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# Time and Mass in Relativity*

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**Summary**

The particle dynamics in the world-manifold is taken as the foundation of the world-geometry, in a Hamiltonian formalism with a scalar mass Hamiltonian, not involving the value of the mass. The Riemannian metric is obtained from the linear relation between the covariant and contravariant vectors of the mechanical momentum of a particle involved in the formalism.

The basic differential equations for the world-lines of the particles allow to define a new kind of parameter called the duration, with the same dimension as the Einstein gravitational constant. The proper-time parameter is defined in terms of the duration and the mass Hamiltonian, before the introduction of the Riemannian metric.

The metric tensor $g(x)$ comes in as a mechanical field related to the inertial properties of the particles, which determines the mass Hamiltonian of the free particles. The symmetry of $g(x)$ is a consequence of the mass Hamiltonian formalism.

Our classical Hamiltonian formalism is naturally related to the wave equations of the relativistic quantum mechanics, and leads to a generalization of those wave equations.

## 1. Proper-Time and Duration

In General Relativity the problem of time appears in a far more complicated way than in Special Relativity, because of the impossibility of identifying the time with one of the four coordinates $x^\mu$ of a Riemannian space-time, without introducing strong restrictions on its topology, which do not seem quite justified. As a matter of fact, the clock-time of an observer is given by its proper time $s$, which is not one of the coordinates of the space-time, but a scalar quantity obtained from the $ds^2$ of the normal hyperbolic Riemannian metric.

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* Dedicated to Professor GUIDO BECK on the occasion of his 70th birthday.

*Thus the problem of time is related to the scalar parametrization of the world-lines of the particles. The interval s cannot be used as a parameter on world-lines with* $ds^2 = 0$ *and it becomes imaginary on those with* $ds^2 < 0$.

The proper time is defined in a purely kinematical way. In order to obtain a deeper analysis of the parametrization of the world-lines of the particles, it is necessary to make use of their dynamics, in particular of their mass properties.

The Hamiltonian form of the particle dynamics appears as the most adequate for the discussion of the problem of time, both in the non-relativistic and the relativistic dynamics, because they are differential equations of the first order. We shall use a new type of relativistic Hamiltonian formalism, symmetrical with respect to the four coordinates $x^\mu$ of a particle and thereby also symmetrical with respect to their conjugate momenta $p_\mu$, for the analysis of the problem of time.

The symmetry of the Hamiltonian formalism with respect to the coordinates $x^\mu$ is essential in General Relativity, since it does not allow a preferential status to be given a priori to one of the coordinates, in the analysis of time. This leads to the introduction of a scalar Hamiltonian function related to the mass, instead of the energy Hamiltonians of the non-relativistic theory and of Special Relativity.

*Thus our analysis of time leads to a new approach to the concept of the mass of a particle, in terms of the scalar Hamiltonian* $K(p, x)_e$ *of the particles with the charge e.*

Its nature of Hamiltonian function renders $K(p, x)_e$ a constant of the motion of the particles, because it does not involve any other variable of the particle, besides the four canonical pairs $p_\mu$, $x^\mu$. The situation is now analogous to that of the motion of the particles in a time-independent field of forces in the non-relativistic Hamiltonian dynamics, which admits the energy Hamiltonian as a constant of the motion.

The constant value of $K(p, x)_e$ on a world-line of a particle gives its mass $m$ by means of the equation

$$m^2 = 2K(p, x)_e \quad \text{with} \quad c = 1, \tag{1}$$

corresponding to the non-relativistic equation which gives the energy $E$ of a particle moving on a trajectory, in terms of the value of its Hamiltonian $H(p, x)$ for that motion

$$E = H(p, x). \tag{2}$$

In the non-relativistic dynamics, the mass of a particle appears as a primary concept, and the energy as a quantity defined in terms of the mass and the three-momentum, by means of the function $H$. Our approach to the concept of mass in Relativity inverts that situation: *the vector* $p_\mu$ *is now a primary concept, and the mass a quantity defined in terms of the energy and the three-momentum, described by the vector* $p_\mu$, *by means of the function* $K(p, x)_e$.

We shall use as basic equations of motion of a particle the pre-Hamiltonian system

$$dx^0/P^0 = \cdots = dx^3/P^3 = -\,dp_0/X_0 = \cdots = -\,dp_3/X_3, \tag{3}$$

with the $P^\mu$ and $X_\mu$ denoting partial derivatives of $K(p, x)_e$

$$P^\mu = \partial_{p_\mu} K(p, x)_e \quad \text{and} \quad X_\mu = \partial_{x^\mu} K(p, x)_e. \tag{4}$$

The system (3) is covariant, although the $dp_\mu$ are not vector components, because the $X_\mu$ are also not the components of a vector.

$K(p, x)_e$ is a constant of the motion of the particles

$$dK(p, x)_e = 0 \tag{5}$$

because of the general equation of variation of the physical quantities $G(p, x)$ of a particle with charge $e$ resulting from (3)

$$dG(p, x) = (G(p, x), K(p, x)_e)\, dz_e, \tag{6}$$

$dz_e$ denoting the common value of the eight ratios in the differential system (3)

$$dz_e = dx^\mu/P^\mu = -\,dp_\mu/X_\mu \tag{7}$$

and $(B(p, x), C(p, x))$ the four-dimensional Poisson bracket of the functions $B(p, x)$ and $C(p, x)$

$$(B(p, x), C(p, x)) = \partial_{x^\mu} B\, \partial_{p_\mu} C - \partial_{p_\mu} B\, \partial_{x^\mu} C. \tag{8}$$

*The real scalar parameter $z_e$ defined by (7) will be called the duration parameter for the particles with charge $e$.* The introduction of $z_e$ allows to go over from the pre-Hamiltonian system (3) to the four-dimensional Hamilton equations

$$dx^\mu = \partial_{p_\mu} K(p, x)_e\, dz_e \quad \text{and} \quad dp_\mu = -\,\partial_{x^\mu} K(p, x)_e\, dz_e. \tag{9}$$

We have a four-dimensional infinitesimal contact transformation

$$p_\mu \to p_\mu + dp_\mu, \qquad x^\mu \to x^\mu + dx^\mu, \tag{10}$$

defined by the motion of the particles with charge $e$, because

$$(p_\mu + dp_\mu)\, \delta(x^\mu + dx^\mu) - p_\mu\, \delta x^\mu = d(p_\mu\, \delta x^\mu), \tag{11}$$

and $d(p_\mu\, \delta x^\mu)$ is the $\delta$ differential of a function

$$d(p_\mu\, \delta x^\mu) = \delta((P^\mu p_\mu - K(p, x)_e)\, dz_e), \qquad \delta z_e = 0 \tag{12}$$

and the vector $p_\mu$ at $x$ can vary without restrictions.

Thus we get the Theorem 1:

THEOREM 1. The motions of the particles with charge $e$, given by the differential system (3), generate a one-parameter group of non-homogeneous contact transformations of the world-manifold, with the parameter $z_e$.

We shall now see that the definition (7) of the duration parameter $z_e$ is analogous to that of the Newtonian time $t$, given by the motion of a mechanical system with a Hamiltonian function $H(p, q)$ depending only on the Lagrangian coordinates $q_r$ and their conjugate momenta $p_r$, with $n$ degrees of freedom, in the non-relativistic mechanics. *Those me-*

*chanical systems are good clocks because their equations of motion can be written as pre-Hamiltonian systems not involving the Newtonian time t*

$$dq_1/P_1 = \cdots = dq_n/P_n = -\,dp_1/Q_1 = \cdots = -\,dp_n/Q_n, \tag{13}$$

*with*

$$P_r = \partial_{p_r} H(p, q) \quad \text{and} \quad Q_r = \partial_{q_r} H(p, q). \tag{14}$$

We can define $dt$ as the common value of the $2n$ ratios in the system (13), because this leads to the correct Hamilton equations

$$dq_r = P_r\, dt \quad \text{and} \quad dp_r = -\,Q_r\, dt. \tag{15}$$

The system (13) gives $2n - 1$ differential equations for the $q_2, \ldots, q_n$ and $p_1, \ldots, p_n$, with $q_1$ taken as the independent variable. Their general solution gives the above $2n - 1$ variables as functions of $q_1$ and arbitrary constants $C_1, \ldots, C_{2n-1}$. Thus $P_1(p, q)$ becomes a function $P(q_1, C)$ and we have the differential equation for $t$

$$dt = dq_1/P(q_1, C). \tag{16}$$

Finally we get

$$t = \int_{q_1^0}^{q_1} dy/P(y, C), \tag{17}$$

$q_1^0$ denoting the value of $q_1$ corresponding to $t = 0$.

The system (13) has the energy $H(p, q)$ as a constant of the motion, corresponding to the constant of the motion $K(p, x)_e$ of the system (3).

*The above discussion shows that $z_e$ is the natural parameter on the world-lines of the particles with charge e. It does not have the dimension of a time.* We have the Theorems 2 and 3:

THEOREM 2. When the velocity of light $c$ is taken as 1, $z_e$ has the dimension of the Einstein gravitational constant $\kappa$. Thus General Relativity involves a natural unit of duration $\kappa$, and allows the existence of a dimensionless parameter $z_e/\kappa$ on the world-lines of the particles with charge $e$.

THEOREM 3. When $c = 1$, $z_e$ has the dimension of $(e/m)^2$.

We shall now introduce a definition of the infinitesimal proper-time $ds_e$ on the world-lines of the particles with charge $e$, in terms of $dz_e$ and the square root of $2K(p, x)_e$:

DEFINITION. The infinitesimal proper-time $ds_e$ corresponding to the infinitesimal displacement $dx^\mu$ on the world-lines of the particles with charge $e$ is taken as

$$ds_e = (2K(p, x)_e)^{1/2}\, dz_e, \tag{18}$$

the sign of the square root being such that it be equal to the mass $m$.

We have the Theorem 4:

THEOREM 4. The definition of $ds_e$ does not require the introduction of a Riemannian metric in the world-manifold. $ds_e$ is real and nonzero for $dx^\mu \neq 0$ on the world-lines of the particles with nonzero real mass. $ds_e = 0$ for any $dx^\mu$ on the world-lines of particles with mass zero. $ds_e$ is imaginary on the world-lines of the particles with $m^2 < 0$, for $dx^\mu \neq 0$.

