ALGEBRAS, QUANTUM THEORY AND PRE-SPACE

by

F.A.M. Frescura and B.J.Hiley

Department of Physics

Birkbeck College

LONDON WC1, UK.

ABSTRACT

The relationship between the algebraic formulation of quantum mechanics, algebraic geometry and pre-space, a notion that arises in Bohm's implicate order, is discussed with particular reference to Schoenberg's contributions in this area. The Heisenberg algebra is examined and it is shown that Dirac's standard Ket can be considered as a primitive idempotent which needs to be introduced into the Heisenberg algebra in order to complete its structure. We discuss the relationship between this idempotent and the vacuum state of the Boson algebra. A fermion algebra based on the method of Cartan is presented. These algebras enable us to generalise the ordinary notions of functions and of Grassmann functions together with their 'differentials' without reference to a continuous space-time manifold. The implications of these algebraic structures to the study of pre-space are discussed.

1. INTRODUCTION.

In a series of papers Mario Schoenberg[1,2] has suggested that there is a deep relationship between quantum theory and geometry. To bring out this relationship, it is necessary to indicate first how the formalism of quantum mechanics and of quantum field theory can be interpreted as special kinds of geometric algebras. Schoenberg has already investigated this matter and has shown that underlying these algebras are extensions of the commutative and the anticommutative Grassmann algebras which have the same structure as the Boson and the Fermion algebras of the annihilation and creation operators. The former have been shown to lead to a symplectic algebra which is the quantum version of the symplectic geometry for the classical phase space, while the latter is related to the Clifford algebras. The symplectic algebra is the algebra of symmetric tensors while the Clifford algebra is the algebra of antisymmetric tensors. A combination of these two kinds of algebra therefore leads to the possibility of a complete algebraic description of geometry in a way that incorporates quantum mechanics in a fundamental way.

While Schoenberg has presented some very suggestive mathematical ideas, he has stated that a fuller development of quantum Physics will probably require a deep revision of our intuitive picture of space and time. In recent years there has been a considerable increase in the investigations into the foundations of quantum mechanics particularly centering around the question of non-locality or non-separability that appears to be basic to quantum theory. This non-locality presents deep problems when contrasted with the usual view that physics is concerned with a local reality in space-time, a view that Einstein repeatedly stressed. However, there is another area of investigation where this basic space-time outlook is called into question, namely the problem of quantising general relativity. Both areas suggest that there is a real need to reconsider our basic notion of space-time.

One of the fundamental problems that lie behind all attempts to develop insights into the problems connected with quantum mechanics stems from the fact that while we know in general how to use its well defined mathematical formalism it is still regarded as a "mysterious, confusing discipline, which none of us really understand" (Gell-Mann[3]). In recent years, David Bohm[4] in particular has given a great deal of thought to the subject and has proposed a very general new outlook which he calls the implicate order. Briefly, to explain the ideas that are involved in this radically new form of description, it is helpful first to contrast the traditional Cartesian approach to physics with the view developed by Bohr[5] in response to quantum mechanics. In the traditional view, it is assumed that there exists a reality in space-time and that this reality is a given thing, all of whose aspects can be viewed or articulated at any given moment. Bohr was the first to point out that quantum mechanics called this traditional outlook into question. To him the "indivisibility of the quantum of action", which was his way of describing the uncertainty principle, implied that not all aspects of a system can be viewed simultaneously. By using one particular piece of apparatus only certain features could be made manifest at the expense of others, while with a different piece of apparatus another complementary aspect could be made manifest in such a way that the original set became non-manifest, that is, the original attributes were no longer well defined. For Bohr, this was an indication that the principle of complementarity, a principle that he had previously known to appear extensively in other intellectual disciplines but which did not appear in classical physics, should be adopted as a universal principle. The Cartesian view was thus limited and had to be replaced by a very different outlook which was to be justified by the principle of complementarity in which complementary views, which at a classical level are contradictory, enter the description of nature in a necessary and essential way.

For many, this implied a limitation to any further intuitive development in physics. But for Bohm, such a view seemed too restrictive and the introduction of the implicate order suggested a way forward without the need for classically contradictory statements. For him the manifestation of outward appearances involved forming explicate orders so that these orders emerged from the implicate order in a well defined way. What then becomes a fundamental form of description is the relation between the implicate and the explicate orders. In this view, space-time itself must be part of an explicate order. When this order is in its implicate form, it is called pre-space (Bohm and Hiley[6]). In this view, the space-time manifold is not a priori given. Rather it is to be abstracted from a deeper pre-space. In this pre-space, the notion of locality is not primary but is a relationship in pre-space which, in an appropriate explicate order, becomes a local order in the explicate space-time. Exactly what links this explicate order with our classical view of space-time is not completely understood yet, but working from the S-martix approach, Chew[7] and Stapp[8] have suggested that the soft photons are to play an essential role. Whatever the precise nature of these processes we see that the appearance of non-locality arises from the implicate pre-space through relationships that cannot be made local. Thus we see the possibility of the non-locality in quantum mechanics arising in a new and subtle way.

Clearly a description of the implicate order cannot be based upon particles or fields acting locally in an a priori given space-time. We prefer to consider a view more akin to that of Whitehead which regards process as the primary form. He used the more neutral term 'activity'. In this view, an object will arise as an abstraction from a quasi stable relatively invariant feature of the basic underlying activity.

But how are we to describe this activity mathematically? Following Bohm's[9] original suggestion, Frescura and Hiley[10] have already given a detailed account of how we are led to the conclusion that activity could best be described in terms of an algebra. Here the binary relation of the addition of two processes gives rise to the possibility of a qualitatively new process. For example, at the mechanical level, the addition of two harmonic processes acting at right angles to each other can give rise to a circular process. The product relation describes the order of succession. With such an algebraic structure, any automorphism, inner or outer, can produce a new description of the same overall properties. In other words, the laws of physics are invariant to these inner automorphisms. It is this feature that Bohm interprets as forming different explicate orders from the same implicate order. Furthermore, quantum mechanics can be regarded as being essentially algebraic and it is this feature that will be exploited in the implicate order. Of course, it is also this algebraic aspect that lies behind the work of Schoenberg[1,2] and which itself opens up new possibilities.

The present formulation of quantum mechanics in the Schroedinger picture with its state vectors in Hilbert space and operators acting on these vectors hides the essential algebraic features. The Heisenberg picture comes closer to a complete algebrisation, but lacks certain features which we will bring out as in a later section. Furthermore, the Schrodinger approach makes a sharp separation between the operator and operand. However, such a separation is not necessary. Indeed exploiting a suggestion of Riesz[11,12], we have already shown how the spinor can be regarded as part of the Clifford algebra, namely a minimal left ideal in the algebra. We have also shown how within the Heisenberg algebra the "state functions" too can be regarded as minimal left ideals in the Heisenberg algebra. Of course, this is exactly what Schoenberg was exploiting in his work on Quantum Theory and Geometry. For us, the importance of his method lies in the fact that there was no longer any distinction between operator and operand. The operand (an element of the minimal left ideal) was merely a distinguished element in the algebra. Furthermore, these ideals can be regarded as quasi-stable and semi-autonomous forms within the algebra which suggests its appropriateness for the description of particle-like or even geometric-like features that arise from the implicate order. The algebra carries the implicate order while the explicate order is contained in the various representations.

2. THE IMPLICATE ORDER PARADIGM.

Here we want to motivate directly the algebraic approach by recalling Bohm's[4] beautiful illustration of the type of process that is typical of the implicate order. Consider two transparent concentric cylinders with a gap between them containing glycerol. If a spot of dye is placed in the glycerol and the inner cylinder rotated a few times, the dye becomes mixed with the glycerol and can no longer be seen. If the inner cylinder is now turned back the same number of turns, the spot of dye will actually reappear in its original form, provided of course that the diffusion is small. The dye can thus be regarded as being enfolded into the glycerol. That is, the spot of dye is implicit in the glycerol. In order to make it visible again, i.e. manifest, we must unfold it, or explicate it, by turning the inner cylinder.

