ELECTROMAGNETISM AND GRAVITATION.(*)

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Electromagnetism and geometry.

1. In the present section we shall consider the Maxwell equations in a space endowed only with a structure of differentiable manifold, but without affine connection and metric. The number of dimensions of the space will not be assumed to be four, but taken as an integer n. Our aim will be the fixation of the geometry of our manifold by the requirements of the development of the electromagnetic theory. Thus we can get a clearer foundation for geometry than in the general theory of relativity. We shall see that the affine connection, the local indefinite minkowsian metric, even the dimension of the space-time manifold are grounded in electromagnetism. The Einstein equations for the metric tensor $g_{\mu\nu}(x)$ come in as equations for the determination of the dielectric and magnetic permeability properties of space. We have already indicated the relation between those properties and the metric in our communication to the Kyoto Conference in September 1965.(1) The possibility of an electromagnetic foundation of geometry was discussed by us in an unpublished lecture at the Blumenau meeting of the Associação Brasileira para o Progresso da Ciencia in July 1966. The discussion of the present lecture includes some new developments.

It is well known that the Maxwell equations can be written in a space endowed only with a structure of differentiable manifold with a sufficiently large dimension, provided two fields be used: a field $F_{\mu\nu}(x)$ described by an antisymmetric covariant tensor of the second order satisfying the homogeneous Maxwell equation

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0 \tag{1a}$$

(∂_μ = partial derivative with respect to the coordinate x^μ)

(*) Lecture given at the Institut Henri Poincaré in 15/11/ 1967.

and a second field described by a contravariant antisymmetric tensor density of the second order $\mathfrak{F}^{\mu\nu}(x)$ satisfying the inhomogeneous Maxwell equation

$$\partial_\nu \mathfrak{F}^{\mu\nu} = \mathfrak{J}^\mu \tag{1b}$$

$\mathfrak{J}^\mu$ denoting the charge and current vector density. The equations (1a) and (1b) involve only ordinary partial derivatives, not covariant ones.

We can now immediately see that the case of the dimension $n=4$ is distinguished by the possibility of replacing the tensor density $\mathfrak{F}^{\mu\nu}$ by an antisymmetric covariant tensor $\hat{F}_{\mu\nu}$

$$\hat{F}_{\mu\nu} = 1/2\ \varepsilon_{\mu\nu\rho\sigma} \mathfrak{F}^{\rho\sigma} \tag{2}$$

$\varepsilon_{\mu\nu\rho\sigma}$ being the Ricci antisymmetric symbol with four indices, because of the four-dimensionality of the differentiable manifold. We shall postulate that the field can be described by the two tensors F and $\hat{F}$. This already gives an electromagnetic foundation for the four-dimensionality. It will be seen later that $\hat{F}$ is the minkowskian dual of F.

Equations (1a) and (1b) are not sufficient for the determination of F and $\mathfrak{F}$. The construction of the electromagnetic theory requires a one-one linear correspondence between the vector spaces of the F and the $\mathfrak{F}$ corresponding to the same point of the manifold

$$\mathfrak{F}^{\mu\nu}(x) = \mathfrak{b}^{\mu\nu,\rho\sigma}(x)\, F_{\rho\sigma}(x) \quad ; \quad F_{\mu\nu}(x) = \mathfrak{b}_{\mu\nu,\rho\sigma}(x)\, \mathfrak{F}^{\rho\sigma}(x) \tag{3}$$

$\mathfrak{b}^{\mu\nu,\rho\sigma}$ is a fourth order tensor density describing the properties of dielectricity and magnetic permeability of space. [redacted] It is known that $\mathfrak{b}^{\mu\nu,\rho\sigma}$ must be symmetric with respect to the two pairs of indices $(\mu\nu), (\rho\sigma)$

$$\mathfrak{b}^{\mu\nu,\rho\sigma} = \mathfrak{b}^{\rho\sigma,\mu\nu} \tag{4}$$

$\mathfrak{b}$ is antisymmetric with respect to the two indices of the same pair

$$\mathfrak{b}^{\mu\nu,\rho\sigma} = -\mathfrak{b}^{\nu\mu,\rho\sigma} \quad ; \quad \mathfrak{b}^{\mu\nu,\rho\sigma} = -\mathfrak{b}^{\mu\nu,\sigma\rho} \tag{5}$$

It follows from the one-one nature of the correspondence (3) that we may define $\mathfrak{b}_{\mu\nu,\rho\sigma}$

by the condition

$$\mathfrak{b}^{\mu\nu,\rho\sigma}\,\mathfrak{b}_{\kappa\lambda,\rho\sigma} = 1/2\,\delta^{\mu\nu}_{\kappa\lambda} \quad , \quad \left(\delta^{\mu\nu}_{\kappa\lambda} = \delta^{\mu}_{\kappa}\delta^{\nu}_{\lambda} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\kappa}\right) \qquad (6)$$