**1a.** We shall now discuss the gauge transformations of the $p_\mu$ induced by those of the electromagnetic potentials $A_\mu(x)$

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_{x^\mu}\phi(x), \tag{1}$$

$\phi(x)$ denoting an arbitrary differentiable scalar field. We have

$$p_\mu \rightarrow p_\mu + e\,\partial_{x^\mu}\phi(x) \quad \text{with} \quad c = 1. \tag{2}$$

It follows from (1) and (2) that the $P_\mu$

$$P_\mu = p_\mu - eA_\mu(x) \quad \text{with} \quad c = 1 \tag{3}$$

are gauge invariant.

The $P_\mu$ will be called the components of the covariant mechanical momentum, and the $P^\mu$ of the system (1–2) the components of the contravariant mechanical momentum. *It follows from the system* (1–7) *that at any point x of a world-line of a particle the vector $P^\mu$ is tangent to the world-line*

$$P^\mu = dx^\mu/dz_e. \tag{4}$$

The Hamiltonian $K(p, x)_e$ must be gauge invariant, in order that the mass of a particle be independent of the choice of the potentials $A_\mu(x)$ for a given electromagnetic field. Thus we get the Theorem 5:

THEOREM 5. $K(p, x)_e$ is a function of the $x^\mu$ and the $P_\mu$, as a consequence of its gauge invariance.

*We shall postulate now that e comes in in $K(p, x)_e$ only as a constant of interaction with the electromagnetic field, through the vector $eA_\mu(x)$. This assumption will be called the Charge Postulate.* We have the Theorem 6:

THEOREM 6. It follows from the Charge Postulate that there is a function $N(y, x)$ of the covariant vector $y_\mu$ and the $x^\mu$, not involving $e$, such that

$$K(p, x)_e = N(P, x). \tag{5}$$

Hence the $P^\mu$ are partial derivatives of $N(P, x)$

$$P^\mu = \partial_{P_\mu} N(P, x). \tag{6}$$

THEOREM 7. It follows from Eq. (6) that the components $P^\mu$ of the contravariant mechanical momentum are gauge invariant. Since we get from the system (1–7) that

$$dz_e = P_\mu\, dx^\mu / P_\mu P^\mu, \tag{7}$$

the duration parameter $z_e$ and the proper time $s_e$ are also gauge invariant.

We get from the Eq. (5), for $e = 0$, the Theorem 8:

THEOREM 8. The fundamental function $N(y, x)$ is given by the Hamiltonian $K(p, x)_0$ of the free neutral particles $K(p, x)_0$

$$N(y, x) = K(y, x)_0. \tag{8}$$

Thereby $N(y, x)$ is determined by the inertial properties of the neutral particles. *It is the basic inertial function.*

We have also the Theorem 9:

THEOREM 9. The Hamiltonian of the free charged particles can also be taken as $K(p, x)_0$, by choosing the potentials $A_\mu(x) = 0$ when the electromagnetic field $F_{\mu\nu}(x) = 0$ everywhere on the world-manifold.

**1b.** Let us introduce now the symmetric contravariant tensor $g^{\mu\nu}(p, x)$

$$g^{\mu\nu}(p, x) = \partial^2_{P_\mu P_\nu} N(P, x).$$

*We shall assume that the determinant of the* $g^{\mu\nu}(p, x)$ *is* $\neq 0$ *for all the* $x$ *and* $P_\mu$, *in order to be able to solve the Eqs.* (1a–6) *with respect to the* $P_\mu$.

The existence theorem for the solutions of the systems of ordinary differential equations applied to (1–3) shows that there is one and only one motion of a particle of charge $e$ with a given $p_\mu$ at a point $x$ of its world-line, when suitable conditions are satisfied by $N(P, x)$. We have the Theorem 10:

THEOREM 10. When the $P_\mu$ can be expressed in terms of the $P^\nu$ and $x^\nu$, there is one and only one motion of a particle with charge $e$ for a given $dx^\mu/dz_e$ vector at a point $x$ of its world-line, when the conditions for the applicability of the existence theorem to the system (1–3) are satisfied by $N(P, x)$.

We have the Theorem 11:

THEOREM 11. When the $P_\mu$ can be expressed in terms of the $P^\nu$ and $x^\nu$, the Hamilton equations (1–9) can be obtained from the Lagrange equations with the Lagrangian function $L(x, \dot{x})_e$

$$L(x, \dot{x})_e = p_\mu \dot{x}^\mu - K(p, x)_e \quad \text{with} \quad \dot{x}^\mu = dx^\mu/dz_e, \tag{1}$$

given by the relativistic Hamilton Principle

$$\delta \int_{z_1}^{z_2} L(x, \dot{x})_e \, dz_e = 0, \tag{2}$$

with $\delta x^\mu = 0$ at the limits of integration. The $p_\mu$ are now obtained in the usual way from the Lagrangian

$$p_\mu = \partial_{\dot{x}^\mu} L(x, \dot{x})_e. \tag{3}$$

We have the Theorem 12:

THEOREM 12. The Lagrange equations

$$\dot{p}_\mu = \partial_{x^\mu} L(x, \dot{x})_e \tag{4}$$

with the $p_\mu$ given by (3), are differential equations for the world-lines of the particles with charge $e$, parametrized by $z_e$. From a solution $x^\mu = f(z_e, x(0), \dot{x}(0))$ we get the solutions $x^\mu = f(s_e/m, x(0), mx'(0))$ for all the values of $m \neq 0$, with $x'^\mu(s)$ denoting the component $dx^\mu/ds_e$ of the velocity.

When the $\delta x^\mu$ are not taken as zero at the limits $z_1$, $z_2$ of the integral of the Lagrangian in the variational principle, we get, instead of (2), the important equation

$$p_{2:\mu}\,\delta x_2{}^\mu - p_{1:\mu}{}^\mu\,\delta x_1{}^\mu = \delta \int\limits_{z_1}^{z_2} L(x, \dot{x})_e\, dz_e. \tag{5}$$

Hence the integral is an ordinary function of the points $x_1$ and $x_2$ and the values $z_1$, $z_2$ of the duration parameter, when taken along the world-line passing through $x_1$, $x_2$ at $z_1$, $z_2$

$$\int\limits_{z_1}^{z_2} L(x, \dot{x})_e\, dz_e = S(x_2, z_2; x_1, z_1)_e, \tag{6}$$

$S$ is the Hamilton principal function of the Hamilton system (1–9). It follows from Eq. (5) that

$$p_{r:\mu} = (-1)^r\, \partial_{x_r{}^\mu} S(x_2, z_2; x_1, z_1)_e \quad \text{for} \quad r = 1, 2. \tag{7}$$

We get from (6) that the differential $\delta S$ with $\delta z_r \neq 0$ is

$$\begin{aligned}\delta S(x_2, z_2; x_1, z_1)_e = p_{2:\mu}\,\delta x_2{}^\mu &- K(p_2, x_2)_e\,\delta z_2 \\ - p_{1:\mu}\,\delta x_1{}^\mu &+ K(p_1, x_1)_e\,\delta z_1.\end{aligned} \tag{8}$$

From (8) and (7) we get the Theorem 13:

THEOREM 13. The function $S(x, z; x_0, z_0)_e$ satisfies with respect to the variables $x^\mu$, $z$ the Hamilton-Jacobi equation associated to the Hamilton equations (1–9)

$$V_{z_e} + K(V_x, x)_e = 0. \tag{9}$$

The application to Eq. (9) of the Jacobi theorem for the partial differential equations of the first order gives the Theorem 14:

THEOREM 14. The general solution of the Hamilton system (1–9) can be obtained by means of a solution $V(x, z; a)$ of Eq. (9) depending on four arbitrary parameters $a^\mu$, and with a nonzero determinant of the second order derivatives $\partial^2_{x^\mu a^\nu} V(x, z; a)$, by means of the equations

$$p_\mu = \partial_{x^\mu} V(x, z; a) \quad \text{and} \quad b_\mu = -\,\partial_{a^\mu} V(x, z; a), \tag{10}$$

the $b_\mu$ being four arbitrary constants.

It is important to note that the Theorem 14, as the association of the Hamilton-Jacobi equation (9) to the Hamilton system (1–9) do not depend on the possibility of expressing the $P_\nu$ in terms of the $P^\mu$ and $x^\mu$ by means of the equations $P^\mu = \partial_{P_\mu} N(P, x)$.

**1c.** We obtained the Hamilton-Jacobi equation (1b–9) from the Hamilton equations (1–9) by means of the Lagrangian formalism of Sec. 1b. It is well known from the Hamiltonian theory of the nonrelativistic dynamics that the Hamilton-Jacobi equation can be introduced by means of the theory of the contact transformations of the Hamilton equations. We shall now see that the same happens also with the Eqs. (1–9) and (1b–9).

Let us consider the bilinear covariant $\omega_2(d, \delta)$ of the linear differential form $\omega(d)$

$$\omega(d) = p_\mu \, dx^\mu - K(p, x)_e \, dz_e, \tag{1}$$

$$\omega_2(d, \delta) = \delta\omega(d) - d\omega(\delta). \tag{2}$$

It is easily seen that the Hamilton equations (1–9) are equivalent to the equation

$$\omega_2(d, \delta) = 0 \quad \text{for arbitrary} \quad \delta p_\mu \quad \text{and} \quad \delta x^\mu. \tag{3}$$

We shall now introduce the contact transformation defined by a differentiable function $V(x, x'; z_e)$ with a nonzero determinant of the second order derivatives $\partial^2_{x^\mu x'^\nu} V(x, x'; z_e)$

$$p_\mu' \, dx'^\mu - p_\mu \, dx^\mu = dV(x, x'; z_e) - \partial_{z_e} V(x, x'; z_e) \, dz_e. \tag{4}$$

It follows from (4) that with

$$K'(p', x', z_e)_e = K(p, x)_e + \partial_{z_e} V(x, x'; z_e) \tag{5}$$

and

$$\omega'(d) = p_\mu' \, dx'^\mu - K'(p', x'; z_e)_e \tag{6}$$

we have

$$\omega(d) - \omega'(d) = -\, dV, \tag{7}$$

so that

$$\omega_2(d, \delta) = \omega_2'(d, \delta). \tag{8}$$

From (8) and (3) we get the Theorem 15:

Theorem 15. The contact transformation (4) gives the Hamilton equations with the Hamiltonian $K'(p', x', z_e)_e$ and the same duration parameter $z_e$ for the new variables $p_\mu'$ and $x'^\mu$, as a consequence of those with the Hamiltonian $K(p, x)_e$ for the $p_\mu$, $x^\mu$.

