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Abstract

We investigate those non-linear systems which, at a classical level, are decribed by the Lagrangian

$$L = \frac{1}{2} \frac{1}{1+\lambda r^2} \left\{ \underset{\sim}{\dot{x}}^2 - \lambda \frac{(\underset{\sim}{x} \cdot \underset{\sim}{\dot{x}})^2}{1+\lambda r^2} \right\} - V(r) .$$

It is shown that this same Lagrangian corresponds to the nonrelativistic motion of a particle of unit mass, constrained to move on a N-sphere S^N, where a conservative center of force $V(r)$ is defined. After examining some classical properties, a quantization procedure is discussed, which takes into account the geometrical symmetries involved. It is also shown that case $V(r)=\frac{1}{2}\omega^2 r^2$, corresponding to a isotropic harmonic oscillator in S^N, is exactly solvable. Explicit results on the energy spectrum and related eigenfunctions are given for arbitrary values of N.

Some general features of the problem are also discussed.

1. INTRODUCTION

As a former student of Professor Mario Schenberg we are pleased to contribute to a number of this journal that celebrates his 70th birthday.

The work we wish to present in this occasion, is an elementary instance of a major problem, namely, that of the relationship between quantum theory and geometry, a rich and conceptually important problem which occupied a distinguished place among his scientific interests(1). So we hope that he may appreciate the present discussion whose main motivation is connected with the investigation of dynamical symmetries in geometries different from the Euclidean one.

We summarize our problem as follows. We will discuss here the quantization of a point particle in a nonrelativistic motion, constrained to move on a N-sphere S^N, where conservative centers of force are defined. A first treatment of this problem was given by Schrödinger(2) who solved the quantum mechanical version of the perennial Kepler problem in a 3-sphere S^3. More recently the general case of S^N, both in a classical and quantum mechanical context was discussed by Higgs(3) and Leemon(4).

The quantization procedure we will discuss here is based on group theoretical considerations related to the geometric symmetries of the system. Our discussion also illustrates some of the essential features of the quantization in curved spaces.

The plan of the paper is as follows. The content of section 2 is entirely classical and deals with the Lagrangian and Hamiltonian descriptions of the classical system. An important rôle is played by our special parametrization of S^N, which is based on the geometrical concept of central (or gnomonic) projection[3],[5]. In the same section, the constants of motion of our system are derived, by means of Emmy Noether's theorem. Appendix A contains a set of explicit results on the differential geometry of S^N, in terms of the central projection parametrization.

Section 3 deals with the quantization procedure. The difficulty in defining the quantum Hamiltonian due to the ambiguities arising from the ordering problem is overcome by adopting a group theoretical criterium[4],[3], first discussed by Velo and Wess[6] which takes into account the geometrical symmetries of the system. As a consequence, the Hamiltonian operator will be defined by means of the 2nd order Casimir operator of the SO(N+1) or SO(N) rotation group. The first case is realized by the N-dimensional rotator, V(r)=0. The presence of V(r) breaks this symmetry to the SO(N) level. In section 4 we describe the results of the energy eigenvalue problem for the isotropic harmonic oscillator in S^N. A detailed analysis is contained in appendix B. The problem is shown to be exactly solvable for a general value of $N(N\geqslant 1)$. Section 5 contains our main conclusions and final remarks.

2. CLASSICAL DESCRIPTION

Now we shall be concerned with the class of classical nonlinear systems decribed by the Lagrangian

$$L = \frac{1}{2}\,\frac{1}{1+\lambda r^2}\left\{\dot{\underset{\sim}{x}}^2 - \lambda\,\frac{(\underset{\sim}{x}\cdot\dot{\underset{\sim}{x}})^2}{1+\lambda r^2}\right\} - V(r), \quad (2.1)$$

where $\underset{\sim}{x}=(x^1, x^2, \ldots, x^N)$, r denotes the Euclidean distance

$$r = \sum_{i=1}^{N} (x^i)^2 , \qquad (2.2)$$

λ is a positive integer and the dot indicates a time derivative.