It follows, (under certain conditions,) from equation (1a) that there is a covariant vector potential A such that

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \qquad (7)$$

We get from (1b) the set of n [redacted] partial differential equations for the A_μ

$$\partial_\nu\left(\mathfrak{b}^{\mu\nu,\rho\sigma}(x)\left\{\partial_\rho A_\sigma(x) - \partial_\sigma A_\rho(x)\right\}\right) = \mathfrak{J}^{\mu}(x) \qquad (8)$$

Equation (8) shows that in order to determine the potential vector $A_\mu(x)$ we must know the tensor density $\mathfrak{b}^{\mu\nu,\rho\sigma}(x)$. Thereby we need equations for the deter- of the tensor density $\mathfrak{b}^{\mu\nu,\rho\sigma}$ describing the properties of dielectricity and magnetic permeability of space, to be used together with the Maxwell equations (1a) - (1b). The necessity of such equations is generally overlooked, because the electromagnetic field is taken in the frame of a space- time endowed with a metric of a locally minkowskian type, so that the $\mathfrak{b}^{\mu\nu,\rho\sigma}(x)$ can be derived from the metric tensor $g^{\mu\nu}(x)$

$$\mathfrak{b}^{\mu\nu,\rho\sigma} = 1/2\sqrt{-g}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) \qquad (g = \text{determinant of } g_{\mu\nu}) \quad (9)$$

because $\mathfrak{F}^{\mu\nu} = \sqrt{-g}\,F^{\mu\nu}$ with $F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} = 1/2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right)F_{\rho\sigma}$.

[illegible]

Equation (9) shows that the physically acceptable tensor densities $\mathfrak{b}^{\mu\nu,\rho\sigma}$ must satisfy (in the case of n = 4) a [redacted] quadratic equation not involving the $g_{\mu\nu}$

$$\mathfrak{b}^{\mu\nu,\rho\sigma}\,\varepsilon_{\rho\sigma\tau\omega}\,\mathfrak{b}^{\tau\omega,\kappa\lambda} = -\varepsilon^{\mu\nu\kappa\lambda} \qquad (10)$$

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The sign minus in the right hand side of equation (10) came from that of $\sqrt{-g}$ in (9). It is therefore characteristic of the $g_{\mu\nu}$ with the Minkowski signature, the only kind having a negative determinant g.

In the Appendix we prove the following basic theorem:

The equation (10) is a necessary and sufficient condition in order that a tensorial density $\mathfrak{b}^{\mu\nu,\rho\sigma}$ with the symmetry properties (4) - (5) be built with a symmetric tensor $g^{\mu\nu}$ having the minkowskian signature. $g^{\mu\nu}$ is determined up to a scalar factor by $\mathfrak{b}^{\mu\nu,\rho\sigma}$.

It follows from the above theorem that the tensor density field $\mathfrak{b}^{\mu\nu,\rho\sigma}(x)$ describing the properties of dielectricity and magnetic permeability of space, which must satisfy equations (4), (5) and (10), determines up to a scalar factor $s(x)$ a field $g^{\mu\nu}(x)$ that defines a metric with the right Minkowski signature. Therefore the properties of dielectricity and magnetic permeability of space allow to define a physically satisfactory conformal geometry in our four-dimensional differentiable manifold. All the tensors $s(x)\, g^{\mu\nu}(x)$ determine at every point the same metric of the angles, although not the same gauge of length.

We started with the tensor density $\mathfrak{b}^{\mu\nu,\rho\sigma}$, because it is sufficient to write down the Maxwell field equations. For the purpose of the construction of the geometry of the space-time it appears more convenient to assume as a basic postulate that those properties are described by a tensor field $c^{\mu\nu,\rho\sigma}(x)$ such that

$$\mathfrak{b}^{\mu\nu,\rho\sigma}(x) = S(x)\, c^{\mu\nu,\rho\sigma}(x) \qquad \big(S(x) = \text{scalar density}\big) \tag{11}$$

The $\mathfrak{b}^{\mu\nu,\rho\sigma}$ may be taken as elements of a 6x6 matrix, by restricting the indices: $\mu<\nu$ and $\rho<\sigma$. The double indices $\mu\nu$ and $\rho\sigma$ will be arranged as usual in alphabetic order. It follows from equation (10) that the determinant of the matrix $\mathfrak{b}$ is 1. In a similar way we associate a matrix to $c^{\mu\nu,\rho\sigma}$ and denote its determinant by