The Eq. (4) gives the system

$$p_\mu = -\, \partial_{x^\mu} V(x, x'; z_e), \qquad p_\nu' = \partial_{x'^\nu} V(x, x'; z_e). \tag{9}$$

The first group of Eqs. (9) can be solved for the $x'^\nu$ in terms of the $p_\mu$, $x^\mu$, $z_e$. The second group gives then the $p_\nu'$. In order to get the $p_\mu$, $x^\mu$ in terms of the $p_\nu'$, $x'^\nu$ and $z_e$, we must start by the solution of the second group (9) with respect to the $x^\mu$.

When $V(x, x'; z_e)$ satisfies the Hamilton-Jacobi equation with respect to the $x^\mu$ and $z_e$

$$V_{z_e} + K(V_x, x)_e = 0, \tag{10}$$

we get from Eq. (5)

$$K'(p', x', z_e)_e = 0. \tag{11}$$

Hence the new variables $p_\nu'$ and $x'^\nu$ are constants of the motion and $V(x, x'; z_e)$ is a solution of the Hamilton-Jacobi equation (1b–9) depending on 4 arbitrary constants $x'^\nu$, with a nonzero determinant of the second order derivatives $\partial^2_{x^\mu x'^\nu} V(x, x'; z_e)$. *Thus we get the proof of the Theorem* 14, *by means of the Eqs.* (9), *expressing the $p_\mu$ and $x^\mu$ in terms of the* 8 *constants of the motion $p_\nu'$, $x'^\nu$ and $z_e$.*

**1d.** The relativistic Hamiltonian theory of this paper leads to the introduction of an eight-dimensional relativistic phase space $S_8(e)$ for the particles with the charge $e$, with the coordinates $p_\mu$, $x^\mu$. We have the Theorem 16:

THEOREM 16. The eight-dimensional phase space $S_8(e)$ for the particles with charge $e$ is a differentiable manifold endowed with the Hamiltonian structure defined by the invariant antisymmetric bilinear form $\delta p_\mu \, dx^\mu - dp_\mu \, \delta x^\mu$ of its two infinitesimal contravariant vectors $(dp_\mu, dx^\mu)$ and $(\delta p_\mu, \delta x^\mu)$, which is the bilinear covariant of the linear differential form $p_\mu \, dx^\mu$. The antisymmetric bilinear differential form is invariant for the contact transformations

$$p_\mu \, dx^\mu - p_\mu' \, dx'^\mu = \text{exact differential}. \tag{1}$$

We have the Theorems 17 and 18:

THEOREM 17. It follows from the Theorem 15 that $K(p, x)_e$ behaves as a scalar for the contact transformations (1) of $S_8(e)$

$$K'(p', x')_e = K(p, x)_e, \tag{2}$$

$K(p, x)_e$ defines a scalar field in the Hamiltonian geometry of $S_8(e)$. The pre-Hamiltonian differential system (1–3) defines the lines of $S_8(e)$ tangent at all their points to the contravariant eight-dimensional vector $-X_\mu$, $P^\mu$ obtained from the gradient $P^\mu$, $X_\mu$ of $K(p, x)_e$ by means of the invariant bilinear antisymmetric form of the Hamiltonian structure of $S_8(e)$.

THEOREM 18. The Poisson bracket $(B(p, x), C(p, x))$ of two scalar fields of the Hamiltonian geometry of $S_8(e)$, defined by the Eq. (1–8) is the skew inner product of the gradient vectors of $B(p, x)$ and $C(p, x)$ associated to the invariant bilinear antisymmetric form $\delta p_\mu \, dx^\mu - dp_\mu \, \delta x^\mu$.

We have also the Theorem 19:

THEOREM 19. The gauge transformation (1a–2) of the $p_\mu$ is a special contact transformation

$$p_\mu \, dx^\mu - p_\mu' \, dx'^\mu = - e \, d\phi(x) \tag{3}$$

leaving invariant the $x^\mu$.

The Hamiltonian geometry of $S_8(e)$ has a natural Liouville measure of the oriented hypervolumes, because the functional determinant

$D(p', x')/D(p, x) = 1$ for the contact transformations (1), as well known from a general theorem of the theory of the contact transformations. Thus we get the Theorem 20:

THEOREM 20. The outer product of eight infinitesimal vectors $(d_r p_\mu, d_r x^\mu)$ is invariant for the contact transformations (1). It defines the oriented infinitesimal hypervolume of $S_8(e)$ built with them.

*The above discussion shows clearly that our particle approach to General Relativity is naturally associated to a more primary type of geometry of the world-manifold than the Einstein normal hyperbolic Riemannian geometry.*

## 2. The Hamilton-Jacobi Approach to the Mass Formalism

In the preceding sections we developed the mass formalism of the particle dynamics starting from the pre-Hamiltonian differential system (1–3), involving the partial derivatives $P^\mu$ and $X_\mu$ of the mass Hamiltonian $K(p, x)_e$, and finally arrived to the Hamilton-Jacobi partial differential equation of the first order (1b–9) involving only the function $K(V_x, x)_e$. The Eq. (1b–9) admits solutions of the form

$$V^{(m)}(x, z_e) = -\tfrac{1}{2}m^2 z_e + U(x, m) \tag{1}$$

with $U(x, m)$ satisfying a partial differential equation with a formal structure similar to that of the fundamental Eq. (1–1)

$$2K(U_x, x) = m^2. \tag{2}$$

The Eq. (2) is indeed obtained from (1–1) by the substitution of $p_\mu$ by $U_{x^\mu}$.

*The partial differential Eq. (2) will be called the ordinary Hamilton-Jacobi equation for the particles with charge e and mass m in General Relativity.* It gets the usual form

$$g^{\mu\nu}(x)(U_{x^\mu} - eA_\mu(x))(U_{x^\nu} - eA_\nu(x)) = m^2 \tag{3}$$

when the function $N(P, x)$ of Eq. (1a–5) is taken as follows

$$2N(P, x) = g^{\mu\nu}(x)P_\mu P_\nu. \tag{4}$$

We shall introduce (4) in Sec. 4, after the clarification of the mechanical signification of the tensor $g^{\mu\nu}(x)$. In the present section we shall discuss the relation between the Eq. (2) and the pre-Hamiltonian differential system (1–3).

It is well known from the theory of the partial differential equations of the first order that there is the Cauchy system of ordinary differential equations for the characteristic lines associated to such an equation. In the case of the Eq. (2) the Cauchy system is

$$dx^0/P^0 = \cdots = dx^3/P^3 = -dp_0/X_0 = \cdots = -dp_3/X_3 = dU/P^\mu p_\mu \tag{5}$$

together with the condition

$$2K(p, x)_e = m^2. \tag{6}$$

The differential system (5) can be replaced by (1–3) and the equation

$$dU = p_\mu \, dx^\mu. \tag{7}$$

Hence we have the Theorem 21:

THEOREM 21. The solutions of the Cauchy system (5)—(6) of the ordinary Hamilton-Jacobi equation (2) are obtained from those of the pre-Hamiltonian system (1–3) corresponding to the value $m^2$ of the constant of the motion $2K(p, x)_e$ with the $U$ given by (7)

$$U = U_0 + \int_{x_0}^{x} p_\mu \, dx^\mu, \tag{8}$$

the integration being along the world-line of a particle with charge $e$ and mass $m$ given by the system (1–3).

*The Theorem* 21 *shows that the pre-Hamiltonian system* (1–3) *comes in naturally even in the theory of the motions of the particles with given e and m, in connection with the ordinary relativistic Hamilton-Jacobi equation* (2).

The Cauchy differential system shows that a ninth variable $U$ comes also in naturally in the dynamics of a particle with charge $e$ and mass $m$, in connection with the Hamilton-Jacobi equation (2). This point will be clarified in the following Sec. 2a.

The ordinary relativistic Hamilton-Jacobi equation (2) is closely related to the Klein-Gordon wave equation for the particles with charge $e$ and mass $m$ in General Relativity

$$2K(p_{op}, x)_e \psi(x) = m^2 \psi(x), \tag{9}$$

$\psi(x)$ denoting the complex wave function for a particle with spin 0, and the $p_{op;\mu}$ being the linear operators for the components of the momentum of the particle.

We have the Theorem 22:

THEOREM 22. In the relativistic quantum mechanics there are square mass operators related to the square mass functions $2K(p, x)_e$ of the classical theory, involving the operators for the four components of the momentum of a particle. Such a square mass operator is well known in the case of the motion of free particles in Special Relativity, where it is related to one of the Casimir operators of the Poincaré group.

**2a.** The Cauchy system (2–5)—(2–6) can be split into the pre-Hamiltonian system (1–3) and that of the Eqs. (2–7)—(2–6), a Pfaffian system. The Pfaff equation (2–7) has four-dimensional integral manifolds defined by arbitrary differentiable functions $U(x)$ of the $x^\mu$

$$U = U(x), \qquad p_\mu = \partial_{x^\mu} U(x). \tag{1}$$

*The condition* (2–6) *requires* $U(x)$ *to be a solution of* (2–2), *in order to give a four-dimensional integral manifold of the system* (2–7)—(2–6). Thus we get the Theorem 23:

THEOREM 23. The Pfaffian system (2–7)—(2–6) gives a generalization of the ordinary relativistic Hamilton-Jacobi equation (2–2), which gives its four-dimensional integral manifolds of the type (1), because it admits also integral manifolds with dimensionality less than 4.

We have the Theorem 24:

THEOREM 24. The Cauchy system (2–5)—(2–6) defines one-dimensional integral manifolds of the Pfaffian system (2–7)—(2–6), associated to the world-lines of the particles with charge $e$ and mass $m$. They are also one-dimensional integral manifolds of the Pfaffian system

$$dx^0/P^0 = \cdots = dx^3/P^3 = dU/p_\mu P^\mu, \qquad 2K(p, x)_e = m^2, \tag{2}$$

whose integral manifolds belong also to the less restrictive system (2–7)—(2–6).

