To prof M. Schoenberg
from S.A.W.

DIRAC'S EQUATION AND A NEW GROUP OF EXTERNAL SYMMETRY

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Amsterdam, 16th November 1972.

DIRAC'S EQUATION AND A NEW GROUP OF EXTERNAL SYMMETRY.

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It is shown how the handedness of the massless leptons, and the Poincaré invariance of leptonic interactions, can be based on a new group of external symmetry.

1. The description of massive leptons by Dirac's equation is extremely successful. Muons, as well as electrons seem to obey quantumelectrodynamics to the higher degree of accuracy. The usual mass zero limit of Dirac's equation does not give such a satisfactory description of neutrinos, as only half of its degrees of freedom seem to be realized in nature. A reason for discarding part of the dynamical possibilities is not to be found in the context of Dirac's theory.
In what follows, first an equation for massive spin one half fields, which is equivalent to Dirac's equation, is given. The mass zero limit of this equation, however, is different from the usual mass zero Dirac equation. It is studied, especially with regard to its symmetry properties. These - external - symmetries are then taken as a basis for a new theory of spin one half particles. We have to assume an indefinite metric in the many-particle vector space, but we are able to attribute a probability interpretation to our formalism. We then are forcibly led back to Dirac's theory in the massive case, and to a description equivalent to Weyl's theory for massless particles, as observed experimentally.

2. The Dirac equation for massive particles

$$\gamma^{\kappa} i\partial_{\kappa}\psi = m\psi \tag{1}$$

leads, by the transformation

$$\hat{\psi} = (m^{-1/2} P_{+} + m^{+1/2} P_{-})\psi , \tag{2}$$

where

$$P_{\pm} = \tfrac{1}{2}(1 \pm \gamma^5) , \tag{3}$$

to an equivalent equation for $\hat{\psi}$:

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$$\gamma^{\kappa} i\partial_{\kappa} \hat{\psi} = (P_{-} + m^2 P_{+}) \hat{\psi} \tag{4}$$

Its limit for $m \to 0$ (leaving out the $\wedge$)

$$\gamma^{\kappa} i\partial_{\kappa} \psi = P_{-} \psi \tag{5}$$

is inequivalent to the $m \to 0$ limit of the original equation.

Continuous Poincaré transformations induce linear transformations in the solution space of the c-number equation (5). However, a nonsingular invariant norm does not exist: the continuous Poincaré group is not a symmetry group.

The symmetry group of (5) consists of the 3-dimensional space-rotations and space-time translations (defined as in the Poincaré case), and of the dilatations

$$\psi'(x) = (1 - \ell\varepsilon - \varepsilon P_{-}) \psi\{(1-\varepsilon)x\} \, . \tag{6}$$

Furthermore, it contains the usual time inversion and a spatial inversion:

$$\psi^{P}(t, \vec{x}) = (1 + 2\gamma^{k} i\partial_{k} P_{+}) \psi(t, -\vec{x}) \, . \tag{7}$$

Only for $\ell = 1, 0, -1, -2, -3, \ldots.$ or $\ell = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\ldots$ is the corresponding invariant norm the space integral of a local density:

$$N = \int d\vec{x} \; \psi^{+} \begin{pmatrix} -2\omega & 1 \\ 1 & 0 \end{pmatrix} \omega^{-2(\ell-1)} \, , \quad \text{resp.} \quad N = \int d\vec{x} \; \psi^{+} \begin{pmatrix} -2\Delta & -\omega \\ -\omega & 1 \end{pmatrix} \omega^{-2\ell+1} \psi, \tag{8}$$

where $\omega = \vec{\sigma} . i\vec{\nabla}$, and the choice $\gamma^{0} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\gamma^{k} = \begin{pmatrix} & \sigma^{k} \\ -\sigma^{k} & \end{pmatrix}$, $\gamma^{5} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ is made. Note that N is (positive) definite for halfinteger ℓ, while indefinite for integer ℓ.

The above mentioned continuous group, generated by eight infinitesimal elements, is called $G_{+}^{\uparrow}$. Completed by the spatial inversion we shall denote it by $G^{\uparrow}$ and further completed by time reversal it is called G. This group differs basically from the symmetry group of the mass zero Dirac equation, which is the conformal group.

3. Since the equation (5) (or its counterpart with P_{+}) exhibits a handedness, we attempt to describe the neutrinos of nature by this mass zero limit rather than by the Dirac equation.