A possible model for the N-dimensional system described by (2.1) consists of a point particle, of unit mass, constrained to move on a N-sphere S^N (of curvature $\lambda = R^{-2}$, embedded in an Euclidean (N+1)-dimensional space) where a conservative center of force is defined[(3),(5)]. The N-sphere is parametrized by a set of N real variables

$$-\infty \leq x^i \leq \infty , \quad i = 1, 2, \ldots, N$$

whose precise geometrical meaning is the following: they are the Cartesian coordinates of a point P' belonging to a given tangent plane Π^N to S^N, P' being obtained from a point $P \in S^N$ by means of a <u>central projection</u>. As shown in Ref.(5), the metric of S^N, in a parametrization in terms of the coordinates $(x^1, x^2, \ldots, x^N)$ so defined, is given by

$$(ds)^2 = \frac{(d\underset{\sim}{x})^2}{1+\lambda r^2} - \lambda \frac{(\underset{\sim}{x}\cdot d\underset{\sim}{x})^2}{(1+\lambda r^2)^2} . \qquad (2.3)$$

Accordingly, for the kinetic energy T of a particle of unit mass, one has

$$T = \frac{1}{2}\left(\frac{ds}{d\tau}\right)^2 = \frac{1}{2}\frac{1}{1+\lambda r^2}\left\{\dot{\underset{\sim}{x}}^2 - \lambda \frac{(\underset{\sim}{x}\cdot\dot{\underset{\sim}{x}})^2}{1+\lambda r^2}\right\} . \qquad (2.4)$$

Now, if a conservative center of force in S^N is introduced by means of a function V(r) of the Euclidean distance r one gets, for the Lagrangian L=T-V, precisely the expression given by Eq. (2.1).

It is not difficult to show that the Euler-Lagrange equations corresponding to (2.1) can be written in the form[5]

$$\ddot{\underset{\sim}{x}}_{\tau}^{2} - 2\lambda \frac{\underset{\sim}{x}\cdot\dot{\underset{\sim}{x}}_{\tau}}{1+\lambda r^2}\dot{\underset{\sim}{x}}_{\tau} + \frac{1}{r}\frac{dV}{dr}(1+\lambda r^2)^2\underset{\sim}{x} = 0, \quad (2.5)$$

a system of N ordinary highly nonlinear differential equations.

<u>Remark 2.1</u>. The central projection, as is well known, defines a <u>geodesic mapping</u>[7] of a hemisphere $\frac{1}{2}S^N$ in Π^N.

<u>Remark 2.2.</u> Let us consider, in Π^N, the motion parametrized by a time variable t and described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\underset{\sim}{x}}_{t}^{2} - V(r). \quad (2.6)$$

The corresponding equations of motion are, of course, given by

$$\ddot{\underset{\sim}{x}}_{t} + \frac{1}{r}\frac{dV}{dr}\underset{\sim}{x} = 0. \quad (2.7)$$

It may be shown that the following theorem holds[5]: If the motions on Π^N and on S^N are related by a central projection and are parametrized by time variables t and τ respectively whose differentials satisfy the relation

$$dt = (1+\lambda r^2)d\tau \quad (2.8)$$

then the equations of motion on S^N, Eq.(2.4), are obtainable from the equations of motion on Π^N, Eq.(2.6) by applying Eq.(2.8) ("local time-reparametrization").

The relevant rôle played by the concept of central projection in our problem follows directly from the two theorems above. Several consequences of the local time-reparametrization theorem have been discussed by the author in Ref.(5). Among them, we mention the existence, at a classical level, of constants of motion for the Kepler and isotropic harmonic oscillator problem in S^N formally identical to those existing for the corresponding Euclidean problem, an important result first derived by Higgs[3].

<u>Constants of motion</u>. Noether's theorem in the following form will now be applied to derive, from the Lagrangian Eq. (2.1), the constants of motion of the system: If, for infinitesimal transformations $\underset{\sim}{x} \to \underset{\sim}{x} + \delta \underset{\sim}{x}$, the Lagrangian changes by a total derivative, $\delta L = \frac{dX}{dt}$, then $C = \frac{\partial L}{\partial \dot{\underset{\sim}{x}}} \delta \underset{\sim}{x} - X$ is a constant of motion, $\dot{C} = 0$.

i) For any $V(r)$ and infinitesimal rotations

$$\delta x^i = \omega^{ij} x^j \qquad (i,j = 1,2,\dots,N)$$

where ω^{ij} denotes an infinitesimal antisymmetric tensor, one has $\delta L = 0$. It follows immediately that

$$L^{ij} = x^i \frac{\dot{x}^j}{1+\lambda r^2} - x^j \frac{\dot{x}^i}{1+\lambda r^2} \qquad (2.9)$$

are constants of motion, $\dot{L}^{ij} = 0$.