We have the Theorem 25:

THEOREM 25. The definition of the duration parameter $z_e$ given by the Eq. (1a–7) can be extended to any one-dimensional integral manifold of the Pfaffian system (2) and we get

$$dU = P^\mu p_\mu \, dz_e. \tag{3}$$

We can now extend the definition (1–18) of the proper time parameter $s_e$ in terms of $z_e$ to any one-dimensional integral manifold of the Pfaffian system (2) by taking

$$ds_e = (\sqrt{2K(p, x)_e}/P_\rho P^\rho) P_\mu \, dx^\mu. \tag{4}$$

Let us consider now a world-line defined by real differentiable functions $x^\mu(y)$ of a real parameter $y$. We define $z_e(y)$ on the world-line as an arbitrary differentiable function $f(y)$ with $df/dy \neq 0$. Thus we can take $P^\mu(y) = dx^\mu(y)/df(y)$ and determine the $p_\mu(y)$ by the equations $P^\mu(y) = \partial_{p_\mu} K(p, x(y))$ and $U(y)$ by the equation

$$U(y) - U(0) = \int_0^y p_\mu(\mathbf{y}) \, dx^\mu(\mathbf{y}).$$

*When it is possible to choose the function $z_e(y)$ such that $2K(p(y), x(y))_e = m^2$ for all the values of $y$, we get a one-dimensional integral manifold of the Pfaffian system (2) associated to the given world-line, which may not be a world-line of a particle of charge $e$ and mass $m$ moving in the electromagnetic field of potentials $A_\mu(x)$ involved in $K(p, x)_e$.*

**2b.** At the present level of the theory of the mass formalism of the relativistic dynamics of a particle, there is the scalar $P_\mu P^\mu$ built with the components of the covariant and contravariant vectors of the mechanical momentum, but not identified to the square of the mass $m$ of the particle, which is given by $2K(p, x)_e$, as a consequence of the Fundamental Postulate expressed by the Eq. (1–1). *We shall now introduce the new postulate*

$$2K(p, x)_e = P_\mu P^\mu. \tag{1}$$

It follows from (1) and the Eq. (1a–6) that the function $N(P, x)$ satisfies the Euler equation for the homogeneous functions of degree 2 of the $P_\mu$

$$P_\mu \, \partial_{P_\mu} N(P, x) = 2N(P, x). \tag{2}$$

We have the Theorem 26:

THEOREM 26. The postulate of identification of $P_\mu P^\mu$ and the square mass function $2K(p, x)_e$ renders $N(P, x)$ a homogeneous function of degree 2 of the $P_\mu$. The $P^\mu$ are now homogeneous functions of degree 1 of the $P_\rho$. $K(p, x)_e$ becomes a homogeneous function of degree 2 of $e$ and the four components $p_\mu$.

It follows from the definition (1a–3) of $P_\mu$ that

$$P^\mu p_\mu = P^\mu P_\mu + eA_\mu(x) P^\mu \tag{3}$$

so that by taking into account (1) we get

$$P^\mu p_\mu = 2K(p, x)_e + eA_\mu(x) P^\mu. \tag{4}$$

Thus we get the Theorem 27:

THEOREM 27. On the one-dimensional integral manifolds of the Pfaffian system (2a–2) we have

$$dU = m \, ds_e + eA_\mu(x) \, dx^\mu, \tag{5}$$

so that the variation of $U$ from $x_1$ to $x_2$ is given by the integral of the relativistic Lagrangian for a particle of charge $e$ and mass $m$ along the corresponding interval of the world-line

$$U(2) - U(1) = \int_{x_1}^{x_2} (m + eA_\mu(x) v^\mu) \, ds_e, \tag{6}$$

$v^\mu$ denoting the velocity vector with respect to the proper-time parameter.

We have the Theorem 28:

THEOREM 28. It follows from the Theorem 27 that the one-dimensional integral manifolds of the Pfaffian system (2a–2) associated to arbitrary world-lines by the procedure of Sec. 2a are related to the Feynman path approach to the quantum mechanics, when $m$ is real and nonzero,. because of the relation between the variation of $U$ and the integral of the Lagrangian.

**2c.** We shall now introduce the Pfaff equation

$$dV = p_\mu \, dx^\mu - K(p, x)_e \, dz_e \tag{1}$$

associated to the Hamilton-Jacobi equation (1b–9) for the particles with the charge $e$ and all the values of the mass $m$. We have the Theorem 29:

THEOREM 29. Any solution $V(x, z_e)$ of (1b–9) gives a five-dimensional integral manifold of the Pfaff equation (1) defined by the equations

$$V = V(x, z_e), \qquad p_\mu = \partial_{x^\mu} V(x, z_e). \tag{2}$$

The Pfaff equation (1) may be seen as a generalization of (1b–9), because it admits also integral manifolds with dimensionality less than 5.

We have also the Theorem 30:

THEOREM 30. The Cauchy differential system of (1b–9)

$$dz_e = dx^0/P^0 = \cdots = -dp_0/X_0 = \cdots = -dp_3/X_3 = dV/(P^\mu p_\mu - K(p,x)_e) \tag{3}$$

defines one-dimensional integral manifolds of the Pfaff equation (1), associated to the world-lines with a given value $e$ of the charge and all the real values of $m^2$.

The integral manifolds of the Pfaffian system

$$dz_e = dx^0/P^0 = \cdots = dx^3/P^3 = dV/(P^\mu p_\mu - K(p, x)_e) \tag{4}$$

have the same property with respect to the less restrictive Pfaff equation (1). The integral manifolds of the Cauchy system (3) are one-dimensional integral manifolds of (4), but not all those integral manifolds of (4) can be obtained from the solutions of (3).

The Cauchy system (3) can be replaced by the Hamilton equations (1–9) taken together with the Pfaff equation (1). The solution of the differential system (3) can be obtained immediately from that of the Hamilton equations (1–9), since the Pfaff equation (1) gives the variation of $V$ along the world-lines of the particles with charge $e$

$$V(x_2, z_{e,2}) - V(x_1, z_{e,1}) = \int_{z_{e,1}}^{z_{e,2}} (P^\mu p_\mu - K(p, x)_e)\, dz_e. \tag{5}$$

It follows from (1b–1) that the quantity under the integral in (5) is simply the Lagrangian $L(x, \dot{x})_e$. Thus we see that

$$V(x_2, z_{e,2}) - V(x_1, z_{e,1}) = S(x_2, z_2; x_1, z_1)_e \tag{6}$$

$S$ denoting the Hamilton principal function of the system (1–9), defined by the Eq. (1b–6).

The Pfaffian system (4) can be replaced by the first group of Hamilton equations (1–9) taken together with the Pfaff equation (1).

**2d.** The mass formalism of the preceding sections deals with the motions of particles with a given charge $e$ and all the possible values of $m^2$. It can be easily extended into a mass-charge formalism dealing also with particles with different values of the charge $e$, by assuming the existence of a new pair of conjugate variables $u$ and $p_u$, with $p_u$ giving the value of the electric charge $e$ and $u$ not appearing in the new Hamiltonian function $\bar{K}(p, x)$, obtained from $K(p, x)_e$ by the substitution of $e$ by $p_u$

$$\bar{K}(p, x) = K(p, x)_{p_u}, \tag{1}$$

*$\bar{K}(p, x)$ is a function of the nine variables $p_u$, $p_\mu$ and $x^\mu$, not involving the variable $u$.* It follows from (1) that $K(p, x)_e$ is simply the value of $\bar{K}(p, x)$ for $p_u = e$.

The Pfaff equation (2b–1) is now replaced by

$$d\bar{V} = p_u\, du + p_\mu\, dx^\mu - \bar{K}(p, x)\, dz, \tag{2}$$

with $z$ playing now the role of a duration parameter. The analog of the differential system (2b–3) for the Pfaff equation (2) is

$$\begin{aligned} dz &= du/\bar{P}^u = dx^0/\bar{P}^0 = \cdots = dx^3/\bar{P}^3 \\ &= -\, dp_u/\bar{U} = -\, dp_0/\bar{X}_0 = \cdots = -\, dp_3/\bar{X}_3 \\ &= d\bar{V}/(\bar{P}^u p_u + \bar{P}^\mu p_\mu - \bar{K}(p, x)) \end{aligned} \tag{3}$$

with

$$\bar{U} = \partial_u \bar{K}(p, x), \quad \bar{P}^u = \partial_{p_u} \bar{K}(p, x); \quad \bar{X}_\mu = \partial_{x^\mu} \bar{K}(p, x), \quad \bar{P}^\mu = \partial_{p_\mu} \bar{K}(p, x). \tag{3a}$$

*By taking only the equations of* (3) *not involving $d\bar{V}$, we get the system of Hamilton equations analogous to* (1–9) *for the mass-charge formalism.* Since $\bar{K}(p, x)$ does not contain $u$, we have $\bar{U} = 0$ and the Hamilton equation $dp_u = -\,\bar{U}\, dz$ gives $dp_u = 0$. Thus we get the Theorem 31:

THEOREM 31. In the present mass-charge formalism there are two constants of the motion $\bar{K}(p, x)$ and $p_u$, whose values on a world-line of a particle give $\frac{1}{2}m^2$ and $e$, because the Hamiltonian function $\bar{K}(p, x)$ does not involve the variables $z$ and $u$.

Since $\bar{K}(p, x)$ involves $p_u$ only through the $p_\mu - p_u A_\mu(x)$, we have

$$\bar{P}^u = -\, A_\mu(x) \bar{P}^\mu, \tag{4}$$

so that

$$du = -\, A_\mu(x)\, dx^\mu. \tag{5}$$

We have the Theorem 32:

THEOREM 32. The variable $u$ is not gauge invariant, as shown by (5). The gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_{x^\mu}\Phi(x)$ induces the transformations

$$u \to u - \Phi(x) \quad \text{and} \quad p_u \to p_u, \qquad p_\mu \to p_\mu + p_u\, \partial_{x^\mu}\Phi(x). \tag{6}$$

The gauge transformation corresponds to a change of coordinates in the five-dimensional differentiable manifold $S_5$ with the coordinates $u$, $x^\mu$

$$x^\mu \to x^\mu, \qquad u \to u - \Phi(x) \tag{7}$$

with the $p_u$, $p_\mu$ transforming as the components of a covariant vector of $S_5$.