One thus might hope to obtain in a natural way a description of the handedness of neutrinos. In the following the point of view is taken that the symmetry group $G^{\uparrow}$ is the basic group of external symmetry. Time reversal,

which completes it to the symmetry group of (5), will follow as a consequence in the context in which it is used.
Taking then the group $G^{\uparrow}$ as starting point, one can construct, in the way always used for the Poincaré group, unitary representations, and covariant operator fields. The concept of mass does not exist for the group $G^{\uparrow}$ (for the Poincaré group it is a Casimir operator). The group $G^{\uparrow}$ admits as Casimir operators the absolute value of the helicity, and a quantity c, the ratio between the infinitesimal time translation operator - the energy - and the absolute value of the momentum: the "lightcone" is, also for the group $G^{\uparrow}$, a natural feature, as for the Poincaré group. Only here many cones are a priori possible, corresponding to different values of the Casimir operator c.

An irreducible unitary representation of the group $G^{\uparrow}$ can be constructed, for helicity $\frac{1}{2}$ in absolute value, and a given c. A basis in the representation space is spanned by eigenstates of energy, momentum and helicity; both helicities ($\pm\frac{1}{2}$) occur due to the spatial inversion in the group $G^{\uparrow}$. A priori both types of statistics are possible; we prescribe Fermi statistics, and extend the representation to the corresponding Fock space. Two of these unitary representations, with the same value of the Casimir operator c, are necessary to construct a local covariant two-component field, just as in the Poincaré case particles and antiparticles with the same mass are needed. The requirement of locality of the charge density in terms of $\varphi, \dot{\varphi}, \varphi^{+}, \dot{\varphi}^{+}$ again leads to the above mentioned restrictions on the dilatation dimension ℓ (which can always be taken to be real). On the other hand, locality of the equal time anticommutator restricts ℓ differently: $\ell = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$. Only $\ell = 1$ leads to both a local set of equal time anticommutators and a local charge density. The equal time anticommutators are in this case

$$\begin{aligned} \{\varphi(t,\vec{x}), \varphi^{+}(t,\vec{y})\} &= 0 \\ \{\varphi(t,\vec{x}), \dot{\varphi}(t,\vec{y})\} &= i\delta(\vec{x}-\vec{y}) \\ \{\dot{\varphi}(t,\vec{x}), \varphi^{+}(t,\vec{y})\} &= -i\delta(\vec{x}-\vec{y}) \end{aligned} \qquad (9)$$

Note that we took $x^0 = ct$, c the value of the Casimir operator mentioned above.
The anticommutators (9) lead to an indefinite metric in Fock space, a property most important for what follows. The equation of motion of the two component field φ, which contains both helicities due to space-inversion covariance, is the d'Alembert equation, written again with the

value of c corresponding to the representations contained in φ . In the context of the d'Alembert equation, the indefinite metric arising from the imposed Fermi statistics, is not astonishing.
The connection with the 4-component field ψ above can be readily estab.ished:

$$\psi = \begin{pmatrix} \varphi \\ i\dot{\varphi} - \sigma^k i\partial_k \varphi \end{pmatrix} . \tag{10}$$

Thus we achieved a quantization of equation (5).

4. The equation of motion of the free twocomponent field φ follows from a Klein-Gordon type Lagrangian

$$\mathcal{L}_0 = -\varphi^+ i\overleftarrow{\partial}_k i\partial^k \varphi \ .$$

This Lagrangian allows all the operators of $\overset{\uparrow}{G}_+$ as symmetries; its dilatation dimension is 4, leaving the action invariant under dilatation. The quantization rules take the "canonical" form.

$$\begin{aligned} \{\varphi(t,\vec{x}), \pi_\varphi(t,\vec{y})\} &= i\delta(\vec{x}-\vec{y}) \\ \{\varphi^+(t,\vec{x}), \pi_{\varphi^+}(t,\vec{y})\} &= -i\delta(\vec{x}-\vec{y}) \end{aligned} \tag{11}$$

(all other pairs of operators anticommute at equal times).
Another Lagrangian, differing from the preceding one by a divergence, is

$$\mathcal{L}^{(0)} = -(\varphi^+ i\overleftarrow{\partial}_k \sigma^k)(\sigma'^\lambda i\partial_\lambda \varphi) \tag{12}$$

where $\sigma^k \equiv (1, \sigma^k)$, $\sigma'^\lambda \equiv (1, -\sigma^l)$ and therefore

$$\sigma^k \sigma'^\lambda + \sigma^\lambda \sigma'^k = 2g^{k\lambda} \ .$$

The form of the commutation rules (11) is maintained, though with a different meaning for $\pi_\varphi, \pi_{\varphi^+}$. Note further that $L^{(0)+} \neq L^{(0)}$, so that $\pi_{\varphi^+} \neq (\pi_\varphi)^+$.
For convenience use shall be made of $L^{(0)}$, rather than L_0 . It is manifestly invariant under $\overset{\uparrow}{G}_+$, only invariant up to a divergence under space inversion.
The dilatation invariance of the system can be broken most simply by a term proportional to $\varphi^+\varphi$ in $L^{(0)}$, called a "mass" term. The corresponding field still leads to an indefinite metric in Fock space, which pre-