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C. Equation (11) shows the $\mathcal{C}$ - matrix is the product of the C - matrix by S. Hence

$$C\,S^6 = 1 \quad ; \quad \mathcal{C}^{\mu\nu,\rho\sigma}(x) = C(x)^{-1/6}\, C^{\mu\nu,\rho\sigma}(x) \tag{12}$$

$\mathcal{C}^{\mu\nu,\rho\sigma}(x)$ is completely determined by the tensor $C^{\mu\nu,\rho\sigma}(x)$ and we can now determine completely the tensor $g^{\mu\nu}(x)$ of the expression of $\mathcal{C}^{\mu\nu,\rho\sigma}(x)$, given by the basic theorem, by imposing the condition

$$C(x) = g(x)^{-3} \tag{13}$$

in order that

$$C^{\mu\nu,\rho\sigma}(x) = 1/2\left(g^{\mu\rho}(x)\, g^{\nu\sigma}(x) - g^{\mu\sigma}(x)\, g^{\nu\rho}(x)\right) \tag{14}$$

Thus we get a physically satisfactory metric field $g^{\mu\nu}(x)$. $C^{\mu\nu,\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is the metric of the $F_{\mu\nu}$ corresponding to the metric $g^{\mu\nu} V_\mu V_\nu$ of the covariant vectors V_μ.

The above electromagnetic metric $g_{\mu\nu}$ gives us immediately the affine connection whose components $\Gamma^\rho_{\lambda\mu}$ are the Christoffel symbols $\left\{ {\rho \atop \lambda\mu} \right\}$ built with those $g_{\mu\nu}$

$$\Gamma^\rho_{\lambda\mu} = \left\{ {\rho \atop \lambda\mu} \right\} = 1/2\, g^{\rho\alpha}\left(\partial_\lambda g_{\mu\alpha} + \partial_\mu g_{\lambda\alpha} - \partial_\alpha g_{\lambda\mu}\right) \tag{15}$$

This affine connection Γ determines the (electromagnetic) covariant differentiation of the tensors.

Our philosophy that the geometry of the space-time is based on electromagnetism leads to the identification of the electromagnetic $g_{\mu\nu}(x)$ with the actual metric tensor of the physical space-time. On assuming that the actual metric of the space-time is determined by the Einstein gravitational equations

$$R_{\mu\nu} - 1/2\, g_{\mu\nu}\, R = 8\pi K\, T_{\mu\nu} \qquad (T_{\mu\nu} = \text{total stress tensor of matter} \,;\, K = G/c^4) \tag{16}$$

these will be the equations for the determination of the electromagnetic $g_{\mu\nu}(x)$ and (also of) the tensor $C^{\mu\nu,\rho\sigma}(x)$ describing the properties of dielectricity and magnetic permeability of space. Equations (1) and (16) are thus the complete set of (differential) equations of the electromagnetic theory in a space endowed only with an a priori structure of differentiable manifold.

Our electromagnetic construction of a Riemannian geometry suggests naturally that the differential equations for the $g_{\mu\nu}$ of the dielectricity and magnetic permeability of space are of the form

$$R_{\mu\nu} - 1/2\, g_{\mu\nu} R + K g_{\mu\nu} = U_{\mu\nu} \qquad (K = \text{constant}) \tag{17}$$

$U_{\mu\nu}$ being some symmetric tensor of divergence 0 according to the electromagnetic covariant differentiation. The choice of the expression in the left hand side of (17) is justified, because it [redacted] is a covariant combination of the $g_{\mu\nu}$ and their first and second order derivatives and leads to the right number of independent equations, taking into account the Hilbert condition imposed by the arbitrariness of the curvilinear coordinates by the fact of having covariant divergence 0. Since we want that the metric of the [redacted] space-time be that defined by the electromagnetism, the natural choice of $U_{\mu\nu}$ is [redacted] $U_{\mu\nu} = \alpha\, T_{\mu\nu}$ with α constant, since thus the condition of divergence 0 of $U_{\mu\nu}$ corresponds to the laws of conservation of total energy and momentum. The experimental confontration shows that we can take $K = 0$ and $\alpha = 8\pi\kappa$, κ being the gravitational constant of the Einstein equations (16). Thus we can get the Einstein equations from electromagnetic considerations. The curvature of the universe endowed with the electromagnetic geometry accounts for the gravitational effects.

The relation between the properties of dielectricity and magnetic permeability of space and the gravitational field are intuitively seen in the case gravitational of the deflexion of light rays. From a naive optical point of view such a deflection appears as due to a variation of the refractive properties of space, thereby to a variation of the dielectricity and magnetic permeability.