We have also the Theorem 33:

THEOREM 33. It follows from (5) that $-du$ can be identified to the basic linear differential form of the electromagnetic theory $A_\mu(x)\,dx^\mu$, which is not an exact differential when the electromagnetic field $F_{\mu\nu}(x) \neq 0$.

The Theorem 32 shows in a very clear way the natural relation between the relativistic dynamics of the charged particles and the differential geometry of the five-dimensional manifold $S_5$. *In particular it justifies the choice of $p_u$ as a fifth conjugate momentum, analogous to the four $p_\mu$.*

The possibility of the introduction of the pair of canonically conjugate variables $p_u$ and $u$ may be seen as a classical relativistic germ of the theory of the isospin of the particles, $p_u$ being related to $T_3$ and $u$ to an azimuthal angle in the $T_1$, $T_2$ plane of the isospin space.

2e. The Postulate of identification of $P_\mu P^\mu$ and $2K(p, x)_e$ has important consequences for the mass-charge formalism, by rendering the Hamiltonian $\bar{K}(p, x)$ a homogeneous function of degree 2 of $p_u$ and the $p_\mu$. We have the Theorem 34:

THEOREM 34. The homogeneity of degree 2 of $\bar{K}(p, x)$ in the five variables $p_u$ and $p_\mu$ renders the differential system (2d–3) invariant for the one-parameter group of transformations

$$x^\mu \to x^\mu \quad \text{and} \quad u \to u; \quad p_\mu \to kp_\mu \quad \text{and} \quad p_u \to kp_u \quad \text{with} \quad k \neq 0, \tag{1a}$$

$$\bar{V} \to k\bar{V} \quad \text{and} \quad z \to z/k. \tag{1b}$$

The Pfaff equation (2d–2) is also invariant for the transformations of the group (1a)—(1b).

The transformations with $k > 0$ constitute a subgroup, and those with $k < 0$ are the products of the former ones by the special transformation of the group (1a)—(1b) corresponding to $k - 1$.

We have the Theorem 35:

THEOREM 35. The transformations (1a)—(1b) leave invariant the world-lines of the particles given by the solutions of (2d–3), as a consequence of the first Eq. (1a), but the same world-line is associated to a particle with mechanical momentum $kP^\mu$, having the charge $ke$ and the square mass $k^2m^2$, as a consequence of the second group of Eqs. (1a) and the homogeneity of degree 2 of $\bar{K}(p, x)$ in the variables $p_u$ and $p_\mu$, as well as the interpretation rule for $p_u$

$$P^\mu \to kP^\mu, \qquad e \to ke, \qquad m^2 \to k^2m^2. \tag{2}$$

The second Eq. (1b) shows that the product $e\,dz_e$ is invariant for the transformations (1a—1b) on the same world-line, and thereby independent of the value of the charge of the particle. *The value of $ds_e$ on a world-line is invariant for the transformations* (1a)—(1b) *with* $k > 0$ *and changes only its sign for those* with $k < 0$.

We have also the Theorem 36:

THEOREM 36. *The transformation* (1a)—(1b) *corresponding to* $k = -1$ *gives a classical form of the theorem of charge conjugation.* It associates to any motion of a particle with charge $e$ and mass $m$ in an electromagnetic field $F_{\mu\nu}(x)$ a motion in the same field of a particle with the same mass $m$ and the opposite charge $-e$ on the same world-line, with the reversal of the mechanical momentum $P^\mu$, as well as of the differentials $dz_e$ of the duration and $ds_e$ of the proper-time.

The Theorems 34, 35 and 36 lead to the Theorem 37 for the pre-Hamiltonian systems (1–3) corresponding to different values of the charge of the particles:

THEOREM 37. (a) The differential system (1–3) for the particles with charge $e$ is transformed into that corresponding to the charge $ke$ with $k \neq 0$ by the transformation

$$x^\mu \to x^\mu, \qquad p_\mu \to kp_\mu \tag{3}$$

as a consequence of the homogeneity of degree 2 of $K(p, x)_e$ in the five variables $e$ and $p_\mu$ resulting from the Postulate of identification of $P_\mu P^\mu$ and $2K(p, x)_e$. The world-line for a particle with mass $m$ and charge $e$, corresponding to a solution of (1–3), becomes that given by a solution of the differential system for the value $ke$, of the type (1–3), for a particle with the square mass $k^2m^2$, the mechanical momentum at the same point in the latter motion being $kP^\mu$. It follows from the definition (1–7) of $dz_e$ that the corresponding $dz_{ke}$ at the same point of the world-line is $dz_e/k$. We get at the same point of the world-line for the two motions

$$ds_{ke} = (|k|/k)\, ds_e. \tag{4}$$

(b) *The transformation* (3) *with* $k = -1$ *gives a motion of a particle with mass* $m$ *and charge* $-e$ *associated to that of a particle with charge* $e$ *and mass* $m$ *on the same world-line, with opposite mechanical momentum vectors for the two particles at the same point of their common world-line. Thus the kinetic energy of the particle with charge* $-e$ *will be negative when that of the corresponding particle with charge* $e$ *is positive, on the same world-line.*

The part (b) of the Theorem 37 shows that the pre-Hamiltonian system (1–3) admits also physically inacceptable solutions with negative kinetic energies, as a consequence of the identification of $P_\mu P^\mu$ and $2K(p, x)_e$. They can however be reinterpreted as describing the motions of antiparticles with charge $-e$ and positive kinetic energies, in the same way as in the relativistic quantum mechanics, by using the corresponding solutions of the equations of type (1–3) with charge $-e$ and positive kinetic energies obtained from them by means of the transformation (3) with $k = -1$. *Thus we get a classical approach to the particle-antiparticle duality related to the reversal of the duration and the proper-time, at a very primary level of the construction of geometry.*

### 3. The Mass/Charge Formalism

The pre-Hamiltonian system (1–3), involving the parameter $e$, is the basis of the mass formalism dealing with the motions of all the particles of charge $e$, for all the real values of the square mass $m^2$ of the particles. The differential system (2c–3) of the mass-charge formalism does not involve the parameter $e$ and deals with the motions of the particles with any real values of $m^2$ and $e$. *Both the mass and the mass-charge formalisms do not strictly require the postulate of identification of $P_\mu P^\mu$ and $2K(p, x)_e$, although compatible with it.*

We shall now derive from the mass-charge formalism and the postulate $P_\mu P^\mu = 2K(p, x)_e$ a mass/charge formalism for the relativistic dynamics of a particle, not involving any parameter in the equations of motion, which is applicable to all the charged particles, and very similar to the mass formalism of Sec. 1. We have the Theorem 38:

THEOREM 38. We get from the differential system (2c–3) and the Postulate $P_\mu P^\mu = 2K(p, x)_e$ a pre-Hamiltonian system not involving any parameter, with the Hamiltonian $I(q, x)$ and the variables $q_\mu$ and $x^\mu$

$$dx^0/Q^0 = \cdots = dx^3/Q^3 = -dq_0/X_0' = \cdots = -dq_3/X_3' \tag{1}$$

by taking

$$q_\mu = p_\mu/p_u \quad \text{and} \quad I(q, x) = \bar{K}(p, x)/p_u{}^2 \quad \text{for} \quad p_u \neq 0, \tag{2}$$

$$Q^\mu = \partial_{q_\mu} I(q, x) \quad \text{and} \quad X_\mu' = \partial_{x^\mu} I(q, x). \tag{3}$$

$2I(q, x)$ is a constant of the motion of the particles, whose value on a world-line gives the ratio $m^2/e^2$

$$2I(q, x) = m^2/e^2. \tag{4}$$

For a particle with charge $e \neq 0$, the square mass $m^2$ and the momentum are

$$m^2 = 2e^2 I(q, x) \quad \text{and} \quad p_\mu = eq_\mu. \tag{5}$$

We shall call $2I(q, x)$ the square mass/charge function, as a consequence of the Eq. (4). We have the Theorem 39:

THEOREM 39. $I(q, x)$ is a function of the coordinates $x^\mu$ and of the gauge invariant vector of components $q_\mu - A_\mu(x)$.

We have also the Theorem 40:

THEOREM 40. The natural parameter of the mass/charge formalism for the world-lines of the particles is $w$, whose differential is the common value of all the ratios in the differential system (1). Thus we get the Hamilton equations of the mass/charge formalism

$$dx^\mu = \partial_{q_\mu} I(q, x)\, dw \quad \text{and} \quad dq_\mu = -\partial_{x^\mu} I(q, x)\, dw. \tag{6}$$

We have

$$dw = p_u\, dz. \tag{7}$$

We have also the Theorem 41:

THEOREM 41. (a) By means of the Einstein gravitational constant $\kappa$ we can obtain from $w$ a parameter $T$ with the dimension of a length

$$dT = dw/\sqrt{\kappa}, \tag{8}$$

*$T$ is a relativistic analog of the Newtonian absolute time for the motions of the charged particles.*

(b) The square mass/charge function has the dimension of $\kappa^{-1}$, which is a natural unit for the physical quantity $(m/e)^2$.

The Hamilton-Jacobi equation associated to the Hamilton system (6)

$$Y_w + I(Y_x, x) = 0 \tag{9}$$

is the Hamilton-Jacobi equation of the mass/charge formalism. The basic Pfaff equation of the mass/charge formalism is

$$dY = q_\mu \, dx^\mu - I(q, x) \, dw. \tag{10}$$

The solutions $Y(x, w)$ of the partial differential Eq. (9) define five-dimensional integral manifolds of the Pfaff equation (10)

$$Y = Y(x, w), \qquad q_\mu = \partial_{x^\mu} Y(x, w). \tag{11}$$

We have the Theorem 42:

THEOREM 42. The relativistic dynamics of the charged particles is associated to the five-dimensional differentiable manifold $\tilde{S}_5$ with the coordinates $w$ and $x^\mu$, as a consequence of the Postulate of identification of $P_\mu P^\mu$ and $2K(p, x)_e$, through the Hamilton-Jacobi equation (9) of the mass/charge formalism.

