Applications of the concept of strength of a system of partial differential equations

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The concept of "strength" of a system of field equations was introduced by Einstein, and is of such generality that one can compare vastly different systems of field equations. We review here this concept in arbitrary number of dimensions and apply it to some of the well-known equations of physics. We calculate the strength, in arbitrary dimensions, of massless Klein–Gordon equations, Maxwell equations (in both potential and field formulation), and Einstein equations. We also determine the strength of massless Dirac equation and Weyl's neutrino equation for the case of four dimensions. It turns out that the strength for all these equations is identical for space–time dimensionality of four. Other possible applications of this concept are indicated.

Einstein in the formulation of his unified field theory was faced with the problem of showing that the system of field equations he had developed was rich enough (in number of solutions) to describe a variety of situations in electromagnetic and gravitational theories without being too vague. The first such attempt he made was when he published his "Generalized Theory of Gravitation" as Appendix II to his Princeton Lectures on Relativity.[1] The unsatisfactory nature of Einstein's analysis was pointed out to him by E. Strauss, and Einstein therefore gave a revised discussion in the fifth edition of his book.[2] In a further edition[3] published by Princeton University Press (1953), Einstein completely revised his approach and introduced the concept of "strength" of a system of field equations. Here he obtains the result that potential and field formulations of Maxwell equations have different strengths[4] for the dimension four. He apparently realized the paradoxical nature of this result. Consequently, in the following (and the last revised) edition[5] of this book he rewrote this portion completely and presented the concept of strength in a very neat and concise form. The potential formulation of Maxwell's equations is, however, not discussed here. Lucid as Einstein's discussion has been, one would expect a plethora of papers applying this concept to other problems in physics. Actually, there is only one such discussion in the literature[6] generalizing Einstein's discussion to an arbitrary number of dimensions d (Einstein had confined himself to the case of four dimensions). Unfortunately, however, this paper contains errors which give a completely wrong picture, of the method of approach. Considering also some of the new applications that we have in mind we give here a complete review of this method and apply it to some well known massless wave equations. Plan of the paper is the following. We first consider simple cases in lower dimensions to give credibility to the terms we introduce and thus arrive at a heuristic definition of the concept of strength of a system of field equations. Next we consider some of the well-known equations of physics in arbitrary number of dimensions.

If we have the first order differential equation

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\phi(x,t)=0,$$

we know its solutions have the form $\phi(x,t)=F(x+t)$; for the second order equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\phi(x,t)=0,$$

the solutions are somewhat more general of the type $\phi(x,t)=F_1(x+t)+F_2(x-t)$. We say that the first system determines the solutions more strongly than the second system. This statement generalizes to the d-dimensional analogs of these equations ($\partial_\mu=\partial/\partial x_\mu$):

$$\sum_{\mu=1}^{d}\partial_\mu\phi(x_1,\ x_2,\ldots,\ x_d)=0, \tag{1}$$

$$\sum_{\mu=1}^{d-1}(\partial_\mu^2-\partial_d^2)\phi(x_1,x_2,\ldots,x_d)=0. \tag{2}$$

We are interested in obtaining a numerical characterization of the concept of "strength" that would enable us to compare different systems of equations as in the above examples. Suppose we have an analytic field function of d variable. We can expand it in Taylor series and the totality of its coefficients describe the field completely. Let us consider the terms of the nth order of differentiation in the Taylor expansion; these are of the form

$$\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_N}\phi,\quad \mu_j=1,\,2,\,\cdots,\,d\ ,\quad \sum_1^N k = n$$

and the total number of such coefficients is

$$\begin{bmatrix} d \\ n \end{bmatrix}\equiv\frac{(d+n-1)!}{(d-1)!\cancel{N!}}\equiv\binom{d+n-1}{n}. \qquad \cdot\, n!$$

If the function satisfies some field equations or constraints, these give several relations between the nth order coefficients, thereby reducing the number of coefficients left free to be assigned arbitrary values. If we denote the number of coefficients thus left free by Z_n, then it is clear that from our viewpoint the quantity of importance is the fraction of the total number of nth order coefficients left free, i.e., $Z_n/[{}^{d}_{n}]$. For instance, if the field equation satisfied by our function is (1), then $(n-1)$-fold differentiation of it gives $[{}^{\,d}_{n-1}]$ relations

$$\sum_{\mu=1}^{d}\partial_\mu\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{n-1}}\phi=0$$

between the nth order coefficients. In particular for $d=2$, these relations are

$$\frac{\partial\phi}{\partial t}=\frac{-\partial\phi}{\partial x},\quad \frac{\partial^2\phi}{\partial t^2}=\frac{-\partial^2\phi}{\partial x\,\partial t}=\frac{\partial^2\phi}{\partial x^2},\quad \text{etc.},$$

and when substituted back into the Taylor series yield the solutions