ii) Iff $V(r)=0$ and for infinitesimal transformations of the kind $(\epsilon^i \ll 1)$

$$\delta x^i = (1+\lambda r^2)\epsilon^i$$

one gets

$$\delta L = -\lambda \epsilon^i L^{ij} \dot{x}^j = \frac{d}{dt}\left(-\lambda \epsilon^i L^{ij} x^j\right),$$

where we used the constancy of L^{ij}. From Noether's theorem it follows that

$$p^i(1+\lambda r^2) + \lambda L^{ij} x^j = const.$$

By substituting the explicit form of the canonical momentum in this expression one gets

$$\pi^i \equiv \frac{\dot{x}^i}{1+\lambda r^2} = const, \quad \dot{\pi}^i = 0. \qquad (2.10)$$

In terms of the $\underset{\sim}{\pi}$ we have just defined, the Lagrangian Eq. (2.1) can be written as

$$L = \frac{1}{2}\left(\underset{\sim}{\pi}^2 + \lambda \underset{\sim}{L}^2\right) - V(r), \qquad (2.11)$$

where

$$\underset{\sim}{\pi} = \frac{\underset{\sim}{\dot{x}}}{1+\lambda r^2} \qquad (2.12)$$

$$\underset{\sim}{L}^2 = \frac{1}{2} L^{ij} L^{ij} \qquad (2.13)$$

and

$$L^{ij} = x^i \pi^j - x^j \pi^i . \qquad (2.14)$$

Now, the canonical momentum $\underset{\sim}{p} = \frac{\partial L}{\partial \underset{\sim}{\dot{x}}}$ is given by

$$\underset{\sim}{p} = \frac{\underset{\sim}{\dot{x}}}{1+\lambda r^2} - \lambda \frac{\underset{\sim}{x}\cdot\underset{\sim}{\dot{x}}}{(1+\lambda r^2)^2}\underset{\sim}{x} \qquad (2.15)$$

and it is related to the vector $\underset{\sim}{\pi}$, Eq.(2.12) by

$$\underset{\sim}{\pi} = \underset{\sim}{p} + \lambda \underset{\sim}{x}(\underset{\sim}{x}\cdot\underset{\sim}{p}). \qquad (2.16)$$

It easily seen that the classical Hamiltonian can be written in two alternative forms:

$$H = \underset{\sim}{x}\cdot\underset{\sim}{\dot{p}} - L = \frac{1}{2}(1+\lambda r^2)\left\{\underset{\sim}{p}^2 + \lambda(\underset{\sim}{x}\cdot\underset{\sim}{p})^2\right\} + V(r) \qquad (2.17)$$

or

$$H = \frac{1}{2}\left(\underset{\sim}{\pi}^2 + \lambda \underset{\sim}{L}^2\right) + V(r) . \qquad (2.18)$$

In Appendix A some results of the differential geometry of S^N are explicitly given, in terms of the central projection parametrization discussed earlier.

3. QUANTIZATION

The quantization of our system presents some peculiarities that actually are common to a whole class of constrained motions too. Once the classical Hamiltonian is found, we are faced with the problem of ordering ambiguities in defining the quantum Hamiltonian, as a Hermitian operator. Although the determination of the Hamiltonian operator is not unique, there is a well defined quantization procedure, due to Weyl[8], which is generally accepted as the standard quantization method. If desired, it might be applied to the classical Hamiltonian in the form of Eq. (2.17). However, as it was mentioned before, there is an alternative method, based on the geometrical symmetries of the system, which provides a more elegant solution to our quantization problem. Its starting point is the expression, Eq.(2.18), for the classical Hamiltonian:

$$H = H_0 + V(r) = \frac{1}{2}\left(\underset{\sim}{\pi}^2 + \lambda \underset{\sim}{L}^2\right) + V(r), \qquad (3.1)$$

where the operator $\underset{\sim}{\pi}$ is defined by the simple symmetrization of its classical counterpart, Eq.(2.16), namely

$$\underset{\sim}{\pi} = \underset{\sim}{p} + \frac{\lambda}{2}\left\{\underset{\sim}{x}(\underset{\sim}{x}\cdot\underset{\sim}{p}) + (\underset{\sim}{p}\cdot\underset{\sim}{x})\underset{\sim}{x}\right\}. \qquad (3.2)$$

It immediately follows that, for $\underset{\sim}{p}$ Hermitian, $\underset{\sim}{\pi}$ is also Hermitian.