The linear differential form $q_\mu \, dx^\mu$ and its bilinear covariant $\delta q_\mu \, dx^\mu - dq_\mu \, \delta x^\mu$ play a central role in the theory of the contact transformations of the Hamilton equations (6) of the mass/charge formalism. We have the Theorem 43:

THEOREM 43. The mass/charge formalism leads to the introduction of a relativistic eight-dimensional phase space $S_8$, a differentiable manifold with the coordinates $q_\mu$, $x^\mu$, associated to the motions of the charged particles. The antisymmetric bilinear differential form $\delta q_\mu \, dx^\mu - dq_\mu \, \delta x^\mu$ gives a natural structure of Hamiltonian manifold to $S_8$, leading to a Liouville measure of the hypervolumes, with the dimension of $e^4$.

It is interesting to note that the mass/charge formalism leads to a single phase space $S_8$ for particles with any nonzero values of the charge. In the mass formalism there is a phase space $S(e)_8$ for each given value of $e$, because the gauge transformation (1a–2) of the $p_\mu$ depends on the value of the charge $e$ of the particle. In the case of $S_8$ the gauge transformation of the $q_\mu$

$$q_\mu \to q_\mu + \partial_{x^\mu} \Phi(x) \quad \text{for} \quad A_\mu(x) \to A_\mu(x) + \partial_{x^\mu} \Phi(x) \tag{12}$$

does not require a family of phase spaces $S_8$, because it does not involve a particular value of the electric charge.

**3a.** We shall now make use of the Einstein gravitational constant $\kappa$ in order to modify the mass/charge formalism. Let us introduce the new variables $r_\mu$ and $X$ and the function $J(r, x)$

$$r_\mu = \sqrt{\kappa}\, q_\mu, \qquad X = \sqrt{\kappa}\, Y \quad \text{and} \quad J(r, x) = \kappa I(q, x). \tag{1}$$

*It follows from the definitions* (1) *that $r_\mu\, dx^\mu$ and $X$ are scalars with the dimension of a length and $J(r, x)$ a dimensionless scalar.*

We get from the Eqs. (3–6) the Hamilton system of the modified mass/charge formalism, with the Hamiltonian $J(r, x)$ and the parameter $T$ defined by (3–8)

$$dx^\mu = \partial_{r_\mu} J(r, x)\, dT \quad \text{and} \quad dr_\mu = -\, \partial_{x^\mu} J(r, x)\, dT. \tag{2}$$

We get from (3–10) the basic Pfaff equation of the modified mass/charge formalism

$$dX = r_\mu\, dx^\mu - J(r, x)\, dT \tag{3}$$

and from (3–9) its Hamilton-Jacobi equation

$$X(x, T)_T + J(X_x, x) = 0. \tag{4}$$

*The Einstein constant $\kappa$ appears to be a basic classical link between the mass, the charge and the length, allowing the identification of non-geometric physical quantities to geometrical ones.* As a matter of fact $\kappa$ comes in in the Einstein theory in the identification of the energy-momentum tensor $T_{\mu\nu}$ and the geometric Einstein tensor $S_{\mu\nu}$ leading to the Einstein field equation of General Relativity

$$S_{\mu\nu} = 8\pi\kappa T_{\mu\nu} \quad \text{with} \quad S_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R, \tag{5}$$

$R_{\mu\nu}$ and $R$ denoting the contracted Riemann-Christoffel tensors built with the Riemannian metric of the world-manifold.

*The dimension of $\kappa$ is defined by the condition that $\kappa m$ be a length. Thereby $e\sqrt{\kappa}$ is also a length. Thus we get the length $r_\mu\, dx^\mu$ from the charge $q_\mu\, dx^\mu$.* We have the Theorem 44:

THEOREM 44. In the modified mass/charge formalism, the basic linear differential form $r_\mu\, dx^\mu$ has the dimension of a length and its bilinear covariant $\delta r_\mu\, dx^\mu - dr_\mu\, \delta x^\mu$ too. The phase space $S_8'$ of that formalism has an intrinsic Hamiltonian structure defined by the antisymmetric bilinear differential form $\delta r_\mu\, dx^\mu - dr_\mu\, \delta x^\mu$, giving a Liouville measure of its hypervolumes with the dimension of the fourth power of a length. The constant $\kappa$ allows a geometrization of the phase space of the relativistic dynamics of the charged particles.

We have the Theorem 45:

THEOREM 45. The Hamilton equations (2) give the infinitesimal transformation of a one-parameter continuous group of non-homogeneous contact transformations of the world-manifold, with a parameter $T$ having the dimension of a length, as a consequence of the equation

$$d(r_\mu\, \delta x^\mu) = \delta((R^\mu r_\mu - J(r, x))\, dT) \quad \text{with} \quad R^\mu = \partial_{r_\mu} J(r, x). \tag{6}$$

This group is related to the motions of all the charged particles in a given electromagnetic field, whereas that of the mass formalism defined by the Hamilton equations (1–9) is associated only to the motions of the particles with a given charge $e$.

The linear differential form $dX - r_\mu\, dx^\mu$ with the dimension of a length is related to a five-dimensional manifold $\bar{S}_5$ with coordinates $X$ and $x^\mu$, associated to the geometry of the world-manifold. We have the Theorem 46:

THEOREM 46. The non-homogeneous contact transformation of the world-manifold defined by the equation

$$r_\mu\, dx^\mu - r_\mu'\, dx'^\mu = dW(x, x'), \tag{7}$$

with the condition that the determinant of the derivatives of the second order $\partial^2_{x^\mu x'^\nu} W(x, x')$ be $\neq 0$, corresponds to the homogeneous contact transformation of $\bar{S}_5$ defined by the Eqs. (7) and

$$X - X' = W(x, x'), \tag{8}$$

which give

$$dX - r_\mu\, dx^\mu = dX_\mu' - r_\mu'\, dx'^\mu. \tag{9}$$

Thereby the group of non-homogeneous contact transformations of the world-manifold associated to the mass/charge formalism becomes a geometric group of $\bar{S}_5$.

It is interesting to note that the introduction of $X$ as a variable of the charged particles is done in an automatic way by the Cauchy differential system of the Hamilton-Jacobi equation (4)

$$dT = dx^0/R^0 = \cdots = -dr_0/\tilde{X}_0 = \cdots = dX/(R^\mu r_\mu - J(r, x)) \tag{10}$$

with $R^\mu$ defined by the second Eq. (6) and

$$\tilde{X}_\mu = \partial_{x^\mu} J(r, x). \tag{11}$$

Thus we get the Theorem 47:

THEOREM 47. The introduction of the new variable $X$ for the charged particles follows from the association of a one-dimensional integral manifold of the Pfaff equation (3) to each solution of the Hamilton system (2), by means of the Cauchy differential system (10) of the Hamilton-Jacobi equation (2) of the modified mass/charge formalism.

**3b.** The modified mass/charge formalism for the charged particles leads the introduction of a five-dimensional differentiable manifold $\hat{S}_5$ with the coordinates $T$ and $x^\mu$, instead of the $\tilde{S}_5$ of the Theorem 42. We have the Theorem 48:

THEOREM 48. The five-dimensional differentiable manifold $\hat{S}_5$ belongs actually to the geometry of the world-manifold $S_4$, as a kind of geometric extension, because the $x^\mu$ are the same as in $S_4$, and the variable $T$ has the dimension of a length.

The behaviour of $T$ as an independent variable with respect to the world-coordinates $x^\mu$ on the world-lines of the charged particles results from the fact that

$$2J(r, x)\, dT^2 = ds^2, \tag{1}$$

the differential $ds$ of the proper time being taken as

$$ds = (2\bar{K}(p, x))^{1/2}\, dz. \tag{2}$$

We shall see in Sec. 4 that

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu, \tag{3}$$

with the adequate choice of $K(p, x)_e$, the $g_{\mu\nu}(x)$ being functions of the coordinates $x^\mu$. Thereby we have at the point $x$ of a world-line

$$\kappa\, dT^2 = (e/m)^2 g_{\mu\nu}(x)\, dx^\mu\, dx^\nu. \tag{4}$$

Thus we get the Theorem 49:

THEOREM 49. For given $x^\mu$ and $dx^\nu$ the differential $dT$ depends on the choice of the ratio $e^2/m^2$ of the moving charged particle, for a given electromagnetic field $F_{\mu\nu}(x)$ of the world-manifold, endowed with the Riemannian metric $g_{\mu\nu}(x)$ obtained from the particle dynamics.

## 4. The Dynamical Foundation of the Riemannian Metric

We must now introduce a new Postulate for the determination of the square mass function $2K(p, x)$, which will also give the dynamical foundation of the Einstein equation (3b–3), and thereby of the Riemannian metric of the world-manifold $S_4$.

Our mass formalism of the particle dynamics is distinguished by the existence of a covariant Hamiltonian mechanical momentum vector $P_\mu$ and a contravariant Newtonian mechanical momentum vector $P^\mu$, related by means of the function $K(p, x)_e$. *Thereby the problem of the determination of $K(p, x)_e$ is basically connected to that of the relation between $P^\mu$ and $P_\nu$.*

We shall now introduce the Postulate A:

POSTULATE A. In all the motions of the particles through a point $x$ of $S_4$ there is a one-one linear correspondence

$$P^\mu = g^{\mu\nu}(x) P_\nu, \tag{1}$$

with the $g^{\mu\nu}(x)$ not depending on the particular motion in consideration but only on $x$. The nonzero determinant of the components of the tensor $g^{\mu\nu}(x)$ is negative at all $x$. The $g^{\mu\nu}(x)$ are everywhere continuous and have continuous derivatives up to the third order.