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$\mathfrak{b}^{\mu\nu,\rho\sigma}$ <u>and the tensor duality.</u>

2. The tensor density $\mathfrak{b}^{\mu\nu,\rho\sigma}$ defines a linear operator D on the antisymmetric covariant tensors $F_{\mu\nu}$ as vectors of a six-dimensional vector space

$$\hat{F}_{\mu\nu} = (D F)_{\mu\nu} = 1/2\, \varepsilon_{\mu\nu\rho\sigma} \mathfrak{b}^{\rho\sigma,\tau\omega} F_{\tau\omega} \quad ; \quad \hat{F}_{\mu\nu} = D^{\rho\sigma}_{\mu\nu} F_{\rho\sigma} \text{ with } D^{\rho\sigma}_{\mu\nu} = 1/2\, \varepsilon_{\mu\nu\tau\omega} \mathfrak{b}^{\rho\sigma,\tau\omega} \tag{18}$$

<u>The condition</u> (10) <u>means simply that $-D^2$ is the operator unity 1_{op} on the F</u>

$$D^2 F = - F \quad ; \quad D^2 = - 1_{op} \tag{19}$$

The Ricci tensor density $\varepsilon^{\kappa\lambda\mu\nu}$ defines a [redacted] symetric bilinear form $\left(F^{(1)}, F^{(2)}\right)$ of the six- vectors F

$$\left(F^{(1)}, F^{(2)}\right) = \left(F^{(2)}, F^{(1)}\right) = 1/4\, \varepsilon^{\kappa\lambda\mu\nu} F^{(1)}_{\kappa\lambda} F^{(2)}_{\mu\nu} \tag{20}$$

It is easily seen that the symmetry condition (4) means that

$$\left(F^{(1)}, D F^{(2)}\right) = \left(F^{(2)}, D F^{(1)}\right) \tag{21}$$

The symmetry condition (4) implies that the bilinear form $1/2\, \mathfrak{b}^{\kappa\lambda,\mu\nu} F^{(1)}_{\kappa\lambda} F^{(2)}_{\mu\nu}$ is symmetric in $F^{(1)}$ and $F^{(2)}$. It is $\left(F^{(1)}, D F^{(2)}\right)$.

It follows from (19) and (21) that

$$\left(D F^{(1)}, D F^{(2)}\right) = - \left(F^{(1)}, F^{(2)}\right) \tag{22}$$

F is a simple bivector when it is the outer product $\underbrace{\text{of}}_{}$ $U^{(a)} \wedge U^{(b)}$ two four - vectors U : $F_{\mu\nu} = U^{(a)}_{\mu} U^{(b)}_{\nu} - U^{(a)}_{\nu} U^{(b)}_{\mu}$. A necessary and sufficient condition for F to be a simple bivector is $(F , F) = 0$. <u>It follows from</u> (22) <u>that DF is a simple bivector when F has this property.</u> When $F^{(1)}$ and $F^{(2)}$ are simple bivectors , a necessary and sufficient condition for the existence of three four - vectors $U^{(a)}$, $U^{(b)}$ and $U^{(c)}$ such that $F^{(1)} = U^{(a)} \wedge U^{(b)}$ and $F^{(2)} = U^{(a)} \wedge U^{(c)}$ [redacted] is that $\left(F^{(1)}, F^{(2)}\right) = 0$. <u>Therefore</u> $D F^{(1)}$ <u>and</u> $D F^{(2)}$ <u>will be two simple bivectors built in the above way with three four - vectors W when</u> $F^{(1)}$ <u>and</u> $F^{(2)}$ [redacted] <u>are simple bivectors built in the same way with three four- vectors</u> U .

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It follows from equations (18) and (9) that

$$D F = {}^*F \text{ with } {}^*F_{\mu\nu} = 1/2 \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, g^{\rho\tau} g^{\sigma\omega} F_{\tau\omega} \tag{23}$$ [redacted]

*F is the dual of F corresponding to the metric $g_{\mu\nu}$, with the ordinary definition , $g_{\mu\nu}$ being now a symmetric tensor associate to $\mathfrak{b}^{\mu\nu,\rho\sigma}$. <u>D is the-</u>

refore the ~~the~~ linear operator on the F corresponding to the minkowskian kind of tensor duality. The tensor density $\mathcal{C}^{\mu\nu,\rho\sigma}$ defines directly the usual duality of the antisymmetric tensors F by means of (18), because of the conditions (4), (5) and (10). Thus we have obtained the geometric interpretation of $\mathcal{C}^{\mu\nu,\rho\sigma}$ and of those conditions.

The tensor $\hat{F}$ defined by equation (2) is simply the *F corresponding to the tensor density $\mathcal{C}$. The electromagnetic field is described by the pair of dual tensors F, *F related through the linear operator D of square -1_{op} satisfying the symmetry condition (21).