From the canonical quantization relations (with $\hbar=1$), one gets from Eq.(3.2), the following commutators:

$$[x^i, x^j] = 0, \quad [x^i, \pi^j] = i\left(\delta^{ij} + \lambda x^i x^j\right)$$
$$[\pi^i, \pi^j] = i\lambda L^{ij}. \qquad (3.3)$$

As a consequence of them, the operators

$$L^{ij} = x^i \pi^j - x^j \pi^i \qquad (i,j = 1,2,\dots,N)$$

are generators of the SO(N) group:

$$[L^{ij}, L^{kl}] = i\left(\delta^{ik} L^{jl} + \delta^{il} L^{kj} - \delta^{kj} L^{il} - \delta^{lj} L^{ki}\right). \qquad (3.4)$$

Furthermore, by writing

$$\pi^i = \lambda^{1/2} L^{0i} \qquad (i = 1,2,\dots,N)$$

it is easy to show that the $\frac{1}{2}N(N+1)$ operators, π^i and L^{ij}, generate the Lie algebra of the SO(N+1) group. Consequently,

$$H_0 = \frac{1}{2}\left(\underset{\sim}{\pi}^2 + \lambda \underset{\sim}{L}^2\right) = \frac{\lambda}{2}\left(L^{0i}L^{0i} + \underset{\sim}{L}^2\right)$$
$$= \frac{\lambda}{2} C\left(SO(N+1)\right), \qquad (3.5)$$

where C denotes the 2nd order Casimir operator of the SO(N+1) orthogonal group.

The Hamiltonian operator H_0, Eq.(3.5), represents physically a N-dimensional rotator, that is, a system consisting of a free particle ($V(r)=0$) constrained to move in a N-sphere S^N, supposed fixed. As is well known, its eigenfunctions are hyperspherical harmonics[9] i.e., homogeneous polynomials in the variables $x^1, x^2, \dots, x^N$, of degree $k = 0,1,2,\dots$. They result to be angular functions only. The corresponding eigenvalues are given by

$$E(N,k) = \frac{\lambda}{2} k(k+N-1). \qquad (3.6)$$

It is opportune to point out that for a N-dimensional rotator, a k-level is d(N,k)-fold degenerate where

$$d(N,k) = (2k+N-1)\,\frac{(k+N-2)!}{(N-1)!\,k!}\,. \qquad (3.7)$$

In the presence of a potential $V(r)$, of course, the $SO(N+1)$ summetry is broken. However, the operator H, Eq.(3.1), is still invariant under the $SO(N)$ orthogonal group generated by the L^{ij} operators. Therefore, to solve the wave equation we are led to usual method of separation of variables, the angular part of the eigenfunctions being given by the hyperspherical harmonics of the $SO(N)$ group.

Remark 3.1. It is worthwhile to remark that the Hermiticity condition for a generic operator $\mathcal{O}$ should now read

$$\int_{S^N} \sqrt{g}\, dx^1 dx^2 \ldots dx^N\, \psi^* \mathcal{O} \varphi = \int_{S^N} \sqrt{g}\, dx^1 dx^2 \ldots dx^N\, \varphi (\mathcal{O}\psi)^*, \qquad (3.8)$$

where the invariant measure of S^N is to be used (see Appendix A). As a consequence of Eq.(3.8), the following result holds: the operators

$$\begin{cases} \underset{\sim}{x} = \underset{\sim}{x}\cdot \\ \underset{\sim}{p} = -i\left(\underset{\sim}{\nabla} - \frac{1}{2}(N+1)\,\frac{\underset{\sim}{x}}{1+\lambda r^2}\right) \end{cases} \qquad (3.9)$$

are Hermitian realizations of the canonical commutation rules.

Remark 3.2. By using Eqs. (3.2) and (3.9), the operator H_0 , Eq.(3.1), can be written as

$$H_0 = -\frac{1}{2}(1+\lambda r^2)\left[\underset{\sim}{\nabla}^2 + \lambda\,(\underset{\sim}{x}\cdot\underset{\sim}{\nabla})^2 + \lambda\,(\underset{\sim}{x}\cdot\underset{\sim}{\nabla})\right]. \qquad (3.10)$$

Hence, as indicated in Appendix A

$$H_0 = -\frac{1}{2}\,\Delta$$

where Δ is the Laplace-Beltrami operator for S^N. This may be considered as a nontrivial result.