Suppose now that we put a series of spots of dye into the glycerol. The first is put in at x_1, then after turning the cylinder n times the second is put in at x_2 and so on up to r dots, say. Then turning the inner cylinder in the opposite direction will produce a series of dots appearing and then disappearing at the points x_r,

$x_{r-1}, \ldots x_1$. This then gives the impression of an object moving across the field of view.

Symbolically, this kind of enfolding can be described in terms of multiplications within some algebraic structure. Suppose M_1 is an enfoldment which transforms some explicate point e_1 of our space into the implicate structure e_1'. We shall adopt the convention that enfoldments are produced by a left multiplication. Then we can write

$$e_1' = M_1 e_1 \tag{1}$$

or, in matrix form

$$e_1'(i,k) = \Sigma_j \; M_1(i,j) \; e_1(j,k) \tag{2}$$

Notice that when e_1 and e_1' are regarded as elements of the same minimal left ideal of our algebra, then in the continuous limit equation (2) becomes analogous to a Green's function:

$$\psi(x') = \int G(x,x')\psi(x) \; d^3x. \tag{3}$$

Suppose the enfolded point e_1' is now unfolded by right multiplication with M_2 to produce e_2

$$e_2 = e_1' \; M_2.$$

The [illegible] relation of e_1 to e_2 is

$$e_2 = M_1 e_1 M_2. \tag{4}$$

In order to illustrate how quantum mechanics might emerge from such a description of successive enfoldings each followed by an unfolding, let us make two assumptions. First consider the special case in which $M_2=M_1^{-1}$, i.e. the unfolding is simply the reciprocal of the enfolding, so that

$$(5) \qquad e_2 = M_1\, e_1\, M_1^{-1}.$$

This form is already reminiscent of the kind of relation that features so prominently in the transformation theory of quantum mechanics. Now let us suppose that the transformation in question is in fact an infinitesimal transformation. This means that M_1 can be written in the form

$$M_1 = 1 + K$$

If for convenience we assume also that the infinitesimal matrix K depends on a single parameter τ, say, so that the transformed point e_2 traces out some trajectory as τ varies, then K can be written in the form

$$K = iH\,\Delta\tau$$

where the factor i has been introduced only to ensure that H is Hermitian, in accordance with the usual form of the theory. The equation of motion (5) then becomes

$$e_2 = (1+iH\Delta\tau)e_1(1-iH\Delta\tau)$$

$$= e_1+i[H,e_1]\Delta\tau$$

i.e. $(e_2-e_1)/\Delta\tau = i[H,e_1]$.

In the limit as $\Delta\tau \rightarrow 0$ we can write symbolically

$$(6) \qquad de/d\tau = i[H,e]$$

which is, of course, just Heisenberg's equation of motion. A finite transformation is evidently then generated by

$$M_1 = e^{iH\tau}$$

so that

$$e_2 = e^{iH\tau}\ e_1\ e^{-iH\tau}.$$

Schroedinger's equation can be obtained from (6) simply by assuming that the e_i can be considered as a product of an element of a minimal left ideal with an element of a minimal right ideal. This assumption is equivalent in the quantum theory to the requirement that the state in question be a pure state. In terms of the Dirac notation, the minimal left ideals correspond to the Kets while the minimal right ideals correspond to the Bras, as we shall indicate later.

3. THE RELATIONSHIP BETWEEN THE MINIMAL IDEALS AND THE BRAS AND KETS OF DIRAC.

In the previous section we intimated that the minimal left (right) ideals of some suitable dynamical algebra could be identified with the Kets (Bras) of quantum mechanics. Since this identification is not well known we outline a justification for this claim.

In the Dirac formalism sets of objects, $\{|\psi\rangle\langle\phi|\}$ define the set of dynamical operators. Under addition and multiplication, these form a full matric algebra over the complex field. Choosing a complete set of orthonormal eigenkets $\{|a_i\rangle\}$ of some operator A, we can define a set of operators $e_{ij} = |a_i\rangle\langle a_j|$ such that any operator can then be written as

(7) $$B = \Sigma_{ij} B_{ij} e_{ij},$$

where the coefficients B_{ij} are complex numbers. The completeness relation can then be written as

$$1 = \Sigma_i e_{ii}.$$

The e_{ii} here are primitive idempotents and are usually identified with the projection operators. In this way then, the algebra of dynamical operators can be given a complete representation in terms of the Bras and Kets. According to this approach, it is the Bras and Kets which are given the primary status. All else is defined in terms of them.

A different approach however is possible. The algebra itself can be taken as the fundamental entity, with all else defined in terms of it. Indeed, if the concept of the quantum process be taken as basic, then such an approach is not only possible, but also necessary (Bohm[9,13]). This leads us to propose therefore that the algebra itself be elevated to the primary status and be given fundamental relevance. The Bras and Kets must then be regarded as special or secondary features within the algebra, to be abstracted from the algebraic structure through some relevant discriminating characteristic.

But how can this be done? The prominence given in quantum mechanics to the irreducible representations of the dynamical operators supplies the clue. Consider a simple algebra which has exactly one irreducible representation over the complex numbers. This irreducible representation must necessarily appear as an irreducible component within the regular representation of the algebra. But the representation of any matric algebra A is irreducible if and only if it is obtained from the regular representation of A on a minimal invariant subspace I of A as representation space. We thus expect these minimal invariant subspaces to be important. Now, an invariant subspace I of A is minimal if and only if it is generated by some primitive idempotent element in A. These spaces in fact can be generated in two ways from given primitive idempotent elements e, namely by multiplication from the left or by multiplication from the right. Spaces of the type $I_L=Ae$, generated by left multiplication, are called minimal left ideals. Those of the type $I_R=eA$, generated by right multiplication, are called minimal right ideals. The primitivity requirement on e then gives

$$I_R I_L = eAAe = Ze$$

where Z is the centre of A. Thus, for given e, I_R can be regarded as the vector space dual to I_L. It is evident now from the way that representations are constructed on the spaces I_R and I_L that these play a role in the theory of algebras entirely analogous to that of the Bra and Ket spaces in quantum theory. We can therefore regard the Bra space and the Ket space respectively as minimum right or left ideals in some suitable algebra.

This structure is more easily appreciated when expressed in terms of a particular representation. Suppose ψ is an element of the minimal left ideal generated by the primitive idempotent e. Then since ψ is in *Ae*, there must exist some element B in *A* such that

(8) $$\psi = Be.$$

Now introduce a matric basis e_{ij} of the simple algebra *A*. Since the idempotent e is primitive, we can always choose the basis e_{ij} in such a way that $e=e_{kk}$ (no sum on k) for some fixed k. So, from (7) and (8), we have

(9) $$\psi = (\Sigma_{ij}\ B_{ij}e_{ij})e_{kk} = \Sigma_i\ B_{ik}e_{ik} = \Sigma_i\ \psi_i e_{ik}$$

which is recognised immediately as having the form of a quantum state vector. In the case when i and j range over a continuously infinite index set, the coefficients ψ_i become the familiar wave functions $\psi(x)$. It is also immediately evident that every Ket space in the algebra constructed in this way will automatically contain a complete set of eigenvectors for each observable B. Further, this construction automatically guarantees the completeness relation, which is equivalent to the requirement

$$1 = \Sigma_k e_{kk}.$$

(This observation in fact provides an explanation of the completeness requirement in quantum theory.) In this way, therefore, all the vector space properties associated with the Ket space are completely reproduced in their algebraic counterpart.