vents a probability interpretation in the usual way.
It is, however, possible to introduce a new metric in the vector space of states, which satisfies the following requirements:

a) the new conjugation of operators defined by the new metric - denoted by $\dagger$ as distinguished from the old + - is locally connected with the old one.
b) the operators of $\overset{\uparrow}{G}_{+}$ (except the already broken dilatation) are unitary also with respect to the new metric.
By a) we prepare the way for a similar discussion for spinor fields in local interaction.
Indeed, a collection of such new conjugations is possible, given by

$$\left(\varphi^{\dagger} \quad -i\left(\pi_{\varphi^{+}}\right)^{\dagger}\right) = \left(\varphi^{+} \quad -i\left(\pi_{\varphi^{+}}\right)^{+}\right) \frac{1}{\alpha^{n}} \begin{pmatrix} -2\sigma^{k} i\overleftarrow{\partial}_{k} & m^{2} \\ 1 & 0 \end{pmatrix}^{n} \tag{13}$$

where n is an arbitrary positive or negative integer, and the arbitrary real constant α has the dimension of a mass.
Note that

$$\begin{pmatrix} -2\sigma^{k} i\overleftarrow{\partial}_{k} & m^{2} \\ 1 & 0 \end{pmatrix}^{-1} = \frac{1}{m^{2}} \begin{pmatrix} 0 & m^{2} \\ 1 & 2\sigma^{k} i\overleftarrow{\partial}_{k} \end{pmatrix}.$$

As a consequence the conjugates of φ^{+} and π_{φ} are also local expressions in terms of φ and $\pi_{\varphi^{+}}$.
The existence of such mew metrics, locally connected with the original one - the "local interpretability" - is made possible by the mass term. However, also for the massless case this line of thought will lead to results, to be discussed in section 6.

5. The choice of interpretable metrics as found in section 4 for free massive particles is further restricted in the presence of electromagnetic interactions.
The gauge invariance of the massive Lagrangian $L^{(o)} - m^{2}\varphi^{+}\varphi$ leads, in the usual way, to the introduction of four potentials, A_{k} and A_{o}. The free part of the Lagrangian pertaining to them has to be $\overset{\uparrow}{G}$ - invariant as well as gauge-invariant, and it turns out to be identical to that of electromagnetism, however, with a light velocity in general different

from the c of the particle field. We therefore make the additional assumption that both limiting velocities are equal, which ensures that all one-photon states, as well as all one-particle states are eigenstates of the S-matrix.

We furthermore assume that no interaction term in the Lagrangian breaks dilatation invariance, so that this invariance is violated by the mass term only.

In the original indefinite metric - associated with + conjugation in the incoming fields - the S-matrix can be formulated in terms of the interaction Hamiltonian as a functional of the incoming (free) fields. Now again one is invited to find a new metric, associated with a new local conjugation, in such a way that rotations and translations are represented by unitary operators, and with respect to which the scattering operator is unitary. The external symmetry operators in the space of asymptotic states being the free ones, one is led by the first conditions to exactly the same choice of new conjugations as was dealt with in the preceding section. Examination of the possible interaction Hamiltonians - corresponding to minimal interaction and Pauli-type terms - yields that the latter are, for no choice of n, hermitean in the new sense, while the minimal coupling term only is hermitean for $n = 1$. A metric corresponding to this value of n indeed is positive definite. It turns out that the S-matrix obtained via this metric is the usual S-matrix of quantumelectrodynamics.

Therefore it is shown that a $\overset{\uparrow}{G}$-covariant spinor field, which by breaking dilatation invariance acquires mass, and which is gauge-invariantly coupled to the electromagnetic field (without further breaking of dilatation invariance) leads to the S-matrix of quantumelectrodynamics, with its additional symmetries up to full Poincaré symmetry. The only acceptable metric happens to have the property, that it is positive definite and that Poincaré transformations which are unitary with respect to it leave the S-matrix invariant.

6. The question arises whether the original field, where dilatation covariance is not broken by a mass term, also permits introduction, in a local way, of a new conjugation, so as to lead to a positive definite metric, accessible to physical interpretation ("principle of local interpretability").

Such a new conjugation should clearly be impossible if one requires the complete group $G^{\uparrow}$ to be represented in a unitary way with respect to the new metric: again part of the symmetry group has to be given up.

