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The quasi-normal modes of a black hole represent solutions of the relevant perturbation equations which satisfy the boundary conditions appropriate for purely outgoing (gravitational) waves at infinity and purely ingoing waves at the horizon. For the Schwarzschild black hole the problem reduces to one of finding such solutions for a one-dimensional wave equation (Zerilli's equation) for a potential which is positive everywhere and is of short-range. The notion of quasi-normal modes of such one-dimensional potential barriers is examined with two illustrative examples; and numerical solutions for Zerilli's potential are obtained by integrating the associated Riccati equation.

1. Introduction

It is known that the evolution of an arbitrary perturbation of the metric coefficients of the Schwarzschild black hole can be fully described in terms of the reflexion (R) and the transmission (T) coefficients of the one-dimensional barrier represented by Zerilli's potential (Zerilli 1970; see also Chandrasekhar 1975). Nevertheless, the notion of *quasi-normal modes* of a black hole has been introduced in the literature in analogy with the normal modes of oscillation of a star. In the context of a black hole these quasi-normal modes are defined as proper solutions of the perturbation equations belonging to certain complex characteristic frequencies which satisfy the boundary conditions appropriate for purely ingoing waves at the horizon and purely outgoing waves at infinity.

It does not appear that the quasi-normal modes of a black hole serve the same purposes as the normal modes of oscillation of a star. Consider, for example, the spherically symmetric perturbations of an initially static configuration either in the Newtonian (Eddington 1918, 1919) or in the relativistic (Chandrasekhar 1964) framework. In either framework, the determination of the characteristic frequencies leads to a two-point boundary-value problem of the classical Sturmian type for a self-adjoint second order differential equation. Consequently, the associated proper solutions (i.e. the normal modes) form a *complete set* in the sense that any arbitrary spherically symmetric perturbation of the star (compatible with the boundary conditions of the problem) can be expressed as a linear superposition of the normal modes. And, therefore, the evolution of any such perturbation can be followed in terms of the normal modes and the characteristic frequencies to which they belong. Also, it follows that if any of the modes should belong to a (purely) imaginary