Similar observations apply to the identification of the Bra space with a minimal right ideal of the algebra. Choosing $I_R=eA$, where e is the same idempotent that is used to generate I_L, ensure that I_L and I_R are dual vector spaces.

Quantum theory deals principally with two kinds of dynamical variables: those related to 'extrinsic' properties such as position, momentum, orbital angular momentum, etc., and those representing internal degrees of freedom, such as the intrinsic spin, isospin, and the various internal quantum numbers of the elementary particles. The method we have outlined is easily implemented for this last kind of variable. A detailed treatment of some particular cases involving the Pauli and the Dirac Clifford algebras have been given in Frescura and Hiley[10]. However, technical difficulties are encountered with variables of the former type. These difficulties are related to the infinite dimensionality of the dynamical algebras involved and to the closely related problem of defining their 'bondaries', or 'limits'. We now indicate briefly in a particular case how some of these difficulties may be overcome.

Consider the polynomial Heisenberg algebra $L_q(1)$, which is the (associative) algebra generated by the Heisenberg position and momentum operators, Q and P, satisfying the usual commutation relations. For convenience, we shall use the operator $D=2\pi iP/h$ in place of P. $L_q(1)$ may then be regarded as generated from the set {1,Q,D} subject to the relation

(10) $$[D,Q] = 1$$

It is not difficult to show that this algebra contains no non trivial primitive idempotents. It thus contains no non trivial minimal left or right ideals. If we are to carry out the program outlined above therefore, the algebra $L_q(1)$ will have to be extended to incorporate suitable idempotent operators. How this can be done may be gleaned from an examination of the Dirac version of the quantum theory. Of particular relevance is the concept of the 'standard Ket'.

Suppose $\{A_j\}$ is some given complete set of dynamical variables.

Let the corresponding eigenket basis spanning the Ket space be $\{|a_{ij}\rangle\}$. We can then associate a complex function with every Ket $|P\rangle$ in the usual manner

$$\psi(a_{ij}) = \langle a_{ij}|P\rangle$$

This means that in the basis $\{|a_{ij}\rangle\}$, each function ψ uniquely defines $|P\rangle$ and vice versa, so that we may write

$$(11) \qquad |P\rangle = |\psi,\{A_r\}\rangle.$$

Suppose now that $f(A_r)$ is any function of the observables $\{A_r\}$. Then $f(A_r)$ is itself an operator in its own right and we have

$$\langle a_{ij}|f(A_j)|\psi,\{A_r\}\rangle = f(a_{ij})\langle a_{ij}|\psi,\{A_r\}\rangle$$

$$= f(a_{ij})\psi(a_{ij}) = f.\psi(a_{ij})$$

or $$f(A_j)|\psi,\{A_r\}\rangle = |(f.\psi),\{A_r\}\rangle,$$

for all $|\psi,\{A_r\}\rangle$. In particular, for $\psi=1$ we have

$$f(A_j)|\{A_r\}\rangle = |f,\{A_j\}\rangle$$

which we can write as

$$(12) \qquad f(A_j)|\rangle = |f\rangle.$$

The Ket $|\rangle$, which Dirac writes as $\rangle$, is called the standard Ket for the complete set $\{A_j\}$. It is clear that the standard Ket corresponding to different complete sets will, in general, be different so that there exists a multiplicity of standard Kets. Thus, in the Dirac formalism, every Ket can be expressed uniquely as a product

$$|\psi\rangle = \psi(A_j)\,|\rangle,$$

where ψ is some well defined complex function. Conversely also, each complex function ψ uniquely defines a Ket $\psi(A_r)\,|\rangle$. Similarly, every Bra can be written as a product $\langle|\phi(A_r)$, where $\langle|$ is the standard Bra for the complete set $\{A_r\}$ of dynamical operators for the system. Hermitian conjugation then gives

$$^{+}:\psi(A) \longrightarrow \psi^{*}(A^{+})$$

$$:|\rangle \longrightarrow \langle|$$

and $$:\langle| \longrightarrow |\rangle$$

so that for a Hermitian operator A

$$^{+}:\psi(A)\,|\rangle \longrightarrow (\psi(A)\,|\rangle)^{+} = \langle|\psi^{*}(A).$$

It is evident from the above that the existence of the Ket space in the Dirac theory is guaranteed by the existence of the single element $|\rangle$. For, given the element $|\rangle$, every Ket in the Ket space S_K is generated from $|\rangle$ by some element $\psi(A)$ of $\mathcal{D}_A$, where $\mathcal{D}_A$ is the non denumerably infinite quasi algebra of the dynamical variables proposed by Dirac. Thus we must have

(13) $D_A|\rangle = S_k$.

In the Dirac theory then, the possibility of constructing a Ket space rests crucially on the existence of the fundamental Ket $|\rangle$.

To see how this helps in the case of the Heisenberg algebra, we note that the action of the derivative d on $|\rangle$ follows from

$$d|\psi\rangle = |d\psi(q)/dq\rangle = \psi'(q)|\rangle$$

so that

$$d|\rangle = |(d/dq)1\rangle = 0|\rangle = 0$$

i.e.

$$d|\rangle = 0.$$

In the dual Bra space we find

$$\langle|d = 0.$$

Introduce now an element e in D_q which projects every element in the Ket space S_k into the standard Ket so that

$$e|\psi\rangle = \psi(0)|\rangle$$

for all $|\psi\rangle$ in S_k. It is not difficult to verify that e is idempotent. Consider now the effect of the product de on each ket $|\psi\rangle$ in S_k:

$$(de)|\psi\rangle = d(e|\psi\rangle) = d(\psi(0)|\rangle) = (0)\ d|\rangle = 0$$

or

$$de = 0.$$

Also

$$(eq)|\psi\rangle = e(q\psi(q)|\rangle) = e|\phi\rangle = \phi(0)|\rangle$$

where $\phi(q) = q\psi(q)$, so that $\phi(0) = 0$. Hence

$$de = 0.$$

The products qe and ed are well defined algebraic elements in D_q which are neither zero nor scalars. However, rather than expanding them in terms of some basis of D_q, we shall retain this product notation for them as it relates them directly to the quasi algebra generated by the set $\{1,d,q,e\}$. It should be noted here that the element e cannot be expressed as a polynomial function of the operators d and q and so is not an element of the sub quasi-algebra generated by $\{1,d,q\}$.

Returning to the polynomial Heisenberg algebra $L_q(1)$, it is now evident how $L_q(1)$ must be extended if it is to contain Ket and Bra spaces within its own structure: we must include among the generators of the algebra an idempotent element analogous to the element e of the above discussion. We thus arrive at the quasi-algebra $B(1)$, a generalisation of the Heisenberg algebra, which is the algebra generated over the complex field from the set $\{1,D,Q,E\}$ with

(14) $$[D,Q] = 1,\ E^2 = E,\ DE = 0,\ EQ = 0.$$

This is in fact one of the algebras introduced by Schoenberg[1]. Details can be found in Schoenberg[1,2] and in Frescura and Hiley[14]. We here list only some of its more important features. First, define a set of elements E_{mn} by

(15) $$E_{mn} = (m!n!)^{-1/2}\, Q^m E D^n.$$

We follow the usual convention of defining $Q^0=1=D^0$, so that $E_{00}=E$. It can then be shown that

(16) $$E_{ij}E_{mn} = \delta_{jm}\, E_{in}.$$

It follows from this relation that the E_{mn} are linearly independent. It can further be shown that

(17) $$1 = \Sigma_r\, E_{rr}$$

and

(18) $$Q = \Sigma_n\, (n+1)^{1/2}E_{n+1,n}\ ,\qquad D = \Sigma_n\, (n+1)^{1/2}E_{n,n+1}.$$

All the generators of $B(1)$ can thus be written in terms of the E_{mn}. The elements E_{mn} thus constitute a basis for $B(1)$. In fact, relation (16) is the defining relation for a matric basis in a full matric algebra. We have shown therefore that $B(1)$ is a full matric algebra.