To solve the wave-equation

$$\left(H_0 + V(r)\right)\psi(x^1, x^2, \dots, x^N) = E\,\psi(x^1, x^2, \dots, x^N) \tag{3.12}$$

we write, according to the foregoing discussion, the separation Ansatz

$$\psi(x^1, \dots, x^N) = f(r)\, P^l(\theta_1, \theta_2, \dots, \theta_p, \varphi). \tag{3.13}$$

The second factor in the right hand side of this equation represents a hyperspherical harmonic of degree l, in the angular variables $\theta_1, \theta_2, \dots, \theta_p, \varphi$ (with p=N-2) defined as follows[8]:

$$\begin{aligned}
x^1 &= r\cos\theta_1 \\
x^k &= r\left(\prod_{i=1}^{k-1}\sin\theta_1\right)\cos\theta_k\,, \quad k = 2, \dots, p \\
x^{p+1} &= r\left(\prod_{i=1}^{p}\sin\theta_i\right)\cos\varphi \\
x^{p+2} &= r\left(\prod_{i=1}^{p}\sin\theta_i\right)\sin\varphi \\
r &= \sum_{i=1}^{N}(x^i)^2; \quad 0 \leq \theta_i \leq \pi\,, \quad 0 \leq \varphi \leq 2\pi\,.
\end{aligned} \tag{3.14}$$

After a lenghty calculation, one gets the radial differential equa tion for $f(r)$:

$$(1+\lambda r^2)\Big\{-\frac{1}{2}(1+\lambda r^2)f''_r - \frac{1}{2r}(N-2+1+\lambda r^2)f'_r + \frac{1}{2}\frac{l(l+N-2)}{r^2}f\Big\} + \big(V(r)-E\big)f = 0. \tag{3.15}$$

It is convenient to perform the following change of the independent variable r:(3),(5)

$$\lambda^{1/2} r = \tan\chi \quad . \qquad (\chi \equiv \theta_1) \tag{3.16}$$

Accordingly, the following equation for $f(\chi)$ is obtained

$$f''_\chi + (N-1)\cot\chi\, f'_\chi - l(l+N-2)\operatorname{cosec}^2\chi\, f + \frac{2}{\lambda}\big(E-V(\chi)\big)f = 0. \tag{3.17}$$

The integration of Eq.(3.17) will be discussed in detail in Appendix B for the special case of a isotropic harmonic oscillator center in S^N defined by the potential

$$V(r) = \frac{1}{2}\omega^2 r^2 . \tag{3.18}$$

In terms of the variable χ , defined by Eq.(3.16), this potential can be rewritten as .

$$V(\chi) = \frac{1}{2}\frac{\omega^2}{\lambda}\tan^2\chi \cdot \tag{3.19}$$

Notice that this potential vanishes for $\chi=0$ ("North pole") and for $\chi=\pi$ ("South pole") but it is infinite at the "equator" $\chi=\frac{\pi}{2}$. It follows that we have two attractive centers, one in each pole and an infinite reflecting barrier at the equator. Hence, the motion will be necessarily confined to a hemisphere where the particle will move under the sole influence of the respective attractive center.

Also as a consequence, the following boundary condition must hold:

$$f\left(\frac{\pi}{2}\right)=0 \ .$$

The normalization condition for $\psi(x^1,x^2,\ldots,x^N)$ is now

$$\int_{S^N}\sqrt{g}\ dx^1 dx^2 \ldots dx^N\,|\psi|^2=1 \qquad (3.21)$$

where (see Appendix A)

$$\sqrt{g}=(1+\lambda r^2)^{-\frac{1}{2}(N+1)}$$

Assuming that the hyperspherical harmonics in Eq. (3.13) are normalized to unity in S^N, and having in mind Eq.(3.19) it is easily seen that the normalization condition can be written as

$$\lambda^{-\frac{N}{2}}\int_0^{\frac{\pi}{2}} d\chi\,(\sin\chi)^{N-1} f^2(\chi)=1 \ . \qquad (3.22)$$

Remark 3.1. The Kepler problem in S^N, $V_K=-\frac{\mu}{r}$ $(\mu>0)$, presents a completely different behavior since

$$V_K=-\mu\,\lambda^{1/2}\cot\chi\ .$$

Hence, one has two centers of force, an attractive one at the North pole a repulsive one at the South pole, the equator being a line of zeros so that the motion is not confined to an hemisphere in this case. For a recent discussion of the Kepler problem in S^N the reader in referred to Refs. (3),(4).

4. THE ISOTROPIC HARMONIC OSCILLATOR IN S^N.

The eigenvalue problem defined by Eq.(3.17), with boundary condition Eq.(3.20) has been investigated for the case of the oscillator potential given by Eq.(3.19). In Appendix B, we will give a detailed discussion of it and derive the energy spectrum and eigenfunctions. We state in this section the main results.