We have the Theorem 50:

THEOREM 50. The Postulate A and the Eq. (1a–6) give

$$\partial_{P_\mu} N(P, x) = g^{\mu\nu}(x) P_\nu, \tag{2}$$

so that

$$g^{\mu\nu}(x) = \partial^2_{P_\mu P_\nu} N(P, x). \tag{3}$$

*The integrability of the system of partial differential Eqs.* (2) *imposes the symmetry of the tensor* $g^{\mu\nu}(x)$

$$g^{\mu\nu}(x) = g^{\nu\mu}(x) \tag{4}$$

*not required by the Postulate A.*

We get from (2) and (4) the Theorem 51:

THEOREM 51. It follows from (2) and (4) that

$$2N(P, x) = g^{\mu\nu}(x)P_\mu P_\nu + S(x) \quad \text{with} \quad S(x) \text{ an arbitrary scalar} \tag{5}$$

the tensor $g^{\mu\nu}(x)$ being symmetric. On the other hand, it follows from (1) and (4) that

$$P_\mu P^\mu = g^{\mu\nu}(x)P_\mu P_\nu. \tag{6}$$

The identification of $P_\mu P^\mu$ and $2N(P, x)$ requires that the scalar field $S(x)$ be taken as zero. Thus $N(P, x)$ becomes a quadratic form of the $P_\mu$

$$2N(P, x) = g^{\mu\nu}(x)P_\mu P_\nu \quad \text{for} \quad S(x) = 0. \tag{7}$$

We have the Theorem 52:

THEOREM 52. When the scalar field $S(x) = 0$ there is a mass/charge formalism, as a consequence of the Postulate A and the symmetry condition (4), with

$$2I(q, x) = g^{\mu\nu}(x)(q_\mu - A_\mu(x))(q_\nu - A_\nu(x)) \tag{8}$$

because

$$2\bar{K}(p, x) = g^{\mu\nu}(x)(p_\mu - p_u A_\mu(x))(p_\nu - p_u A_\nu(x)) + S(x) \tag{9}$$

as a consequence of (5).

We have the Theorem 53:

THEOREM 53. The Postulate A renders the vector space of the $P_\mu$ at the point $x$ of the world-manifold into a pseudo-euclidean vector space with the indefinite metric quadratic form $g^{\mu\nu}(x)P_\mu P_\nu$, the condition of symmetry (4) being imposed. Because of the four-dimensionality of the world-manifold $S_4$, the negative sign of the determinant of the $g^{\mu\nu}(x)$ restricts the signature of $g^{\mu\nu}(x)P_\mu P_\nu$ to be either $-2$ or 2, the kind of pseudo-euclidean geometry being essentially the same in both cases, which are exchanged by the change of the sign of the quadratic form.

*When* $S(x) = 0$, $g^{\mu\nu}(x)P_\mu P_\nu$ *defines the mass-metric of the* $P_\mu$, *with* $g^{\mu\nu}(x)P_\mu P_\nu$ *giving the value of* $m^2$ *for the particles having the mechanical momentum* $P_\mu$ *at the point* $x$ *of their world-lines.*

We shall introduce the Postulate B:

POSTULATE B. The signature of the quadratic form $g^{\mu\nu}(x)P_\mu P_\nu$ is $-2$.

We have the Theorem 54:

THEOREM 54. The indefiniteness of the quadratic form $g^{\mu\nu}(x)P_\mu P_\nu$ allows $m^2$ to take all the real values on the world-lines of the particles with a given charge $e$, passing at any world-point $x$. The Postulate B associates the negative values of $m^2$ to the tachyons.

We have

$$P_\mu = g_{\mu\nu}(x)P^\nu \quad \text{with} \quad g_{\mu\nu}(x)g^{\nu\rho}(x) = \delta_\mu{}^\rho. \tag{10}$$

*The tensor* $g_{\mu\nu}(x)$ *defines a pseudo-euclidean metric of the Minkowski type in the vector space of the contravariant vectors* $V^\mu$ *at the point* $x$ *of the world-manifold, as a consequence of the signature* $-2$ *of the quadratic form* $g_{\mu\nu}(x)V^\mu V^\nu$. Thus we get the Theorem 55:

THEOREM 55. The Minkowskian metric of the $V^\mu$ at $x$ defined by the mechanical tensor $g_{\mu\nu}(x)$ leads to a normal hyperbolic Riemannian metric of the world-manifold, by taking the square of the $ds$ of an infinitesimal displacement from $x$ to $x + dx$ on a line as the value of the quadratic form $g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$ for the contravariant vector of components $dx^\mu$

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu\, dx^\nu. \tag{11}$$

In the case of a world-line of a particle we get from the first Hamilton equation (1–9)

$$dx^\mu = P^\mu\, dz_e \quad \text{and} \quad g_{\mu\nu}(x)\, dx^\mu\, dx^\nu = g_{\mu\nu}(x)P^\mu P^\nu\, dz_e{}^2 \tag{12}$$

since $g_{\mu\nu}(x)P^\mu P^\nu = g^{\mu\nu}(x)P_\mu P_\nu$ we have

$$ds^2 = g^{\mu\nu}(x)P_\mu P_\nu\, dz_e{}^2. \tag{13}$$

When $P_\mu P^\mu$ is identified to $2K(p, x)_e$, the Eq. (13) becomes

$$ds^2 = 2K(p, x)_e\, dz_e{}^2 \tag{14}$$

and by taking into account the definition (1–18) of $ds_e$ we get

$$ds^2 = ds_e{}^2. \tag{15}$$

Thus we get the Theorem 56:

THEOREM 56. The Riemannian metric of the world-manifold defined by (11) gives the square proper-time differential $ds_e{}^2$ for the infinitesimal displacements $dx^\mu$ on the world-lines of the particles, when the Postulate of identification of $P_\mu P^\mu$ and $2K(p, x)_e$ is valid, therefore when $S(x) = 0$.

The Theorem 56 shows that the introduction of the normal hyperbolic Riemannian metric of the world-manifold, with the mechanical $g_{\mu\nu}(x)$ as the metric tensor, leads to a generalization of the dynamical definition of the proper-time to any differentiable world-line, for $S(x) = 0$. We shall introduce the following definition:

DEFINITION. The $ds^2$ of the Riemannian metric given by (11) is the square of the proper-time differential corresponding to the infinitesimal displacement $dx^\mu$ on any world-line of $S_4$, when $S(x) = 0$.

We have the Theorem 57:

THEOREM 57. The distinction of the time-like infinitesimal displacements with $ds^2 > 0$, the space-like infinitesimal displacements with $ds^2 < 0$ and the null infinitesimal displacements with $ds^2 = 0$ renders the world-manifold into a space-time. The space-time structure is based on the existence of the mass-metrics in the vector spaces of the mechanical momentum vectors $P_\mu$ and $P^\mu$ at all the points $x$ of the world-manifold for $S(x) = 0$

$$g^{\mu\nu}(x)P_\mu P_\nu = m^2 \quad \text{and} \quad g_{\mu\nu}(x)P^\mu P^\nu = m^2. \tag{16}$$

The existence of the space-time structure is also related to that of a mass/charge formalism of the particle dynamics for $S(x) = 0$.

It is noteworthy that the second group of Hamilton equations (1–9) does not come in in the mechanical construction of the normal hyperbolic Riemannian metric of the world-manifold, only the first group

$$dx^\mu = g^{\mu\nu}(x)P_\nu \, dz_e. \tag{17}$$

**4a.** The Theorems 8 and 9 show that the inertial properties of the particles are described by the function $N(p, x)$, because it gives the Hamiltonian $K(p, x)_0$ of the free neutral particles, whose motions are purely inertial. Since $N(p, x)$ is determined by the two fields $g^{\mu\nu}(x)$ and $S(x)$, as a consequence of the Eq. (4–5), we have the Theorem 58:

THEOREM 58. The two fields $g^{\mu\nu}(x)$ and $S(x)$ describe the inertial properties of the particles. When we take $S(x) = 0$, those inertial properties are completely described by the tensor field $g^{\mu\nu}(x)$. The Riemannian metric of the world-manifold is therefore determined by inertial properties of the particles.

We have also the Theorem 59:

THEOREM 59. The inertial properties of the particles determine also the affine connection of the world-manifold given by the Christoffel symbols $\begin{Bmatrix} \lambda \\ \mu \ \ \nu \end{Bmatrix}$ and the Riemann-Christoffel curvature tensor $R^\mu_{\nu,\rho\sigma}$

$$\begin{Bmatrix} \lambda \\ \mu \ \ \nu \end{Bmatrix} = \tfrac{1}{2} g^{\lambda\rho}(x)(\partial_{x^\mu} g_{\rho\nu}(x) + \partial_{x^\nu} g_{\rho\mu}(x) - \partial_{x^\rho} g_{\mu\nu}(x)), \tag{1}$$

$$R^\mu_{\nu,\rho\sigma} = \partial_{x^\rho} \Gamma_{\nu\sigma}{}^\mu - \partial_{x^\sigma} \Gamma_{\nu\rho}{}^\mu + \Gamma_{\nu\sigma}{}^\alpha \Gamma_{\alpha\rho}{}^\mu - \Gamma_{\nu\rho}{}^\alpha \Gamma_{\alpha\sigma}{}^\mu \text{ with } \Gamma_{\mu\nu}{}^\lambda = \begin{Bmatrix} \lambda \\ \mu \ \ \nu \end{Bmatrix}, \tag{2}$$

as well as the Einstein symmetric tensor $S_{\mu\nu}$ of the field Eqs. (3a–5).

The Einstein system (3a–5) is a set of partial differential equations for the inertial tensor $g^{\mu\nu}(x)$ of the particles. It may also be seen as a system for the determination of the inertial function $N(y, x)$, when $S(x) = 0$.

The function $\bar{K}(y, x)$ of the nine variables $y_u$, $y_\mu$, $x^\mu$, defined by the Eq. (4–9), is a polynomial of degree 2 of the five $y$, with coefficients built with the $g^{\mu\nu}(x)$, $S(x)$ and $A_\mu(x)$. We have the Theorem 60:

THEOREM 60. When $S(x) = 0$, the Einstein equations for the $g^{\mu\nu}(x)$ and the electromagnetic partial differential equations for the $A_\mu(x)$ constitute a set of partial differential equations for the determination of the polynomial $\bar{K}(y, x)$ of the $y$. When $S(x)$ is not constant we need also another partial differential equation for $S(x)$, in order to determine the function $\bar{K}(y, x)$. *The determination of $\bar{K}(y, x)$ leads therefore to a unification of the theories of the inertial fields $g^{\mu\nu}(x)$ and $S(x)$ with the electromagnetic field.*

*The above results show that the Hamiltonian Function $\bar{K}(p, x)$ of the mass-charge formalism is a key element of the whole Classical Physics, giving a unified description of the basic geometric-inertial fields and of the electromagnetic field. $\bar{K}(p, x)$ is also related to the Hamiltonian geometry of the world-manifold, besides its Riemannian geometry, and to the Classical statistical mechanics, in its relativistic form. Thus the mass concept appears as the central concept of the whole structure of the Classical Physics.*

**4b.** It follows from the Hamilton equations (1–3) and the Eq. (4–7) that the absolute differential $DP_\mu$, corresponding to the Riemannian metric defined by the inertial tensor $g^{\mu\nu}(x)$, has the value

$$DP_\mu = eF_{\mu\nu}(x)\, dx^\nu \quad \text{with} \quad Dp_\mu = dp_\mu - \left\{ \begin{matrix} \lambda \\ \mu \quad \nu \end{matrix} \right\} p_\lambda\, dx^\nu \tag{1}$$

so that we have a satisfactory relativistic equation for the absolute differential $DP^\mu$

$$DP^\mu = eg^{\mu\nu}(x) F_{\nu\rho}(x)\, dx^\rho. \tag{2}$$

In the case of a free particle

$$DP^\mu = 0 \tag{3}$$

*so that $P^\mu$ undergoes a parallel displacement of the affine connection $\left\{ \begin{matrix} \lambda \\ \mu \quad \nu \end{matrix} \right\}$ along the world-line of the particle.* Since $P^\mu = dx^\mu/dz_e$, we get the differential equation of the world-lines of the particles

$$d^2x^\mu/dz_e^2 + \left\{ \begin{matrix} \mu \\ \rho \quad \sigma \end{matrix} \right\} dx^\rho/dz_e\, dx^\sigma/dz_e = eg^{\mu\nu}(x) F_{\nu\rho}(x)\, dx^\rho/dz_e \tag{4}$$

not involving any particular value of the mass $m$.

