The Origin of Dynamical Symmetries in Nonrelativistic Mechanics.

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The basic assumptions underlying nonrelativistic classical and quantum physics are i) space is absolute, Euclidean and 3-dimensional, ii) time is absolute and flows uniformly. In classical mechanics one further assumes that the solution of a problem consists in determining positions as a function of time. Thus path is a basic concept in mechanics and can be suitably extended to quantum theory. In the following we deduce all dynamical symmetries from this principle.

Newton's equations in generalized co-ordinates are $\mathrm{D}v^i/\mathrm{d}t = F^i$, or

$$\frac{\mathrm{d}^2q^i}{\mathrm{d}s^2} + \Gamma^i_{jk}\frac{\mathrm{d}q^j}{\mathrm{d}s}\frac{\mathrm{d}q^k}{\mathrm{d}s} - \frac{\mathrm{d}\alpha^{-1}}{\mathrm{d}t}\frac{\mathrm{d}q^i}{\mathrm{d}s} = \alpha^{-2}F^i\,. \tag{1}$$

The first term refers to the flat-space (¹) background $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}q^i\,\mathrm{d}q^j$; $\Gamma^i{}_{jk}$ are Christoffel symbols which vanish for Cartesian co-ordinates, $v^i = \mathrm{d}q^i|\mathrm{d}t$, $\mathrm{D}|\mathrm{d}t = v^j\nabla_j$, ∇_j being covariant derivative.

Under a projective correspondence (¹,²) $\bar{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + \delta^i{}_{(j}\partial_{k)}\psi$, $\mathrm{d}\bar{s} = \mathrm{d}s\exp[2\psi]$, the geodesic paths of a Riemannian geometry are unchanged, though the metric and curvature in general change. There exists (²) a co-ordinate system: the affine normal co-ordinates which transform under a projective change as $\bar{y}^j = y^j|f(y)$, $\mathrm{d}\bar{s} = \mathrm{d}s|f^2(y)$. If the curvature is unchanged, then $f(y) = \exp[-\psi] = 1 + \boldsymbol{a}\cdot\boldsymbol{y} = \gamma + \delta s$ and

$$\bar{y}^j = y^j|(\gamma + \delta s)\,, \qquad \bar{s} = (\alpha + \beta s)|(\gamma + \delta s)\,, \qquad \alpha\delta - \beta\gamma = 1\,. \tag{2}$$

The importance of these $SL_{2,R}$ transformations for the symmetries of eq. (1) is clear if we recognize that paths given by (1) may be considered as paths in a Riemannian space (³). In particular for $\boldsymbol{F} = -\nabla\varphi$ it is a conformally flat space (⁴,⁵): $\mathrm{d}S^2 = 2(E - V)\,\mathrm{d}s^2$. We shall return to the significance of these transformations later.

In certain Riemannian spaces one can simulate this additive change in Γ by the

(¹) L. P. EISENHART: *Riemannian Geometry* (Princeton, N. J., 1966).

(²) L. P. EISTENHART: *Non-Riemannian Geometry*, American Mathematical Society, Colloquium VIII (1927).

(³) K. H. MARIWALLA: *Journ. Math. Phys.*, **12**, 96 (1971).

(⁴) See, *e.g.*, J. L. SYNGE: *Handbuch der Physik*, Vol. **3**.1 (Berlin, 1960), p. 139; or ref. (⁵), p. 278.

(⁵) L. P. EISENHART: *Continuous Groups of Transformations* (Dover, N. Y., 1961).

We end on a historical note. VAN DANTZIG who invented the name Lie derivative first applied it in physics to electrodynamics and thermodynamics. Recently, Lie derivatives in the context of the first part of this paper have been widely used ([11-13]). In the present work path invariance together with the recognition of the role of dilatation symmetry has permitted a complete classification of symmetry groups of single-term potentials. The method of infinitesimal canonical transformations in the form given here has enough a constructive potential to yield ([14]) invariance groups for any Hamiltonian system.

([11]) I. ILIEV: *Prikl. Mat. Meh.*, **36**, 125 (1972) (in Russian) and references therein.
([12]) M. IKEDA and M. KIMURA: *Math. Jap.*, **16**, 159 (1971) and references therein.
([13]) W. R. DAVIS: in *Lanczos Festschrift* (London, 1973) and references therein.
([14]) K. H. MARIWALLA: to be published.

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