Requiring only the hermiticity of energy, momentum, angular momentum and charge in the new metric does nog lead to a new local conjugation of all operators concerned. As φ obeys a Klein-Gordon equation, the independent fields are: $\varphi, \varphi^+, \dot{\varphi}, \dot{\varphi}^+$, but another choice is more convenient:

$$\varphi, \; \sigma^{\prime k} i\partial_k \varphi, \; \varphi^+, \; \varphi^+ \sigma^{\prime k} i\overleftarrow{\partial}_k .$$

Only one pair of fields among these four can be taken as each others new conjugates, under the conditions enumerated above. An example of such a pair is

$$\varphi, \; -\varphi^+ \sigma^{\prime k} i\overleftarrow{\partial}_k ,$$

so that

$$\varphi^\dagger = -\varphi^+ \sigma^{\prime k} i\overleftarrow{\partial}_k , \; -(\varphi^+ \sigma^{\prime k} i\overleftarrow{\partial}_k)^\dagger = \varphi . \qquad (14)$$

Such a singular conjugation is also suggested by (13) in the limit $m \longrightarrow 0$. Because of the fact that $\sigma^{\prime k} i\partial_k$ has zero among its eigenvalues, the operator $\varphi^{+\dagger}$ does not exist in this example. Furthermore, as a consequence of the equations of motion

$$\varphi^\dagger \sigma^{\prime k} i\overleftarrow{\partial}_k = 0 \qquad \text{or} \qquad (\sigma^{\prime k} i\partial_k \varphi)^\dagger = 0 . \qquad (15)$$

The metric corresponding to this conjugation therefore has to be such that some operators (φ^+) do not have a $\dagger$ conjugate, while others ($\sigma^{\prime k} i\partial_k \varphi$) have 0 as their $\dagger$ conjugates. This cannot be realized with a non-singular metric in the full vector space.

It can be realized only in a subspace. The metric in this subspace corresponding to (14), (15) turns out to be such that left-handed particle states and right-handed antiparticle states have a positive norm. Right-handed particle states are outside the subspace, whereas left-handed antiparticle states have norm zero. So the metric corresponding to (14),(15) is semi-definite and defined only in a subspace.

Thus only left-handed particle states and right-handed antiparticle states are physical. The right-handed particle states and left-handed antiparticle states do not contribute to matrix elements of physical operators. They are unphysical and play a role similar to scalar and longitudinal photons in quantumelectrodynamics.

Of course, there are three other equivalent metrics where particle and antiparticle and/or left and right are interchanged.

In view of the fact that the new conjugation is not defined for every field, the original conditions, aiming at hermiticity in the $\dagger$ sense of energy, momentum, angular momentum and charge have to be reconsidered. It turns out that indeed the restrictions of these quantities to the subspace with semi-definite norm, are $\dagger$ hermitean with respect to that norm.
The degrees of freedom that do appear in the expressions of these projected physical quantities are then exactly those of one Weyl field, containing particles of one helicity only, and antiparticles of the opposite helicity. The formalism then again admits all the transformations of the Poincaré group as $\dagger$ unitary transformations.
In fact, the most general metric that is possible in this massless free case is slightly more complicated than the one given by (14),(15), though it is still characterized by the property that only particles of one helicity and antiparticles of the other helicity are physical. It is only by considering interactions - in this case weak interactions with an intermediate boson field - that one is able to fix uniquely the conjugation. This is analogous to the massive case, where the electromagnetic interaction prefered one conjugation out of several possibilities characterized by an integer n of the free case (section 5).
A closer consideration of the S-matrix, obtained via the metric given by (14),(15), teaches that it indeed is unitary and equal to the usual S-matrix of weak interactions.

Summarizing, we have shown that a description of quantumelectrodynamics and of leptonic weak interactions, starting from the group $G^{\uparrow}$ rather than the Poincaré group as group of external symmetries, is possible. The vector space of states and the equations of motion can be described in the framework of $G^{\uparrow}$, and these determine in themselves a unique invariant positive definite metric in the space of states. The Poincaré invariance of the phenomena appears as a result, rather than as an initial assumption. The advantage of this point of view is that the handedness of massless spin one half particles comes in in a natural way, whereas in the usual quantum field theories of weak interactions it is put in by hand, by the choice of the coupling parameters.
The use of $G^{\uparrow}$ as the group of external symmetry seems to have further promising consequences for the problem of combining external and internal symmetries, and for the understanding of superselection rules for lepton number. These aspects will be dealt with in a later communication.

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Amsterdam, 16th November 1972.
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ACKNOWLEDGEMENTS.

Many stimulating discussions with J.Hilgevoord, E.A.de Kerf, G.J.Komen and J.Smit are gratefully acknowledged. One of the authors (S.A.W.) takes this opportunity to thank M.Goldhaber and the Brookhaven National Laboratory for their hospitality during a very early phase of this work. It is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.)", which is financially supported by the "Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.)".
A more detailed account will be presented by E.Mendels to the Proc.Kon. Ned.Ak.v.Wet., Amsterdam.