The definition of the tensor duality by means of a tensor density $\mathcal{C}^{\mu\nu,\rho\sigma}$ is more satisfactory than the ordinary definition (23) in terms of a metric tensor $g^{\mu\nu}$, for *F is not changed when $g^{\mu\nu}$ is replaced by $S\, g^{\mu\nu}$: the tensor duality is related to the conformal geometry of the space-time and $\mathcal{C}^{\mu\nu,\rho\sigma}$ too.

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Let us consider a tensor L^{μ}_{ρ} such that $L^{\mu}_{\rho}\, g^{\rho\sigma}\, L^{\nu}_{\sigma} = g^{\mu\nu}$ with determinant L. L^{μ}_{ρ} describes obviously a local Lorentz transformation. The inverse Lorentz transformation is described by the tensor $\underline{L}^{\sigma}_{\nu}$ with $L^{\mu}_{\rho}\,\underline{L}^{\rho}_{\nu} = \delta^{\mu}_{\nu}$. It is easily seen that

$$L^{\alpha}_{\mu}\, L^{\beta}_{\nu}\, \mathcal{C}^{\mu\nu,\rho\sigma}\, L^{\gamma}_{\rho}\, L^{\delta}_{\sigma} = \mathcal{C}^{\alpha\beta,\gamma\delta} \quad \text{and} \quad \underline{L}^{\mu}_{\alpha}\, \underline{L}^{\nu}_{\beta}\, D^{\rho\sigma}_{\mu\nu}\, L^{\kappa}_{\rho}\, L^{\lambda}_{\sigma} = L^{-1}\, D^{\kappa\lambda}_{\alpha\beta} \tag{24a}$$

Hence

$$\underline{L}^{\mu}_{\alpha}\,\underline{L}^{\nu}_{\beta}\, D^{\gamma\delta}_{\mu\nu} = L^{-1}\, D^{\kappa\lambda}_{\alpha\beta}\,\underline{L}^{\gamma}_{\kappa}\,\underline{L}^{\delta}_{\lambda} \quad \text{so that} \quad L_{op} D = L^{-1} D\, L_{op} \quad \text{with} \quad \left(L_{op}F\right)_{\mu\nu} = \underline{L}^{\rho}_{\mu}\,\underline{L}^{\sigma}_{\nu}\, F_{\rho\sigma} \tag{24b}$$

L_{op} denoting the linear operator on the F giving the Lorentz transformation $L^{\alpha}_{\mu} V^{\mu}$ of the V^{μ} and $\underline{L}^{\nu}_{\alpha} U_{\nu}$ of the U_{ν}. In the case of the transformations of the continuous Lorentz group $L = +1$ so that

$$L_{op}\, D = D\, L_{op} \qquad (L = +1) \tag{24c}$$

In the case of an infinitesimal Lorentz transformation $L^{\mu}_{\rho} = \delta^{\mu}_{\rho} + \theta\, g^{\mu\alpha} F^{(L)}_{\rho\alpha}$, θ infinitesimal, and $\underline{L}^{\sigma}_{\nu} = \delta^{\sigma}_{\nu} - \theta\, g^{\sigma\alpha} F^{(L)}_{\nu\alpha}$ so that

$$L_{op} F = F + \theta\, F^{(L)} \times F \quad \text{with} \quad \left(F^{(L)} \times F\right)_{\mu\nu} = g^{\rho\sigma}\left(F^{(L)}_{\mu\rho} F_{\nu\sigma} - F^{(L)}_{\nu\sigma} F_{\mu\rho}\right) \tag{25}$$

We may write

$$L_{op} = 1_{op} + \mathrm{Ad}\, F^{(L)} \quad \text{with} \quad \left(\mathrm{Ad}\, F^{(L)}\right) F = F^{(L)} \times F \tag{25a}$$

The product $F^{(L)} \times F$ is simply the vector product of the Lie algebra of the

Lorentz group. Ad $F^{(L)}$ is the linear operator on the F, taken as the vectors of that Lie algebra, corresponding to the vector $F^{(L)}$ in the adjoint representation.

We get from (24c) for any F the equation

$$D(\mathrm{Ad}\ F) = (\mathrm{Ad}\ F)D \qquad (24d)$$

which shows the important role of D in the Lie algebra of the Lorentz group.

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The above considerations show that the Lorentz group, which defines the local symmetry of the space-time, appears as closely related to the properties of dielectricity and magnetic permeability of space. The Lie algebra of that group is particularly electromagnetic, since its vectors are the antisymmetric tensors $F_{\mu\nu}$ and its structure is defined by the $g_{\mu\nu}$ through the product $F^{(L)} \times F$, hence essentially by the tensor density $\mathfrak{e}^{\mu\nu,\rho\sigma}$. We are thus led to the idea that the system of the possible electromagnetic fields at each point of the space-time is to be viewed as a Lie algebra rather than as a vector space. The local geometry would then be a consequence of the intrinsic properties of the field Lie algebra. The relations between the electromagnetic algebras at different points would then lead to the global (geometric) properties through linear connections etc. We shall develop this point of view in later sections of this paper.

