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Causality Implies the Lorentz Group

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Causality is represented by a partial ordering on Minkowski space, and the group of all automorphisms that preserve this partial ordering is shown to be generated by the inhomogeneous Lorentz group and dilatations.

LET M denote Minkowski space, the real 4-dimensional space-time continuum of special relativity, and let Q denote the characteristic quadratic form on M,

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2,$$
$$x = (x_0, x_1, x_2, x_3) \in M.$$

There is a partial ordering on M given by $x < y$ if an event at x can influence an event at y; more precisely, $x<y$ if $y-x$ is a time vector, $Q(y-x)>0$, oriented towards the future, $x_0 < y_0$. Let $f : M \to M$ be a function that is a one-to-one mapping (*we make no assumptions that f is linear or continuous*). We call f a *causal automorphism* if both f and f^{-1} preserve the partial ordering; in other words,

$$x < y \leftrightarrow fx < fy, \quad \text{all} \quad x, y \in M.$$

The causal automorphisms form a group, which we call the *causality group*.

Let G be the group generated by (i) the orthochronous Lorentz group (linear maps of M that leave Q invariant, and preserve time orientation, but possibly reverse space orientation), (ii) translations of M, and (iii) dilatations of M (multiplication by a scalar).

Theorem. The causality group = G.

Remark 1. The significance of the theorem is that if we interpret the principle of causality mathematically as the set M together with the partial ordering, then the inhomogeneous Lorentz group appears naturally (with dilatations and space reversal) as the symmetry group of M. Therefore the basic invariants of physics, which are the representations of the inhomogeneous Lorentz group, follow natually from the single principle of causality.

Remark 2. It is easy to see that G is contained in the causality group, since the generators of G preserve the partial ordering. The converse is not obvious at first sight, because there seems no reason why a causal automorphism should be linear or even continuous. In fact, the result depends essentially upon space being more than 1-dimensional. If space were 1-dimensional then the causality group would be much larger than G, and the general causal automorphism would map the space and time axes into curved lines, as is shown by the example below. Thus the typical 2-dimensional picture of Minkowski space to be found in most textbooks is misleading.

Remark 3. The condition for f to be a causal automorphism is a global condition, but is equivalent (by an elementary compactness argument using the transitivity of $<$) to the following local condition: given $x \in M$, then there is a neighborhood N of x such that

$$y < z \Leftrightarrow fy < fz, \quad \text{all} \quad y, z \in N.$$

Intuitively this means we need only think of the principle of causality acting in our laboratories for a few seconds, rather than between distant galaxies forever, and still we are able to deduce the Lorentz group.

Remark 4. There is another relation on M given by $x <\cdot\, y$ if light can go from x to y; more precisely $x <\cdot\, y$ if $y - x$ is a light vector, $Q(y - x) = 0$, oriented towards the future, $x_0 < y_0$. The relation $x <\cdot\, y$ is not a partial ordering because it is not transitive,

$$x <\cdot\, y <\cdot\, z \not\Rightarrow x <\cdot\, z.$$

We shall show in Lemma 1 that, in the definition of causal automorphism, it does not matter whether we use $<$ or $<\cdot$ (or both). Intuitively this means that the Lorentz group can be deduced equally well either from causality between heavy particles, or from causality between photons, or from both. Remark 3 also holds for $<\cdot$, although the argument is slightly more complicated due to the lack of transitivity.