For $N>1$, including normalization constants, the eigenfunctions are given by

$$f(\chi)=\mathcal{N}\,(\sin\chi)^{\ell}(\cos\chi)^{\frac{\sqrt{h}}{\lambda}+\frac{1}{2}}F\left(\frac{\sqrt{h}}{\lambda}+\frac{1}{2}(n+\ell+N),-\frac{1}{2}(n-\ell);\ell+\frac{N}{2};\sin^2\chi\right) \tag{4.1}$$

where F stands for the ordinary hypergeometric function, usually denoted by $_2F_1$ and

$$h=\omega^2+\left(\frac{\lambda}{2}\right)^2 \tag{4.2}$$

Furthermore, ℓ and n are nonnegative integers ($\ell, n = 0,1,2,\ldots$).

The normalization constant appearing in Eq.(4.1) is determined according to Eq.(3.22). We obtained the result

$$\mathcal{N} = \left[\frac{2\lambda^{\frac{N}{2}}\,\Gamma(\alpha+\beta+1)\,\Gamma(\alpha+\beta+2k+1)\,\Gamma(\alpha+\beta+k+1)}{k!\,\Gamma^2(\alpha+1)\,\Gamma(\beta+k+1)} \right]^{\frac{1}{2}}.$$

The corresponding energy eigenvalues are given by the positive quantities

$$E(n,N) = \left(n+\frac{1}{2}N\right)\sqrt{h} + \frac{1}{2}\lambda\left(n^2+Nn+\frac{N}{2}\right). \quad (4.4)$$

The case N=1 (circumference) has also been investigated. We have shown that formula (4.4) is still valid for this case, although the eigenfunctions appear in a slightly different way, in groups of opposite parity.

The spectrum given by Eq.(4.1) coincides with that obtained by Leemon[4] using Schrödinger's method[1].

It is clear that when λ goes to zero, we get the well known Euclidean result. On the other hand, in the limit ω tending to zero, we have a free particle in a hemisphere $\frac{1}{2}S^N$. Formula (4.4) shows that although the energy levels are no more equally spaced as they are in the Euclidean limit, the degeneracy pattern is the same as that which prevails in the Euclidean limit. This is a quite interesting result. It allows us to conclude that our system also admits the SU(N) group as degeneracy group, the same group responsible for the accidental degeneracies of the N-dimensional isotropic harmonic oscillator in Euclidean space[10]. Consequently, there must exist a set of Hermitean operators, say N^{ij} $(i,j=1,2,\ldots,N)$, symmetric and traceless, which together with the L^{ij} operators generate the symmetrical irreducible representation of the SU(N) group. However, its explicit construction is still an open problem, for a general value of N.

5. CONCLUSION

We have shown that the nonlinear system which in R^N is described by Lagrangian Eq.(2.1) can be interpreted as a particle system constrained to move in a N-sphere S^N, subject to a conservative center of force V(r). In establishing this result, essential use was made of the properties of the central projection of S^N in a fixed tangent plane π^N.

The quantization of motion in Riemannian manifolds is a problem that has a theoretical interest of its own. In this work, we have illustrated some of its general features by working out the example of a nonlinear oscillator. Its merit is that the example is <u>exactly solvable</u>, allowing, consequently, a clear understanding of how the underlying geometry (in our case that of a N-sphere embedded in a R^{N+1} space) influences the quantum mechanical energy spectrum.

Of course, since the curvature of the universe is extremely feeble, the effect predicted for the spectrum of our oscillator for N=3 is entirely negligible in the present model. Nevertheless, from a theoretical point of view, the result that the degeneracy pattern of the spectrum is the same as that occuring in Euclidean geometry seems to us interesting enough, leading to the conclusion that for the isotropic harmonic oscillator, the degeneracy groups are the same for both geometries, coinciding therefore with the unitary group in N dimensions, SU(N)[10].

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Appendix A

Some Geometric Results on S^N

In this appendix, we give explicit results relative to S^N, based on the central projection parametrization described in the text.

i) Metric:

$$ds^2 = \frac{1}{2}\,\frac{1}{1+\lambda r^2}\left\{(d\underset{\sim}{x})^2 - \lambda\,\frac{(\underset{\sim}{x}\cdot d\underset{\sim}{x})^2}{1+\lambda r^2}\right\}$$

$$= g_{ij}\,dx^i dx^j\,; \qquad r^2 = \delta_{ij}\,x^i x^j .$$

ii) Metric tensor:

