of all the points, ψ_k. The combination, $\psi_{j+1} - \psi_j$, goes into $\sum_k (C_{j+1,k} - C_{j,k})\psi_k$. It is clear that the difference operator has been replaced by another one that is "non-local", i.e. one that connects points from all over the space (as if each point had been "exploded" into the whole space). This is an example of how the form of the equation has been radically altered by an inner automorphism.

To discuss the implications of this fact, we define two neighbourhood operators, N^+ and N^-, which are determined by the equations

$$N^+\psi_j = \psi_{j+1}$$
$$N^-\psi_j = \psi_{j-1} \qquad (40)$$

Under a general automorphism, $\psi_j = \sum_k C_{jk}\phi$, these operators become

$$(N^+)\phi = \sum_{\ell j}\sum C^*_{j+1,k}C_{j,\ell}\phi_\ell \qquad (41)$$

(41)

$$(N^-)\phi = \sum_{\ell j}\sum C^*_{j+1,k}C_{j,\ell}\phi_\ell$$

Now, to turn the argument around, suppose that a certain structure of algebraic equations in a general Heisenberg representation is <u>given</u>. To define the theory physically, we have then to add a set of neighbourhood operators and if these reduce to the canonical form (40), a set of "positions" will be defined uniquely in their proper order. Furthermore, if the physical equations are all "local" in the sense defined above, they will involve only these neighbourhood operators and operators that are diagonal in position. By including in this way a suitable set of neighbourhood operators, the physical significance of the Heisenberg representation will be kept invariant, and the arbitrariness of what is to be meant by the proper "position representation" will be removed.

What we have done is thus to treat the algebraic structure as a pre-space, whose "points" are related to ordinary "points" by the exploding transformation (or equivalently, these points" may be described as being in an implicate order). We have then shown how a unique geometric (or explicate) order can be determined by means of the neighbourhood operators, which are part of the same algebraic structure. So our ordinary physical space (i.e., the space which describes basic entities such as fields and particles) emerges from the pre-space.

It must be emphasized, however, that this is only a beginning of what is needed for a full development of this notion. Firstly, we have to go to the continuous limit, where new physically significant features arise. In particular, one will then require the continuity of all physically significant features, and this is a further restriction, which has no meaning in a discrete array. Then, we will have to relate time as well as space, and incorporate general as well as special relativity, along with modern quantized field theories (including gravitation). We are currently working on these questions, which will be treated in further publications.

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FOOTNOTES

1 We realise that Dirac's q-numbers are more general than the q-numbers considered in this paper. Nevertheless, we hope this limited investigation will help throw some light on the more general problem.

2 Greater detail may be found in Chapter 44 of Misner, Thorn and Wheeler (1973), Patton and Wheeler (1975) and also in Wheeler (1980). In presenting this account we are merely tracing the historic evolution of the ideas without commitment to the actual concepts used in the development.

3 The e_0^1 and e_1^0 will be related to the generators of the translation operators in the position and momentum space in the limit as $n \to \infty$ (see section 4).