By expressing $dz_e$ in terms of $ds$, the Eq. (4) becomes

$$d^2x^\mu/ds^2 + \left\{ \begin{matrix} \mu \\ \rho \quad \sigma \end{matrix} \right\} dx^\rho/ds\, dx^\sigma/ds = (2K(p, x)_e)^{-1/2} eg^{\mu\nu}(x) F_{\nu\rho}(x)\, dx^\rho/ds \tag{5}$$

for $K(p, x)_e \neq 0$. We have the Theorem 61:

THEOREM 61. The differential equations of the world-lines of a particle with charge $e$ and mass $m$ are obtained from (5) by the substitution of $2K(p, x)_e$ by $m^2$, for $m \neq 0$. When $e = 0$, the Eqs. (5) become the ordinary differential equations of the non-null geodesics of the world-manifold, with the metric tensor $g_{\mu\nu}(x)$. The Eqs. (4) with $e = 0$ are the differential equations of all the geodesics of $S_4$, including those with $ds^2 = 0$.

By the introduction of the new parameter $w = ez_e$ we get from (4) a system of differential equations of the world-lines of the particles applicable to all the charged particles

$$d^2x^\mu/dw^2 + \begin{Bmatrix} \mu \\ \rho \quad \sigma \end{Bmatrix} dx^\rho/dw \, dx^\sigma/dw = g^{\mu\nu}(x) F_{\nu\rho}(x) \, dx^\rho/dw \tag{6}$$

because it does not involve $e$.

**4c.** We shall now consider the Hamilton system for the variables $p_\mu$, $x^\mu$ with the Hamiltonian function $M(p, x)_e$ and the parameter $s_e$

$$dx^\mu = \partial_{p_\mu} M(p, x)_e \, ds_e, \qquad dp_\mu = - \partial_{x^\mu} M(p, x)_e \, ds_e, \tag{1}$$

$$M(p, x)_e = (2K(p, x)_e)^{1/2} \tag{2}$$

the Hamiltonian $M(p, x)_e$ and the parameter $s_e$ being real. We have the Theorem 62:

THEOREM 62. The solutions of the system (1) with real values of $s_e$ and $K(p, x)_e$ are the same as those of the Hamilton system (1–9) with $ds_e$ given by (1–18), for $K(p, x)_e > 0$. The values of the constant of the motion $M(p, x)_e$ on the world-lines of the particles satisfying the Eqs. (1) give the corresponding positive values of the masses $m$. $s_e$ is the real proper-time parameter.

The introduction of the Postulates A and B of Sec. 4 gives

$$M(p, x)_e = (g^{\mu\nu}(x)(p_\mu - eA_\mu(x))(p_\nu - eA_\nu(x)) + S(x))^{1/2}. \tag{3}$$

*Thus we obtain an extension to General Relativity of the mass formalism given in* (SCHÖNBERG 1947) *for Special Relativity with* $S(x) = 0$, *which is naturally related to the wave equations of the first order of the relativistic quantum mechanics of the type of the Dirac equation.*

The operator $M_{op} = \gamma^\mu(p_{\mu;op} - eA_\mu(x))$ is a quantum analog of $M(p, x)_e$ for the particles with spin $\frac{1}{2}$, in Special Relativity. The Dirac equation can be written as $M_{op}\psi = m\psi$, corresponding to the classical Hamilton-Jacobi equation for the particles with charge $e$ and mass $m$, $M(U_x, x)_e = m$, obtained from the Hamilton-Jacobi equation associated to the system (1)

$$V_{s_e} + M(V_x, x)_e = 0 \tag{4}$$

by taking

$$V(x, s_e) = - ms_e + U(x). \tag{5}$$

We have the Theorem 63:

THEOREM 63. The type of Hamilton-Jacobi partial differential equation (4) leads to the introduction in Special Relativity of a generalized Dirac equation, with a wave spinor $\phi(x, s)$ not involving any particular value $m$ of the mass of the particle

$$i\hbar\partial_s\phi = M_{op}\phi. \tag{6}$$

The ordinary Dirac equation $M_{op}\psi = m\psi$ corresponding to the particular solution

$$\phi(x, s) = \psi(x) \exp(- is/\hbar). \tag{7}$$

## 5. The Four-Dimensionality of the World-Manifold

We assumed in the preceding sections that the world-manifold is four-dimensional, but this assumption could be easily dropped, by a slight modification of the Postulates A and B in order to get a normal hyperbolic metric for a value $n \geqslant 4$ of the dimensionality of the "world-manifold."

The four-dimensionality gives special properties to the antisymmetric tensors of the second order $E_{\mu\nu}$, even at the pre-Riemannian level of the geometry of the differentiable manifold, as a consequence of the existence of the scalar-density $\varepsilon(E, E')$, bilinear and symmetric with respect to $E_{\mu\nu}$ and $E_{\mu\nu}'$, built with the Levi Civita antisymmetric tensor-density $\varepsilon^{\kappa\lambda\mu\nu}$

$$\varepsilon(E, E') = \tfrac{1}{4}\varepsilon^{\kappa\lambda\mu\nu}E_{\kappa\lambda}E_{\mu\nu}'. \tag{1}$$

At the Riemannian level of geometry, we get the well known duality of the $E_{\mu\nu}$, $^*E_{\mu\nu}$

$$^*E_{\mu\nu} = \tfrac{1}{2}|g|^{1/2}\varepsilon_{\mu\nu\rho\sigma}g^{\rho\kappa}g^{\sigma\lambda}E_{\kappa\lambda}, \quad (g = \text{determinant of } g_{\mu\nu}). \tag{2}$$

The dual $^*F_{\mu\nu}(x)$ of the electromagnetic field $F_{\mu\nu}(x)$ derived from the potentials $A_\mu(x)$ is precisely the field involved in the non-homogeneous Maxwell equations. *This shows already the need of the four-dimensionality for the electromagnetic theory.* We have discussed in considerable detail the role of the tensor-duality in the electromagnetic theory, in our paper (SCHÖNBERG 1971), in which we have also given a construction of the geometry of the world-manifold based on the electromagnetic theory in a general differentiable manifold.

We have the Theorem 64:

THEOREM 64. *In the dynamics of a particle, the four-dimensionality of the world-manifold plays a central role in the theory of the spin, whose basic vector $S^\mu$ for the spin is defined in terms of the angular momentum tensor $M_{\mu\nu}$ and $P_\mu$, by means of the Levi Civita $\varepsilon^{\kappa\lambda\mu\nu}$*

$$S^\mu = \tfrac{1}{2}|g|^{-1/2}\varepsilon^{\mu\nu\rho\sigma}M_{\nu\rho}P_\sigma. \tag{3}$$

The scalar $g_{\mu\nu}(x)S^\mu S^\nu/m^2$ gives the value of the square of the spin of the particle at the point $x$ of its world-line.

In the paper (SCHÖNBERG 1971), we started from the existence of the two electromagnetic tensors $^*F_{\mu\nu}$ and $F_{\mu\nu}$, related by a linear operator $L$ satisfying two conditions

$$^*F = LF \text{ with } L^2 = -1_{op} \text{ and } \varepsilon(LE, E') = \varepsilon(E, LE') \text{ for all } E, E' \tag{4}$$

suggested by Eq. (2), which involves only the tensor $C^{\mu\nu}(x)$ of the conformal geometry of the world-manifold

$$C^{\mu\nu}(x) = |g|^{1/4} g^{\mu\nu}(x). \tag{5}$$

*The Eqs.* (4) *were used to obtain the conformal metric of the world-manifold, which can be associated to a normal hyperbolic Riemannian geometry, as a consequence of the sign minus in the second Eq.* (4). The Eq. (2) leads to the second Eq. (4) only for $g(x) < 0$.

The above discussion leads to the Theorem 65:

THEOREM 65. The four-dimensionality of the world-manifold and the normal hyperbolic type of its Riemannian metric, obtained from the inertial tensor $g^{\mu\nu}(x)$ given by the relation between the $P^\mu$ and $P_\mu$, are determined by the condition of the existence of an electromagnetic tensor $^*F_{\mu\nu}(x)$ satisfying the equation $^*(^*F_{\mu\nu}(x)) = -F_{\mu\nu}(x)$. The inertial tensor $g^{\mu\nu}(x)$ describes also the properties of dielectricity and magnetic permeability of the world-manifold, by means of its associate conformal tensor $C^{\mu\nu}(x)$ of Eq. (5), which defines the relation between the two electromagnetic fields $^*F_{\mu\nu}(x)$ and $F_{\mu\nu}(x)$.

*The results of the present paper and those of* (SCHÖNBERG 1971) *belong to a new level of the Theory of Relativity, not coinciding with either Special Relativity, General Relativity or any of the forms of unified field theories, but related to those other three levels. The new level of Relativity is characterized by a different approach to the geometry of the world-manifold, relating all its properties to physical properties of matter, even its four-dimensionality.*

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