It also follows from (16) that $E=E_{oo}$ is a primitive idempotent and so can be used to generate the Bra and Ket spaces for $B(1)$. In fact, any element of the form

(19) $|\rangle = \Sigma_n a_n E_{on}$.

can be taken as the fundamental Ket. The left ideal it generates then consists of elements of the form

$$|\psi\rangle = \Sigma_{m,p} \lambda_{mo} a_p E_{mp} = \psi(Q)|\rangle$$

where $\psi(Q)$ is the polynomial $\Sigma_m (1/m!) \lambda_{mo} Q^m$.

The algebra $B(1)$ allows only for a single degree of freedom. Generalisation to a larger number is straightforward. For every independent degree of freedom i introduce a set $\{1_i, D_i, Q_i, E_i\}$ of dynamical operators satisfying

$$E_i^2 = E_i, \quad D_i E_i = 0 = E_i Q_i.$$

Operators corresponding to independent degrees of freedom all commute with each other. The algebra $B(n)$ generated in this way is thus the direct product of the subalgebras $B_i(1)$ corresponding to the independent degrees of freedom. The element $E=\prod_{i=1}^{n} E_i$ is then idempotent and primitive and can be used to generate the Bra and Ket spaces. Any element of the form

$$|\rangle = \prod_{i=1}^{n} |\rangle_i$$

where $|\rangle_i$ is a fundamental Ket in $B_i(1)$, can be taken as the fundametal Ket of $B(n)$.

4. THE QUASI ALGEBRA $B(1)$ AS A THEORY OF BOSONS.

From (15) it is evident that Q and D may be represented as

$$(19)\qquad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} ; \qquad D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} .$$

This is immediately recognised as the representation of the creation and annihilation operators a^+ and a of the second quantization formalism for an harmonic oscillator with a single degree of freedom. This is not surprising, of course, since the commutator

$$[a, a^+] = 1$$

is formally identical to that introduced in (10). The quasi algebra of second quantization is thus formally identical to the quasi algebra *B*(1). Since the harmonic oscillator provides a theory of bosons, the quasi-algebra *B(1)* can be said also to be a theory of bosons.

If we write

(20) $$a^{+} = 2^{1/2}\ (Q-D)$$

$$a = 2^{1/2}\ (Q+D)$$

and introduce the idempotent operator V to play a role analogous to E then it is easy to show that the vacuum state takes the form

(21) $$|0\rangle = \Sigma_n\ \nu_n V_{on}$$

where

(22) $$V^{mn} = (m!n!)^{-1/2}\ (a^{+})^{m}\ V\ a^{n}$$

and

(23) $$a^{+} = \Sigma_n\ (n+1)^{1/2}\ V_{n+1,n}$$

$$a = \Sigma_n\ (n+1)^{1/2}\ V_{n,n+1}$$

Hence we obtain a representation of a and a^{+} identical to (19). From the property

(24) $$V|0\rangle = |0\rangle, \qquad V|n\rangle = 0$$

where

$$|n\rangle = (n!)^{-1/2}\ (a^{+})^{n}\ |0\rangle,$$

the operator V is recognised as the vacuum projection operator.

In the quantum theory, the fundamental Bra $\langle 0|$ is related to the fundamental Ket $|0\rangle$ through

$$\langle 0| = |0\rangle^{+}.$$

Thus put

$$(25) \quad \langle 0| = |0\rangle^{+} = (\Sigma_r \ \alpha_r \ V \ a^r)^{+} = \Sigma_r \ \alpha_r^{*} \ (a^{+})^r \ V^{+}$$

We shall further assume that V is itself Hermitian, that is

$$(26) \qquad V^{+} = V.$$

Then

$$(27) \qquad |\psi\rangle = \psi(a^{+})|0\rangle$$

$$= \Sigma_r \ (r!)^{-1/2} \ \psi_r \ (a^{+})^r \ |0\rangle$$

and from (25) and (26) we have

$$\langle\psi| = \langle 0| \ \Sigma_r \ (r!)^{-1/2} \ \psi_r^{*} \ a^r = \langle 0|\psi^{*}(a).$$

The scalar product is then

$$(28) \qquad \langle\phi||\psi\rangle = \langle 0|\phi^{*}(a)\psi(a^{+})|0\rangle$$

$$= \langle\phi|\psi\rangle \ \langle 0||0\rangle.$$

As in the case of the algebraic theory of spinors the scalar product of a Bra with a Ket appears as the coefficient of the algebraic element $\langle 0||0\rangle$.

5. THE RELATION OF E TO V AND OF THE STANDARD KET $|\rangle$ TO THE VACUUM STATE $|0\rangle$.

The idempotent elements E and V can be related in the following way. Put

$$(28) \qquad V = \Sigma_{ij}\ \alpha_{ij}\ E_{ij} = (\Sigma_i\ \lambda_i Q^i)\ E\ (\Sigma_j\ \pi_j D^j)$$

$$= \Lambda\ E\ \Pi$$

for some Λ and Π. But aV=0, and so

$$0 = (D+Q)\ \Sigma_{ij}\ \lambda_i \pi_j\ Q^i E D^j$$

$$= \Sigma_{ij}\ \lambda_i \pi_j\ (j!)^{1/2}\ [i\{(i-1)!\}^{1/2}\ \delta^i_{n+1}\delta^j_m + \{(i+1)!\}^{1/2}\ \delta^i_{n-1}\delta^j_m]$$

for each m and n. In particular, putting n=0 we find

$$\lambda_1 = 0$$

and for each n>0, since not all the π_m are zero, we have

$$\lambda_{n+1} = -\ \lambda_{n-1}/(n+1)$$

from which it follows that

$$\lambda_{2n} = (-1/2)^n\ \lambda_0/n! \qquad \text{and} \qquad \lambda_{2n+1} = 0.$$

Hence

$$\Lambda = \Sigma_n\ \lambda_n Q^n = \Sigma_n\ \lambda_0\ (-1)^n\ (Q^2/2)^n\ /n!$$

so that

$$\Lambda = \lambda_0\ \exp(-Q^2/2).$$

Similarly, we find

$$\Pi = \pi_o \exp(D^2/2).$$

Λ and Π are clearly invertible , so that we have

(29) $$V = \lambda_o\pi_o \exp(-Q^2/2)\ E\ \exp(D^2/2)$$

and

(30) $$E = (\lambda_o\pi_o)^{-1} \exp(Q^2/2)\ V\ \exp(-D^2/2).$$

We now use (30) to define E^+. Assuming that $V^+=V$, then it can be shown that

$$E^+ = (\lambda_o\pi_o/\lambda_o{}^*\pi_o{}^*)\ \exp(-D^2/2)\exp(-Q^2/2)\ E\ \exp(D^2/2)\exp(Q^2/2)$$

and

$$E^+D = 0 = QE^+.$$

This then enables us to define the Bra space in $B(1)$ in terms of the standard Ket via

(31) $$\langle| = |\rangle^+ =$$

$$\Sigma_m\ \alpha_m{}^*(-1)^m(m!)^{-1/2}D^m\ (\lambda_o\pi_o/\lambda_o{}^*\pi_o{}^*)\exp(-D^2/2)\exp(-Q^2/2)E\exp(D^2/2)\exp(Q^2/2).$$

The Bra corresponding to the Ket

$$|\psi\rangle = \psi(Q)|\rangle$$

will be

$$\langle\psi| = \langle|\ \psi^*(Q)$$

so that

(32) $\langle\psi||\phi\rangle = \langle\psi|\phi\rangle\ \langle|\,|\rangle$

which is analogous to equation (28).