4 The symbol $\underline{\underline{\Delta}}$ means it is a definition.

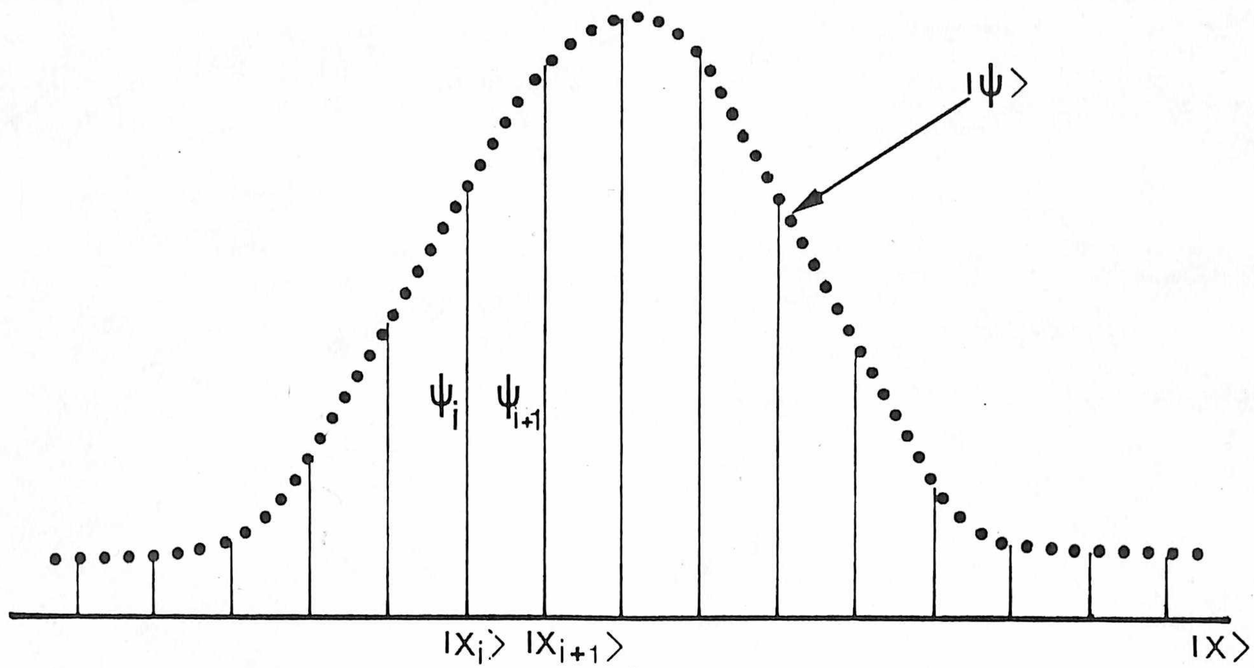

Figure 1.

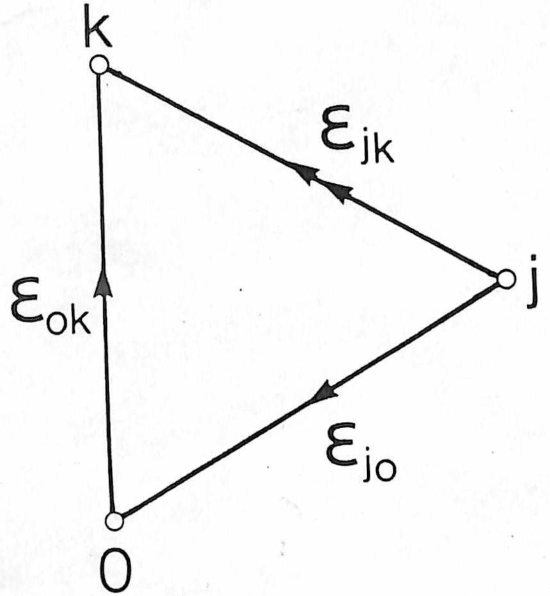
k
ε_{jk}
ε_{ok}
j
ε_{jo}
0

Fig 2a

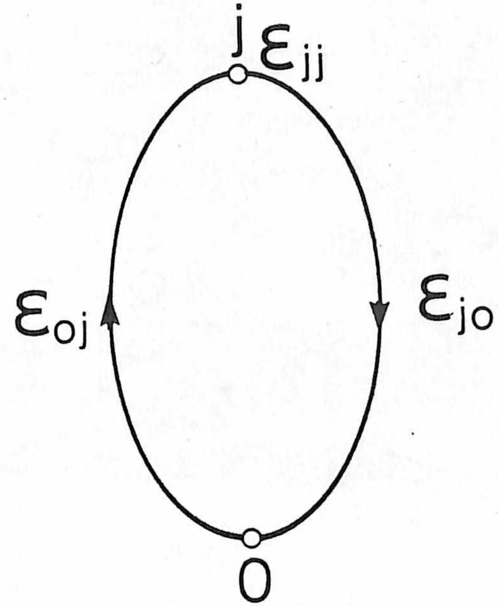
j
ε_{jj}
ε_{oj}
ε_{jo}
0

Fig 2b