Covariant components:

$$g_{ij} = \frac{1}{1+\lambda r^2}\left\{\delta_{ij} - \lambda\,\frac{\delta_{mi}\delta_{nj}\,x^m x^n}{1+\lambda r^2}\right\};$$

contravariant components:

$$g^{ij} = (1+\lambda r^2)\left\{\delta^{ij} - \lambda\,x^i x^j\right\}.$$

iii) Determinant:

$$g = \det\|g_{ij}\| = (1+\lambda r^2)^{-(N+1)} > 0 .$$

iv) By means of Christoffel's formula

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right\}$$

one gets the simple expression

$$\Gamma^i_{jk} = -\frac{\lambda}{1+\lambda r^2} \left(\delta_{jl} \delta^i_k + \delta_{kl} \delta^i_j \right) x^l .$$

v) Ricci tensor: From its definition

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik} \Gamma^m_{lm} - \Gamma^m_{il} \Gamma^l_{km}$$

one obtains

$$R_{ik} = \frac{\lambda}{1+\lambda r^2} (N-1) \left\{ \delta_{ik} - \lambda \, \frac{\delta_{mi} \delta_{nk} x^m x^n}{1+\lambda r^2} \right\} .$$

vi) Scalar curvature: $R = g^{ik} R_{ik}$.

One has

$$R = N(N-1)\lambda .$$

vii) Equations of motion (free case)

From

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 ,$$

one gets

$$\ddot{x}^i - 2\lambda \, \frac{\underset{\sim}{x} \cdot \dot{\underset{\sim}{x}}}{1+\lambda r^2} \dot{x}^i = 0$$

in agreement with Eq.(2.5) for $V(r)=0$.

viii) The Laplace-Beltrami operator: From the definition

$$\Delta = g^{-\frac{1}{2}} \frac{\partial}{\partial x^i} \left(g^{\frac{1}{2}} g^{ij} \frac{\partial}{\partial x^j} \right)$$

one obtains

$$\Delta = (1+\lambda r^2) \left\{ \underset{\sim}{\nabla}^2 + \lambda \left(\underset{\sim}{x} \cdot \underset{\sim}{\nabla} \right)^2 + \lambda \left(\underset{\sim}{x} \cdot \nabla \right) \right\},$$

where $\underset{\sim}{\nabla}$ indicates the flat nabla operator. Δ is, of course, invariant under reparametrizations of S^N.

ix) Invariant measure on S^N:

$$dx = \sqrt{g}\, dx^1 dx^2 \ldots dx^N .$$

x) The (topological) Euler characteristic of S^N:

$$\chi(S^N) = 1 + (-1)^N .$$

Appendix B

Radial Equation: Eigenvalues and Eigenfunctions

The radial equation in the variable χ $\left(0 \leq \chi \leq \frac{\pi}{2}\right)$ for our nonlinear oscillator in S^N, Eqs. (3.17) and (3.19), can be recast into the general form of the hypergeometric equation

$$z(1-z)u''_z + [\gamma - (\alpha+\beta+1)z]u'_z - \alpha\beta u = 0 \qquad (B.1)$$

by means of the Ansatz

$$f(\chi) = (\sin\chi)^{\ell} (\cos\chi)^{-2p} u(\chi) \qquad (B.2)$$

and change of the independent variable

$$z = \sin^2\chi \, . \qquad (B.3)$$

In Eq. (B.2), ℓ is a nonnegative integer and p is defined as the positive quantity

$$p = \frac{1}{4}\left[\left(1+4\frac{\omega^2}{\lambda^2}\right)^{\frac{1}{2}} - 1\right] . \qquad (B.4)$$

The constants α, β and γ in Eq.(B.1) are taken as

$$\begin{aligned} \begin{matrix}\alpha\\ \beta\end{matrix} &= -p + \frac{1}{4}(2\gamma - 1) \mp \frac{1}{2\lambda}\left(\omega^2 + 2E\lambda + \frac{\lambda^2}{4}(N-1)^2\right), \\ \gamma &= \frac{N}{2} + \ell \, . \end{aligned} \qquad (B.5)$$

Our Ansatz above, in terms of the variables of the variable z reads

$$f_i(z) = z^{\frac{1}{2}}(1-z)^{-p} u_i(z), \quad 0 \leq z \leq 1, \quad (i=1,2) \quad \text{(B.6)}$$

where $u_1(z)$ and $u_2(z)$ denote the two linearly independent so lutions of Eq.(B.1):

$$\begin{aligned} u_1(z) &= F(\alpha, \beta; \gamma; z), \quad \text{"type 1"}; \\ u_2(z) &= z^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z), \quad \text{"type 2".} \end{aligned} \quad \text{(B.7)}$$

In (B.7)

$$F(\alpha, \beta; \gamma; z) \equiv {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}$$

with $(\alpha)_k = \dfrac{\Gamma(\alpha+k)}{\Gamma(k)}$ and so on.