The elements $\Lambda = \lambda_o \exp(-Q^2/2)$ and $\Pi = \pi_o \exp(D^2/2)$ which relate E and V in (28), (29) and (30), strictly speaking, are not elements of $\mathcal{B}(1)$. They belong to a larger algebra $\mathcal{B}^c(1)$, which is the topological closure of $\mathcal{B}(1)$. This means that $\mathcal{B}(1)$, though isomorphic to the algebra $\mathcal{B}_*(1)$ generated from $\{1,a^+,a,V\}$, is not identical to it. In fact, from (20), it is evident that $\mathcal{B}(1)$ and $\mathcal{B}_*(1)$ coincide only on the subalgebra generated from $\{1,Q,D\}$ or $\{1,a^+,a\}$,

i.e. $$\mathcal{B}(1) \cap \mathcal{B}_*(1) = \mathcal{L}_q(1).$$

We are now in a position to related the Ket spaces $\mathcal{B}(1)|\rangle$ and $\mathcal{B}_*(1)|0\rangle$. Consider the relation between $|\rangle$ and $|0\rangle$. Equation (18) and equation (21) can be written in the form

$$|\rangle = E\ \Sigma_r\ (r!)^{-1/2}\ \lambda_r\ D^r = E\ A$$

$$|0\rangle = V\ \Sigma_r\ (r!)^{-1/2}\ \beta_r\ a^r = V\ B$$

where A and B are elements of $\mathcal{B}(1)$ and $\mathcal{B}_*(1)$ respectively. But from equation (29)

$$|0\rangle = VB = \lambda_o\pi_o \exp(-Q^2/2)\ E \exp(D^2/2)\ B.$$

Now, as the right ideal $E\mathcal{B}(1)$ is minimal, so also is the right ideal $\exp(-Q^2/2)E\mathcal{B}(1)$. Hence the elements

$$\lambda_0\pi_0 \exp(-Q^2/2)\ E \exp(D^2/2)\ B \quad \text{and} \quad \pi^{-1/4}\exp(-Q^2/2)\ EA$$

are each elements of some minimal right ideal and for some element T of $\mathcal{B}^c(1)$ we can thus write

$$\lambda_0\pi_0 \exp(-Q^2/2)\ E \exp(D^2/2)\ B = \pi^{-1/4}\exp(-Q^2/2)\ EAT$$

that is

(33) $\quad |0\rangle = \pi^{-1/4}\exp(-Q^2/2)\ |\rangle\ T.$

It can be shown similarly

(34) $\quad |\rangle = \pi^{1/4}\exp(Q^2/2)\ |0\rangle\ S$

for some S in $\mathcal{B}^c(1)$.

The Ket space $\mathcal{B}(1)|\rangle$ and the Fock space $\mathcal{B}_*(1)|0\rangle$ are distinct spaces. This is evident from the fact that their generating elements $|\rangle$ and $|0\rangle$ are not elements of the common portion $\mathcal{B}(1)\cap\mathcal{B}_*(1)$ of the algebras $\mathcal{B}(1)$ and $\mathcal{B}_*(1)$, and so they generate distinct though isomorphic structures. Even if we embed $\mathcal{B}(1)$ and $\mathcal{B}_*(1)$ in their common closure $\mathcal{B}^c(1)$, the spaces $\mathcal{B}(1)|\rangle$ and $\mathcal{B}_*(1)|0\rangle$ remain distinct.

The Ket space generated by $|\rangle$ can be identified with the Fock space generated from $|0\rangle$ only if we widen our considerations to include the larger algebra $B^c(1)$. In this larger system we have

$$B^c(1)|0\rangle = B^c(1)|\rangle T.$$

Here also, the Ket space $B^c(1)|\rangle$ and the Fock space $B^c(1)|0\rangle$ are in general distinct though isomorphic ideals of $B^c(1)$, but with this difference, that the elements Λ and Π which relate the two fundamental Kets are now contained within our algebra, $B^c(1)$. It is always possible therefore, in this case, by a special choice of standard Ket to make the space $B^c(1)|\rangle$ coincide with the Fock space $B^c(1)|0\rangle$. We need only replace $|\rangle$ above by $|\rangle T^{-1}$. Then

$$B^c(1)|\rangle = B^c(1)|0\rangle.$$

Alternatively, given some $|\rangle$, it is always possible to find some $|0\rangle$ in $B^c(1)$ so as to ensure the identity of the ideals they generate.

Returning to the algebras $B(1)$ and $B_*(1)$, the left ideals $B(1)|\rangle$ and $B(1)|0\rangle$ can be related to the state spaces of the usual quantum mechanics as follows. It is well known that the eigenstates of the number operator a^+a can be written in the form

$$|n\rangle = (n!)^{-1/2}\,(a^+)^n\,|0\rangle$$

which we regard as an element of the left ideal $B_*(1)|0\rangle$ of $B_*(1)$. Now consider the left ideal $B(1)|\rangle$ of $B(1)$ with its basis vectors $Q^n|\rangle$, $n=0,1,2...$. Since

$$a^+a\ Q^n|\rangle \neq \lambda\ Q^n|\rangle$$

the representation matrix for a^+a in this basis will not assume diagonal form. But the vectors

$$|n\rangle_* = (n!)^{-1/2}\ (a^+)^n\ |0\rangle S$$

in $B(1)|\rangle$ are in one to one correspondence with the basis $|n\rangle$ of the isomorphic vector space $B_*(1)|0\rangle$. Thus

$$|n\rangle_* = (n!)^{-1/2}\ (a^+)^n\ |0\rangle\ S$$

$$= (n!)^{-1/2}(\ 2)^n\ (Q-D)^n\ \pi^{-1/4}\ \exp(-Q^2/2)\ |\rangle$$

where we have used equation (34). But

$$\exp(-Q^2/2)\ (D-Q) = D\exp(-Q^2/2)$$

so that

$$|n\rangle_* = (n!2^n)^{-1/2}\ \pi^{-1/4}\ (-1)^n\ \exp(Q^2/2)\ D^n\ \exp(-Q^2)\ |\rangle$$

$$= (n!2^n)^{-1/2}\ \pi^{-1/4}\ \exp(-Q^2/2)\ H_n(Q)\ |\rangle$$

where $H_n(Q)|\rangle=(-1)^n\ \exp(Q^2)\ D^n\ \exp(-Q^2)|\rangle$ is the algebraic analogue of the nth Hermite polynomial $H_n(x)$ introduced by Schoenberg[1]. Denoting the normalized nth Hermite polynomial by $h_n(x)$, we then have

$$|n\rangle_* = \exp(-Q^2/2)\; h_n(Q)\; |\rangle$$

so that the general element of the ideal $B(1)|\rangle$ takes the form

$$|\psi\rangle = \exp(-Q^2/2)\; \Sigma_r\; \psi_r\; h_r(Q)\; |\rangle$$

which is already well known from the quantum account of the one dimensional oscillator.

6. THE QUASI ALGEBRA *B*(1) AS A THEORY OF FUNCTIONS AND DIFFERENTIATION.

In the purely algebraic approach, no restriction of square integrability need be imposed on the elements of the algebraic Ket space. This means that the probability interpretation no longer has universal application, though it nevertheless remains valid for a wide variety of problems. The essentially new feature in this approach is that the 'state function' becomes a self referent process for which the probability interpretation is not essential. Dirac[15] has shown that this new feature does not change the results of QED but avoids some of the difficulties. However he points out also that this generalisation lacks a physical interpretation. The approach outlined here opens up the possibility of providing an understanding of this generalisation by developing a larger framework within which new questions can be raised.