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Tensor duality and space-time four-dimensionality.

2a. In the n-dimensional metric geometry there is a duality between the antisymmetric covariant tensors of the orders p and n - p

$$*U_{a_1,\dots,a_p} = \sqrt{\pm g}\ \varepsilon_{a_1,\dots,a_n}\ g^{a_{p+1}b_1} \dots\dots g^{a_n b_{n-p}}\ U_{b_1,\dots,b_{n-p}} \big((n-p)!\big)^{-1} \qquad (26_a)$$

The construction of $*U$ requires both the metric tensor g^{ab} and the Ricci symbol $\varepsilon_{a_1,\dots,a_n}$. We may also write

$$*U_{a_1,\dots,a_p} = \pm \left(\pm g\right)^{-1/2}\ \varepsilon^{c_1,\dots,c_p,b_1,\dots,b_{n-p}}\ g_{a_1c_1} \dots g_{a_pc_p}\ U_{b_1,\dots,b_{n-p}} \big((n-p)!\big)^{-1} \qquad (26b)$$

using the Ricci tensor density $\varepsilon^{a_1,\dots,a_n}$ and g_{ab}.

For even $n = 2m$, there is a self-duality of the tensors of order m, since $m = n - m$. A self-duality of the covariant antisymmetric tensors of the second or-

~~[illegible]~~

der exists only in a four-dimensional space, since $2m = 4$. It is interesting to note that the self-duality of the antisymmetric tensors of order m in a 2m-dimensional space is a conformal property, because of the invariance of $^*U$ for the change of gauge $g_{ab} \to S\, g_{ab}$. In all the other cases there is no such invariance.

Equation (26b) shows clearly that the tensor duality is the product of two linear mappings: $U_{b_1,\dots,b_{n-p}} \to \mathcal{U}^{c_1,\dots,c_p} = \left((n-p)!\right)^{-1} \varepsilon^{c_1,\dots,b_{n-p}}\, U_{b_1,\dots,b_{n-p}}$; $\mathcal{U}^{c_1,\dots,c_p} \to {}^*U_{a_1,\dots,a_p} = \pm\, g_{a_1c_1}\dots g_{a_pc_p}\, \mathcal{U}^{c_1,\dots,c_p} (\pm g)^{-1/2}$. The first mapping does not involve the tensor g_{ab}, but only the Ricci tensor density $\varepsilon^{c_1,\dots,c_p,b_1,\dots,b_{n-p}}$. The second mapping is essentially a lowering of p tensorial indices by means of the metric tensor g_{ab}. The dimensionality of the space comes in more characteristically in the first mapping, the raising of indices by means of the Ricci tensor density ε.

In the case of the self-dualities, corresponding to even $n = 2m$, the Ricci tensor density $\varepsilon^{a_1,\dots,a_{2m}}$ allows to build a bilinear form of the antisymmetric covariant tensors $U_{a_1,\dots,a_m}$ similar to $\left(F^{(1)}, F^{(2)}\right)$

$$\left(U^{(1)}, U^{(2)}\right) = (m!)^{-2}\, \varepsilon^{a_1,\dots,a_{2m}}\, U^{(1)}_{a_1,\dots,a_m} U^{(2)}_{a_{m+1},\dots,a_{2m}} = (-1)^m \left(U^{(2)}, U^{(1)}\right)$$

The bilinear form of the $U_{a_1,\dots,a_m}$ is symmetric for even values of m and antisymmetric for odd values of m. The $U_{a_1,\dots,a_m} \to \mathcal{U}^{b_1,\dots,b_m}$ linear mapping has [redacted] different natures in the cases of even and odd values of m, and the tensor self-duality too.

It is easily seen that

$$^{**}U_{a_1,\dots,a_p} = (-1)^{p(n-p)} (\mathrm{sign}\ g)\, U_{a_1,\dots,a_p} \tag{27}$$

$$^{**}U_{a_1,\dots,a_m} = (-1)^m (\mathrm{sign}\ g)\, U_{a_1,\dots,a_m} \quad \text{for } n = 2m \tag{27a}$$

We can introduce a linear operator D on the $U_{a_1,\dots,a_m}$ defined by the condition $D\,U = {}^*U$. It follows from (27a) and the definition of $\left(U^{(1)}, U^{(2)}\right)$ that

$$D^2 = (-1)^m (\mathrm{sign}\ g)\, 1_{op}\ ; \quad \left(U^{(1)}, D\,U^{(2)}\right) = (-1)^{m^2} \left(D\,U^{(1)}, U^{(2)}\right) \tag{27 b}$$

Equation (27 b) shows that $D^2 = 1_{op}$ for the non minkowskian metrics of the four-dimensional space, so that the tensor duality is involutory. Thus we get the

Fundamental theorems of the local geometry.