As $p > 0$ in Eq.(B.6), in order that $f_i(z)$ be finite for $z=1$ (corresponding to $\chi = \frac{\pi}{2}$), one necessarily has

$$u_i(z)\Big|_{z=1} = 0 \qquad (i=1,2) \qquad \text{(B.8)}$$

together with

$$f_i(z)\Big|_{z=1} = 0 \qquad (i=1,2). \qquad \text{(B.9)}$$

<u>Type 1 solutions</u>

Making use of the property

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{z}{1-z}\right) \qquad \text{(B.10)}$$

it is easy to see that a necessary condition for a nonsingular beha vior of $u_1(z)$ for $z=1$ is that the hypergeometric function

F in the r.h.s of (B.10) reduces to a polynomial. This condition, together, with Eq.(B.8), leads to

$$-(\gamma-\beta)=k\ ,\qquad k=0,1,2,\dots\ .\qquad (B.11)$$

Type 2 solutions. By similar arguments, it is shown that type 2 solutions satisfying Eq.(B.8) are polynomials:

$$\beta-1=k\ ,\qquad k=0,1,2,\dots\qquad (B.12)$$

However, for a generic N, solutions $f_2(z)$ must be altogether discarded because of their divergent behavior at $z=0$. Indeed,

$$f_2(z)=z^{1-\gamma+\frac{\ell}{2}}(1-z)^{-\beta}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z).$$

Therefore, $f_2(z)$ is proportional to $z^{1-\gamma+\frac{\ell}{2}}$ times a polynomial in z. But as $1-\gamma+\frac{\ell}{2}=1-\frac{1}{2}(N+\ell)$, the exponent of z is, in general, a negative number and consequently, $f_2(z)$ diverges for z going to 0, with the exception of the case $N=1,\ell=0$. The particular case of S^1, therefore, admits both types of solutions.

Eigenvalues. The energy eigenvalues may be obtained by using Eq.(B.11) and the value of β given by Eq.(B.5), just solving for E. One gets

$$E=\frac{\lambda}{2}\left[\left(2k+\gamma+\frac{1}{2}\right)^2+4p(2k+\gamma)-\frac{1}{4}(N-1)^2\right].\qquad (B.13)$$

By putting

$$n=2k+\ell\ ,\qquad n=0,1,2,\dots\qquad (B.14)$$

Eq.(B.13) can be written in the form[3]

$$E=\left(n+\frac{N}{2}\right)\sqrt{h}+\frac{\lambda}{2}\left(n^2+nN+\frac{N}{2}\right) \qquad (B.15)$$

where

$$h=\omega^2+\left(\frac{\lambda}{2}\right)^2 . \qquad (B.16)$$

The corresponding eigenfunctions are given by

$$f_1(z)=\mathcal{N}\,(1-z)^{\frac{\sqrt{h}}{2\lambda}+\frac{1}{4}}\,z^{\frac{l}{2}}\,F\left(-k,\frac{\sqrt{h}}{\lambda}+\gamma+k;\gamma;z\right). \qquad (B.17)$$

Normalization: To determine the normalization constant $\mathcal{N}$ in Eq.(B.17) by means of the normalization condition Eq.(3.22) we made use of the following property, relating polynomial hypergeometric functions and Jacobi polynomials[10]:

$$F\left(-k,\alpha+1+\beta+k;\alpha+1;\frac{1-x}{2}\right)=\frac{k!}{(\alpha+1)_k}\,P_k^{(\alpha,\beta)}(x), \qquad (B.18)$$

with

$$\alpha+1=\gamma\ , \qquad \beta=\frac{\sqrt{h}}{\lambda}\ , \qquad \frac{1-x}{2}=z\ .$$

The use of (B.18) reduces the normalization integral to that of Jacobi polynomials. One obtains

$$\mathcal{N}=\left[\frac{2\lambda^{\frac{N}{2}}\,\Gamma(\alpha+\beta+1)\,\Gamma(\alpha+\beta+2k+1)\,\Gamma(\alpha+\beta+k+1)}{k!\;\Gamma^2(\alpha+1)\,\Gamma(\beta+k+1)}\right]^{\frac{1}{2}}. \qquad (B.19)$$

<u>The Special case N=1</u>. We have also investigated the case of the circumference. In this case, Eq.(3.16) is still valid with $\ell=0$ and for an angular variable χ varying in the interval $-\pi\leq\chi\leq\pi$, ("Southern hemisphere"). It may be shown that formula Eq.(B.15) still holds. However, as already pointed out, eigenfunctions constructed from both type 1 and 2 hypergeometric functions are possible. They correspond to even and odd values of n , respectively.

A detailed account of this case will appear elsewhere.

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