For the moment, we content ourselves with pointing out some important features of this generalisation. Consider first the algebraic approach in relation to the theory of functions. Suppose that W is the subalgebra of $B(1)$ generated from $\{1,Q\}$ and that K is some Ket space of $B(1)$. Suppose also that $B^c(1)$ is the closure of $B(1)$, W' the closure of W, and K' the closure of K. We have shown that each function f uniquely defines the Ket vector $|f\rangle$ in $B^c(1)$ through the relation

(35) $$|f\rangle = f(Q)|\rangle.$$

The correspondence between f and the Ket vectors $|f\rangle$ is injective since $|f\rangle = |g\rangle$ if and only if $f=g$. Also $|f+g\rangle = |f\rangle+|g\rangle$, so that the correspondence is seen to be an injective vector space isomorphism from the space of functions f into the Ket space K' of $B^c(1)$. The fact that this isomorphism is injective and not surjective on K' means that K' contains elements which have no counterpart in classical analysis. A theory of functions based on the space K' can thus be considered to be a generalisation of the classical analysis.

Consider the product, in the algebra, of two Ket vectors $|f\rangle$ and $|g\rangle$. Then

$$|f\rangle|g\rangle = g(0)|f\rangle \neq |fg\rangle$$

so that a satisfactory account of all the properties associated classically with functions is not possible in terms of the space K' alone. We note however that there is a one to one correspondence between the elements $|f\rangle$ of K' and the elements $f(q)$ of $B^c(1)$, this correspondence being uniquely defined for given fundamental Ket $|\rangle$ by relation (35). It is easily demonstrated that the set $\{f(q)\}$ of all such elements forms a sub quasi-algebra W' of $B^c(1)$. It follows then that for given $|\rangle$ there exists an uniquely defined injective isomorphism from the space of classical functions f into the sub quasi-algebra W' of $B^c(1)$ and that this isomorphism is algebraic in the sense that if f and g are functions, then

$$f(q) \dot{+} g(q) = (f+g)(q) \quad \text{and} \quad f(q)g(q) = fg(q).$$

Through this injective isomorphism, the quasi-algebras W' and $B^c(1)$ provide a basis for an account of the algebraic properties of the functions of classical analysis.

It is clear that through the above isomorphism the quasi-algebra W' will contain the space of classical functions as a proper subset. W' thus contains elements which have no classical counterpart. Hence W' (and also $B^c(1)$) can be said to be a generalisation of the classical theory of functions. In particular, it is well known that W′ contains elements corresponding to the Dirac delta function, and so provides also an account of the Schwartz distributions.

Turning now to differentiation, suppose that $f(Q)$ is in the subalgebra W of $B(1)$. Then

$$f(Q) = \Sigma_n \; f_n Q^n$$

and

$$D\, f(Q) = \Sigma_n \; f_n (DQ^n)$$

$$= \Sigma_n \; f_n (Q^n D + nQn-1)$$

$$= f(Q)\; D + f'(Q)$$

where $f'(Q) = \Sigma_n \; f_n n Q^{n-1}$ is in W and corresponds to the differentiated polynomial function df/dx whenever the element $f(Q)$ of W corresponds to some classical polynomial function $f(x)$. Thus if $f(Q)$ is some element of the closure W′ of W, then

$$D\, f(Q) = f(Q)\; D + f'(Q)$$

where f′ corresponds to the differentiated function df/dx of f. Also, for each element |f⟩ of a ket space K of $B(1)$ we have

$$D\; |f\rangle = Df(Q)|\rangle = f(Q)\; D|\rangle + f'(Q)|\rangle = |f'\rangle.$$

The effect of D on the closure K′ of K can be similarly defined. It is evident that the maps $D:K \rightarrow K$ and $D:W \rightarrow W$ defined respectively according to the reqirements

$$D:|f\rangle \longrightarrow D|f\rangle \quad \text{and} \quad D:f(Q) \longrightarrow Df(Q)$$

induce among those elements of K and of W which correspond to the classical polynomial functions (regarded as operators and as operands respectively) an operation analogous to the Newton-Leibnitz operation of differentiation, D playing the role of the operator d/dx. We can thus interpret D as the algebraic generalisation of Classical differentiation. This allows the immediate extension of the concepts of the differential calculus to a larger class of functions than is permitted by the classical theory. In particular, it supplies immediately a theory of differentiation for the distributions. In this sense, $\mathcal{B}(1)$ and $\mathcal{B}^c(1)$ can be said to constitute a theory of differentiation.

As theories of differentiation, neither $\mathcal{B}^c(1)$ nor $\mathcal{B}(1)$ are equivalent to the classical theory, even in those regions where they may be considered to overlap. They are conceptually quite distinct.

7. THE FERMION ALGEBRA

We have shown how the Boson algebra *B*(n) leads to an algebraic generalization of classical functions and of their differentiation. Since this algebra can be realised through the Boson creation and annihilation operators, it is natural to ask whether there exists an analogous structure underlying the Fermionic creation and annihilation operators. As this would involve the use of anticommutators in place of the commutators defining *B*(n), we anticipate that we shall have to introduce Grassmann variables, and functions of these variables, in the place of the ordinary variables and functions.

Introduce a set of Grassman variables θ_i, i=1,..n, such that

(36) $$\{\theta_i, \theta_j\} = 0.$$

It is well known that the algebra generated by the θ_i in this way is just the algebra of exterior forms in n variables. Its dimension is 2^n. We can form polynomial functions of these variables by taking linear combinations of the generated monomial exterior forms. Because of the finite dimension of this algebra, every polynomial function of them can be expressed in terms of a finite sum of basic monomial elements. No problems of the kind encountered in the case of *B*(n) when defining functions of the Q_i are thus encountered here. The general function of the variables θ_i is therefore just the general element of the space of exterior forms.

Following Cartan[16], we now define a new kind of differential operator Δ^i to operate on these exterior forms. We shall require these operators to be the analogue of the ordinary differential operators D_i which were used in the construction of the algebra $B(n)$. They are therefore to operate on exterior monomial forms of order r to produce an exterior monomial form of order r-1. We shall define the action of the Δ^i first on the exterior two forms. Suppose

$$F = \Sigma_{ij}\ a^{ij}\ \theta_i\ \theta_j$$

where $a^{ij}=-a^{ji}$ is an antisymmetric tensor for the n dimensional space. The derivative of this form will be defined to be

$$(37) \qquad \Delta^i F = \frac{\partial}{\partial \theta^i} F = \Sigma_{ij}\ a^{ij}\ \theta_j$$

so that

$$\Delta^j(\Delta^i F) = a^{ij} = -a^{ji} = -\Delta^i(\Delta^j F).$$

On the space of exterior two forms therefore we have

$$(38) \qquad \{\Delta^i, \Delta^j\} = 0.$$

The rules for differentiating these monomials of order two are:

1. If the variable θ_i does not occur in the monomial, then differentiation with respect to θ_i gives 0, and

2. If the variable θ_i occurs in the monomial, bring it into the first position using anticommutation relations (36) and then differentiate as in (37).

These rules can now be extended to forms of order p: the derivative of

$$\theta^{i_1}\theta^{i_2}\ldots\theta^{i_p}$$

with respect to θ_i is zero if $i \neq i_r$ for some $r=1,..p$; it is

$$\theta^{i_2}\theta^{i_3}\ldots\theta^{i_p}$$

if $i=i_1$. If $i=i_r$ for some $r\neq 1$, then bring the factor θ_{i_r} into the first position by the anticommutation relations (36), and then proceed as above. Scalars and one forms are to be differentiated in the ordinary way. It is not difficult to show that these rules can be applied consistently to exterior forms of all orders and that the anticommutation relations (38) are then valid on the entire space of exterior forms.