Theorem I . The four-dimensionality of the physical spatial differentiable manifold is determined by the condition of self-duality of the vector spaces of the antisymmetric covariant tensors of the second order at the different points of the manifold, which is required by the description of the electromagnetic fields by a pair of dual tensors F and $*F = DF$, the non singular linear operator D describing the spatial properties of dielectricity and magnetic permeability.

Theorem II . The local Minkowski metric and the nature of the physical spatial differential manifold as a space - time are determined by the conditions that the D duality of the F and $*F$ be non involutory : $**F \neq F$ for $F \neq 0$.

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Geometry of the tensor duality.

2b. The self-dualities of the $2m$ - dimensional spaces are closely related to the properties of the flat m- dimensional manifolds of the tangent spaces .The flat m - dimensional manifolds through the point of contact of such a tangent space can be described by the simple covariant m - vectors , which are outer products of m covariant vectors $\vec{U}^{(1)}\wedge\dots\wedge\vec{U}^{(m)}$. The components of the simple covariant m - vectors are the antisymmetrical covariant tensors of order m $\delta^{b_1,\dots,b_m}_{a_1,\dots,a_m} U^{(1)}_{b_1}\dots U^{(m)}_{b_m}$ with $\delta^{b_1,\dots,b_m}_{a_1,\dots,a_m}$ defined as [redacted] $(m!)^{-1}\, \varepsilon^{b_1,\dots,b_m,c_1,\dots,c_m}\, \varepsilon_{a_1,\dots,a_m,c_1,\dots,c_m}$ by means of the Ricci symbols. The linear operator D of the self-duality transforms a simple covariant m- vector into another simple covariant m - vector. Let us now introduce [redacted] a set of basic orthonormed covariant vectors , in order that $g_{ab} = e_a \delta_{ab}$ with $e_a = 1$ or -1 . Now equation (26 a) has the simple form (for $p = m$

$$*U_{a_1,\dots,a_m} = \sqrt{\pm g}\; e_{a_{m+1}}\dots e_{a_{2m}}\; U_{a_{m+1},\dots,a_{2m}} (m!)^{-1} \quad \text{with } \varepsilon_{a_1,\dots,a_{2m}} = 1 \tag{26c}$$

The flat manifolds of maximal dimensionality m lying on the isotropic or null cone of equation $g_{ab} X^a X^b = 0$ (in cartesian coordinates X) are described by the simple covariant m- vectors obtained as outer products of m null linearly independent covariant vectors . The $U_{a_1,\dots,a_m}$ of those flat m - manifolds are eigenvectors of D and any simple m-vector which is eigenvector of D describes such a manifold. The two eigenvalues of D correspond to the two families flat m- dimensional manifolds on the null cone. We may say that the self-duality leaves invariant the flat m- manifolds on the null cone.

The antisymmetric covariant tensor $U_{a_1,..,a_p}$ describes a simple covariant p-vector when it is the outer product $\vec{U}^{(1)} \wedge \ldots \wedge \vec{U}^{(p)}$ of p linearly independent covariant vectors. It describes the (n-p) - dimensional flat manifold of the tangent space through the point of contact obtained by the intersection of the p hyperplanes described by the covariant vectors $U^{(1)},..,U^{(p)}$. The metric tensor allows to associate a contravariant vector $V^a = g^{ab} U_b$ to the covariant vector $\vec{U}$. The p linearly independent contravariant vectors $\vec{V}^{(1)},..,\vec{V}^{(p)}$ determine a p-dimensional flat manifold through the point of contact of the tangent space, which is described by the antisymmetric covariant tensor $^*U_{a_{p+1},..,a_n}$. $^*U_{a_{p+1},..,a_n}$ is therefore a simple covariant (n-p)- vector, the outer product of n-p covariant vectors describing n-p hyperplanes having the p-dimensional flat manifold of the $\vec{V}^{(1)},..,\vec{V}^{(p)}$ as their intersection. The tensor duality transforms therefore simple covariant p-vectors into simple covariant (n-p)-vectors. In the case of the self-duality of the 2m - dimensional spaces, the simple covariant m- vectors are transformed into simple covariant m-vectors.