How do the operators Δ^i behave relative to the θ_i? Suppose F is a homogeneous exterior form of order p. Form the exterior product $\theta_i F$ of θ_i with F. Then it can be shown that

$$\Delta^j(\theta_i F) = \delta^j_i F - \theta_i(\Delta^j F).$$

This result is independent of p and so is true for forms F of all orders p. It is true therefore for the general exterior form. On the whole space of exterior forms then, we have

(39) $$\{\Delta^i, \theta_j\} = \delta^i_j$$

We have thus arrived at the algebra that we are seeking. Take as generators the 2n variables Δ^i, θ_i satisfying

(40) $$\{\theta_i, \theta_j\} = 0, \quad \{\Delta^i, \Delta^j\} = 0, \quad \{\Delta^i, \theta_j\} = \delta^i_j \quad ,$$

and generate from them the algebra $F(n)$. The anticommutation relations (40) are directly the analog of the commutation relations

$$[Q_i, Q_j] = 0, \quad [D^i, D^j] = 0, \quad [D^i, Q_j] = \delta^i_j$$

which define the algebra $L_q(n)$ in the case of Bosons. Our algebra $F(n)$ is therefore entirely analogous to the Boson algebra $B(n)$. In fact, this is just the algebra introduced by Schoenberg[1] from different considerations to complement the Heisenberg algebra of quantum theory. The derivation that we have given of it here shows that it can be regarded as a generalisation of the algebra of differential exterior forms introduced into the classical differential geometry by Cartan. We regard *F*(n) therefore as the complete algebra of differential forms. In fact it has very deep connections with all aspects of the theory of differential forms, including the dual forms, homology, harmonic integrals, and also spinors. We hope to show in detail these connections in a later publication. Here we shall only outline some important features.

This algebra also has very deep connections with the physics of grand unification and of quantum gravity. This is just the algebra introduced in connection with supersymmetry recently (see Rocek[17]). It is a great credit to Schoenberg that he was able to foresee even in the 1950's that this structure would necessarily have an important part to play in the foundations of physics and especially in the theory of space.

The theory of *F*(n) can be developed in complete analogy to the theory of *B*(n) though, of course, with certain important differences. It is well known from the work of Schoenberg[1,2] and of Cartan[18] that *F*(n) is in fact a Clifford algebra for the 2n dimensional phase space spanned by the operators θ_i, Δ^i. It can therefore be represented faithfully by a full set of matrices of order 2^n. This strúcture contains its own idempotents so that, unlike the algebra *B*(n) where the idempotents E_i or V_i had to be imported from the outside, there is no need to enlarge the algebra further. The fermionic Ket spaces are already contained within it. In fact, it is not difficult to show that, for each i, the element

$$P_i{}^i = \Delta^i \theta_i$$

(no sum on the i) is idempotent. Each $P_r{}^r$ is primitive in the subalgebra generated from the two operators θ_r, Δ^r correponding to the rth fermionic degree of freedom. It can be shown that the element

$$P = \prod_{i=1}^{n} P_i{}^i$$

is primitive in *F*(n). It is clear from its form that P plays a distinguished role in the theory, and so we shall refer to it as the principal primitive idempotent for the the operators θ_i, Δ^i. P satisfies the relations

$$P^2 = P, \quad \Delta^i P = 0, \quad P\,\theta_i = 0.$$

Its role in *F*(n) is clearly analogous to E or V in *B*(n) and can be thought of as the fermionic vacuum state.

The connection of this algebra with the Fermions becomes very clear if it is noted that it can be represented by Fermionic creation and annihilation operators according to the scheme

$$\theta_i \leftrightarrow a_i, \quad \Delta^i \leftrightarrow a_i^+, \quad P_i^i \leftrightarrow a_i^+ a_i .$$

The relation of $F(n)$ to the exterior differential forms of Cartan can be seen as follows. The minimal left ideal $I_L = F(n)P$ generated by P is a vector space of dimension 2^n. Its general element can be written in the form

(41) $$F(\theta_i)\ P$$

where $F(\theta_i)$ is the general exterior form of the n variables θ_i. The space I_L therefore can be identified with the space of exterior forms in the same way that $B(n)|\rangle$ could be identified with the space of ordinary functions. The analogy is made still clearer if we write $|F\rangle$ for the general element of I_L and $|\rangle_F$ for P. Then (41) becomes

$$|F\rangle = F(\theta_i)\,|\rangle_F .$$

The right ideal $I_R = PF(n)$ is then the space of dual forms. The affine invariants associated with these forms and their duals can be found by forming products of the type

$\langle G || F \rangle = \langle G,F \rangle$ P. In this way, forms in n variables and their duals may be regarded as the spinors of the Clifford algebra for a 2n dimensional space.

In the context of homology, I_L is the space of chains and I_R the space of co-chains. $\mathcal{D}_1$ and Δ^1 are related to the boundary and co-boundary operators.

The fact that *F*(n) may be regarded as a generalisation of the theory of exterior forms is interesting in its own right. It means that the notion of differentiation can now be raised in a purely algebraic context and is no longer tied to the notion of tangent and cotangent vectors of a differential manifold. This algebra together with the Boson algebra considered as an algebrisation of the theory of ordinary functions and their differentials now becomes independent of the differential manifold which has the notion of locality built in as a basic feature of the descriptive form. By freeing the mathematics from the continuous manifold we have opened up the possibility of being able to use these structures to discuss the notion of prespace which lies at the heart of the description of physics through the implicate order.

In order to demonstrate in what ways the purely algebraic approach can be used to throw new light onto the geometric aspects of these algebras, consider an affine space A_n of dimension n, together with its dual A_n^D. Rather than use the second quantised notation, we shall follow Schoenberg[1] and denote the "annihilation operators" by $\{I_j\}$ and the "creation operators" by $\{I^j\}$. Now consider the algebra generated over some field in which the basis elements $\{I_j\}$ $\{I^j\}$ satisfy

$$\{I_i, I_k\}_+ = 0, \qquad \{I^j, I^k\}_+ = 0$$

and $$\{I^j, I_k\}_+ = 1 = \langle I^j, I_k\rangle 1$$

where $\langle I^j, I_k\rangle$ is an inner product defined in the algebra. The elements Λ of F(n) can then be written in the form

$$\Lambda = \sum_{p,q} (p!q!)^{-1} \lambda^{j_1 \ldots j_p}_{k_1 \ldots k_q} I_{j_1} \ldots I_{j_p} P I^{k_1} \ldots I^{k_q}$$

where $P = \prod_{i=1}^{n} P_i$. If we make an affine transformation of the space A_n together with the corresponding dual transformation on A_n^D, the coefficients $\lambda^{j_1 \ldots j_p}_{k_1 \ldots k_q}$ of Λ will transform as antisymmetric tensors of A_n. The 2^{2n} dimensional vector space underlying the

Fermion algebra F(n) can thus be decomposed into the direct sum

$$F(n) = \bigoplus_{p,q} A(p,q).$$

The symmetric tensors of the affine space can be obtained by considering the quasi algebra generated over a field from the set $\{I_j, I^k\}$ of generators satisfying the relations

$$[I_j, I_k] = 0, \qquad [I^j, I^k] = 0$$

and $$[I^j, I_k] = 1 = \langle I^j, I_k \rangle 1.$$

This algebra is formally identical to the quasi algebra $L_q(n)$ defined in section 3, the I^j corresponding to the D_j and the I_j to the Q_j. Affine transformation of the space A_n together with the corresponding dual transformation of $A_n{}^D$ leaves the above defining relations invariant, demonstrating that $L_q(n)$ is a geometric algebra of the geometry of n dimensions.

If $L_q(n)$ is extended to $B(n)$ as discussed above, and the idempotent element V is taken to transform as a scalar under affine transformation, then $B(n)$ will also be a geometric algebra for the affine geometry of n dimensions.