Let us denote by $\vec{U}^{(p+1)},..,\vec{U}^{(n)}$ n-p linearly independent covariant vectors whose outer product gives the above simple covariant (n-p)-vector $^*U_{a_{p+1},..,a_n}$. The hyperplane corresponding to any of those $\vec{U}^{(r)}$ contains the points of coordinates $V^{(s)a}$ so that $U_a^{(r)} V^{(s)a} = 0$. Hence we have $g^{ab} U_a^{(s)} U_b^{(r)} = 0$ with $s = 1,..,p$ and $r = p+1,..,n$. The $\vec{V}^{(r)}$ and $\vec{V}^{(s)}$ are also orthogonal $g_{ab} V^{(r)a} V^{(s)b} = 0$. The tensor duality transforms therefore a p-dimensional flat manifold into a (n-p) - dimensional flat manifold whose contravariant vectors are orthogonal to all those of the former manifold. . Let us consider now a flat m-dimensional manifold on the null cone $g_{ab} X^a X^b = 0$ in the case of $n = 2m$. Now any contravariant vector of the manifold is orthogonal to itself and to all the others of the manifold. Moreover any contravariant vector orthogonal to all those of the manifold must also belong to the manifold, which is therefore invariant for the tensor duality. The simple covariant m - vectors $U_{a_1,..,a_m}$ on the null cone are eigenvectors of D. Any eigenvector $U_{a_1,..,a_m}$ of D which is a simple m- vector describes a m- dimensional flat manifold of the null cone, because it contains the $\vec{V}^{(r)}$, a set of m linearly independent contravariant vectors orthogonal to all those of the manifold.

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In the above description of the flat manifolds of the tangent space by antisymmetric covariant tensors the components of the tensors are homogeneous grassmannian coordinates in the star of the flat manifolds through the point of contact. The linear duality transformations of the components of $U_{a_1,\dots,a_p}$ into those of $^*U_{a_{p+1},\dots,a_n}$ are induced by the polarity with respect to the null cone of the centre in the star of flat manifolds, whose center is the point of contact.. That polarity leaves invariant the null cone as a system of its flat manifolds of maximal dimension m , in the case of $n = 2m$.

The polarity with respect to the light-cone in the tangent space at a point P as an inner transformation of the light-cone taken as a system of two-dimensional manifolds is described by the equation $\mathcal{F}^{\mu\nu} = \epsilon^{\mu\nu,\rho\sigma} F_{\rho\sigma}$, not involving the g^{ab} .

The polarity with respect to a cone is a projective transformation of the star of flat manifolds through the vertex of the cone. It is the basic transformation underlying the cayleyan projective construction of the metric geometry of the star , which is the angular metric.

The tensor calculus is related to the affine geometry of the centred affine tangent spaces, in which a simple covariant p - vector describes a flat manifold of dimensionality n - p endowed with an outer orientation , passing through the point of contact Now the electromagnetic tensor $F_{\mu\nu}$ is associated to the outer orientation of a two-dimensional flat manifold ,when it is a simple covariant bivector of the space-time.

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The flat manifolds on the cone $g_{ab} X^a X^b = 0$ play an important part in the geometrical theory of the spinors of the orthogonal group of the quadratic form $g_{ab}X^aX^b$, as well known . The rôle of the flat manifolds of maximal dimensionality m is particularly remarkable. For even $n = 2m$ there are two families of such flat manifolds. For odd $n = 2m + 1$, there is only one such family . This corresponds to the fact that there are no semi- spinors for odd n and two kinds of semi-spinors for even n , as a consequence of the association of the flat m -dimensional manifolds on the null cone to spinors for odd n and to semi- spinors for even n . In the cese of even n the existence of two kinds of semi - spinors is related to the two eigenvalues of the operator D.

In this section we have examined in a general way the geometry of the tensor duality, without trying to emphasize the special circumstances characteristic of the physically most important case of $n = 4$. This will be done in detail in the following sections. It will be seen that they are mainly related to the fact that the vector space of the self-dual antisymmetric covariant tensors of the second order is now also that of the Lie algebras £ of the infinitesimal transformations of the orthogonal groups of the four-dimensional flat spaces. We have already found in equation (24d) a remarkable consequence of that fortuitous coincidence of two kinds of vector spaces for $n = 4$.

There are also some special consequences of the fact that the bilinear form $(U^{(1)}, U^{(2)})$ of the antisymmetric covariant tensors of order m is symmetric only for $n = 4p$, the simplest case being $n = 4$. For $n = 4$ the equation $(U,U) = 0$ is a necessary and sufficient condition for $U_{\mu\nu}$ to be a simple bivector. This leads to the isomorphism of $SL(4,R)$ and the real orthogonal group $O(3,3)$ of the quadratic form (U, U) and gives special properties to the projective geometry of the star of flat manifolds associate to each point of the four-dimensional differentiable manifolds, which are similar to those of the projective line geometry in three dimensions.