The elements Γ of $L_q(n)$ can then be written uniquely in the form

$$\Gamma = \sum_{p,q} \frac{a_1! \ldots a_n!\, b_1! \ldots b_n!}{p!\, q!} \cdot \gamma^{j_1 \ldots j_p}_{k_1 \ldots k_q} I_{j_1} \ldots I_{j_p} V I^{k_1} \ldots I^{k_q}$$

where a_r is the number of j's equal to r and b_t is the number of k's equal to t. It is not difficult to see that the vector space underlying the quasi algebra $L_q(n)$ can be decomposed into the direct sum

$$L_q(n) = \bigoplus_{p,q} S(p,q)$$

where S(p,q) are the spaces of the symmetric tensors.

What these results show is that every geometric object commonly associated with an affine space of n dimensions can be regarded as either an element of the Fermion algebra F(n) or of the Boson quasi algebra $B(n)$. Although this example provides no new content, it opens up a radically new view on the relation between geometry and quantum mechanics that Schoenberg was the first to stress. A complete understanding of this new approach will require extensive work, but we can already see some essential features emerging.

The above discussion shows a correspondence between the notions of "creation" and "annihilation" on the one hand and their geometric counterparts of "cogredience" and "contragredience" on the other. However the properties of their generating elements relative to one another are contained totally in their multiplicative properties. Indeed the whole content of these algebras is latent intrinsically in the essential algebraic properties ascribed to the generating set through the commutation relations. Furthermore, there is a complete lack of participation of the tensor indices ascribed to the generating elements . These indices appear in the algebraic context as no more than labels for the generating set. Similarly it follows that the quantum concept of creation and annihilation is also absent as a fundamental concept in the definition of F(n) and $\mathcal{B}$(n), the distinction of creation and annihilation not being involved in any essential way.

The basic indifference of the Boson and Fermion algebras both to tensor indices and to creation and annihilation suggest that these notions are inessential. They should be regarded as secondary, subordinate to the primary notion of algebraic structure. This suggests that we must think of these algebras in a new way. Since we have 2n distinct generating elements $\{I^j, I_k\}$ and their relations with D and Q, or θ and Δ, this suggests that a phase space may be more important than the current interpretation through the configuration space. This is already implicit in the work of Schoenberg[1,2] and has also been extended by Bohm and Hiley both in the non relativistic case[19] and the relativistic case[20].

In the usual quantum mechanics much emphasis is place on group structures which lie at the basis of the symmetry properties of the Lagrangian. For the two basic algebras that we are considering, the respective groups arise in the following manner. Put $\lambda_i = I_i + f_{ij} I^j$, where f^{ij} is an antisymmetric tensor. Then it can easily be shown that

$$[\lambda_i, \lambda_j] = 2f_{ij}.$$

$B(n)$ can thus be regarded as the full matric extension of the symplectic quasi algebra K^{2n} of the phase space. If on the other hand we put $e_i = I_i + g_{ij} I^j$, where g^{ij} is a symmetric tensor, then we find

$$\{e_i, e_j\} = 2\, g_{ij}$$

so that $F(n)$ is intimately related to the Clifford algebras.

It should be noted that this view of the Boson and Fermion algebras is more natural than regarding them merely as geometric algebras for the affine space, the groups contained in these algebras being larger than the affine group for the n-dimensional space. Regarding F(n) as a Clifford algebra for the phase space, the quantum field theory of Fermions is seen to be essentially the theory of spinors of the phase space. Similarly, the quantum theory of Bosons is seen to be essentially that of the symplectic spinors for the phase space. In both cases, by spinors we mean the elements of the minimal left ideals of the respective algebras.

It is of interest to note that the group of transformations on the phase space leaving invariant the form f^{ij} has n(2n+1) parameters and for the invariance of g^{ij} we need n(2n-1) parameters. Thus our algebras carry the translations and rotations of space time. These two groups intersect in the central affine group for the n dimensional space. It would seem from this that the forms f and g together, rather than individually, fully determine the configuration space for a system. This feature is one that is taken up in the study of supersymmetries where the orthosymplectic groups leave invariant a bilinear form that is a combination of the symmetric and the antisymmetric forms f and g. The proposed approach is thus rich enough to carry not only the symmetries of space time, but also supersymmetries.

8. CONCLUSION.

If the Boson and Fermion algebras are given a prominence in our conceptual framework commensurate with that suggested by the quantum theory, then the geometry of space and time will appear as a higher level abstraction from the more fundamental notion of algebraic structure. Regarding B(n) and F(n) as "pregeometries" in the sense that they are geometric algebraic structures of more fundamental importance than the geometry of space and time. Indeed the algebraic approach is devoid of all reference to any underlying space time structure. The algebraic "wave function" provides no notion of the extension of a system in space and time nor does it give a "point to point" account of a particular property of the system. The algebraic quantum theory can only support an interpretation through the implicate order. In this sense it can be regarded as neither a "local" theory nor a " non local" theory. It would be appropriate to regard it as an "a-local" theory, with the notion of locality being abstracted from a distinguished relationship in the implicate order as discussed by Bohm, Hiley and Davies[21].

Thus in the pregeometry, the first level of abstraction through the isolation of a set of generators for these two algebras B(n) and F(n) defines a 2n dimensional vector space identifiable as a phase space for the system. The algebras B(n) and F(n) individually define the metric and symplectic geometries respectively for the phase space from which, as a second order abstraction, the configuration space is distinguished. This allows the concepts of geometric covariance and contravariance, or of quantum creation and annihilation, to be defined. The geometry of space time is then obtained from that of the configuration space as a third order abstraction.

REFERENCES

1. Schoenberg M., Nuovo Cim. (Suppl.), 6, 356 (1957).

2. Schoenberg M., An. Acad. Basil. Ci., 29, 473-479 (1957); An. Acad. Basil. Ci., 30, 1-20, 117-131, 259-280, 429-446 (1958).

3. Gell-Mann, M., in The Nature of Matter, Wolfson College Lectures 1980, Ed. Mulvey J. (Oxford, 1981).

4. Bohm D., Wholeness and the Implicate Order (Routledge and Kegan Paul, London, 1980).

5. Bohr, N., Atomic Physics and Human Knowledge (Science Editions, New York, 1961).

6. Bohm, D., and Hiley, B., Generalisation of the Twistor to Clifford Algebras as a Basis for Geometry, to be published in Revista Brasiliera de Physica, 1984.

7. Chew, G., Found. Phys., 13, 273 (1983); Gentle Quantum Events As The Source Of The Explicate Order, Preprint LBL 15811 (Berkeley, 1983).

8. Stapp, H., Phys. Rev., 28D, 1386 (1983)

9. Bohm, D., Found. Phys., 4, 139 (1973).

10. Frescura, F., and Hiley, B., Found. Phys., 10, 7 (1980).

11. Riesz, M., Comptes Rendus 10me Cong. Math. Scand., 123-148 (Copenhague, 1946).

12. Riesz, M., Comptes Rendus 12me Cong. Math. Scand., 241-259 (Lund,1953).

13. Bohm, D., Proc. Int. Conference on Elemetary Particles (Kyoto, 1965).

14. Frescura, F., and Hiley, B., Found. Phys., 10, 705 (1980).

15. Dirac P., Phys. Rev., 139B, 684 (1965)

16. Cartan, E., Les Systemes Differentiels Exterieurs Et Leurs Applications Geometriques (Hermann & Cie, Paris, 1945)

17. Rocek, M., Introduction to Superspace and Supergravity, from Superspace and Supergravity, Ed. Hawking S.W. and Rocek M. (Cambridge University Press, 1981).

18. Cartan E., The Theory Of Spinors (M.I.T. Press, 1966).

19. Bohm D. and Hiley B., Found. Phys., 11, 179 (1981)

20. Bohm D. and Hiley B., Old And New Questions In Physics, Cosmology, Philosophy And Theoretical Biology, Ed. van der Merwe, A. (Plenum, 1983)

21. Bohm D., Hiley B. and Davies P., Algebraic Quantum Mechanics and Pre-Geometry, To Be Published (1984).