Electromagnetism and Gravitation

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A formulation of the electromagnetic theory in a differentiable manifold devoid of any metric and affine structure is discussed. It is shown that the Maxwell equations in such a manifold involve a tensor describing the properties of the dielectricity and magnetic permeability of space because of the anisotropy of such a general space. It is also shown that this tensor is essentially equivalent to the metric of the angles on the manifold. Thus the necessity of having equations for the determination of this tensor in order to determine the electromagnetic field shows that the Maxwell equations are not a complete set of differential electromagnetic equations. The Einstein gravitational equation appears as complementing the Maxwell set of equations allowing the determination of the dielectricity tensor. Thus a natural fusion of the electromagnetic and gravitational theories is obtained with an electromagnetic foundation for the geometry of the world-manifold.

Discute-se uma formulação da teoria eletromagnética numa variedade diferenciável desprovida de quaisquer métrica e estrutura afim. Mostra-se que as equações de Maxwell em tal variedade envolvem um tensor que descreve as propriedades da dieletricidade e da permeabilidade magnética do espaço devido à anisotropia de tal espaço geral. Mostra-se tambem que êsse tensor é essencialmente equivalente à métrica dos ângulos na variedade. Assim a necessidade de se ter equações para a determinação dêsse tensor, a fim de se determinar o campo eletro-magnético, mostra que as equações de Maxwell não são um conjunto completo de equações diferenciais eletromagnéticas. A equação gravitacional de Einstein aparece complementando o conjunto das equações de Maxwell permitindo então a determinação do tensor de dieletricidade. Obtém-se assim uma fusão natural das teorias eletromagnética e gravitacional dando-se um fundamento eletromagnético à geometria da variedade-universo.

1. Introduction

In this paper we present the main results of our work on the formulation of the electromagnetic theory in a world taken only with a structure of differentiable manifold, without the *a priori* assumption of a Riemannian geometry or even of an affine connection. The first results were communicated at the Kyoto Conference[1] in 1965. Later developments were given in unpublished lectures at the 1966 *Blumenau* meeting of the *Sociedade Brasileira para o Progresso da Ciencia*, at the *Institut Henri Poincaré* in *Paris* (1967) and at the 1969 and 1970 *Symposia de Fisica Teorica* of the *Pontificia Universidade Catolica* of *Rio de Janeiro*[2].

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The goal of our work is to discuss the distinguished rôle of electromagnetism in Physics, in particular in the foundations of the normal hyperbolic Riemannian geometry of the world-manifold and of the gravitational theory. We were inspired by some ideas presented by Dirac, a long time ago, on the special rôle of electromagnetism in Physics, resulting from the fact that all measurements depend directly or indirectly on electromagnetic phenomena.

We started from the description of the electromagnetic field by a pair of antisymmetric covariant tensors $F_{\mu\nu}$ and $*F_{\mu\nu}$, involved in the homogeneous and non homogeneous Maxwell equations, respectively. *This gave us the foundation of the dimensionality $n = 4$ of the world-manifold, because the structure of the Maxwell equations in terms of the F, $*F$ pair is only possible for $n = 4$.* This point is discussed in Section 3.

The relation between the fields F and $*F$ corresponds to the properties of dielectricity and magnetic permeability of space, since F is the (**B**, **E**) field and $*F$ the (**H**, **D**) field. This relation is mathematically expressed by a linear operator L:

$$*F = LF \text{ and } F = -L\ *F, \textit{ so that } L^2 = -1_{op},\ 1_{op} = \textit{unit operator.} \tag{1a}$$

It is also necessary to assume the symmetry condition

$$\varepsilon^{\kappa\lambda\rho\sigma} L^{\mu\nu}_{\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} L^{\kappa\lambda}_{\rho\sigma} \quad \textit{with} \quad (LF)_{\mu\nu} = (1/2) L^{\rho\sigma}_{\mu\nu} F_{\rho\sigma}, \tag{1b}$$

which requires the four-dimensional Ricci symbol $\varepsilon^{\kappa\lambda\mu\nu}$, hence the four-dimensionality.

In General Relativity, $*F$ is taken as the dual of F according to the general definition of the dual of an antisymmetric covariant tensor corresponding to the metric given by the symmetric tensor $g_{\mu\nu}$ of determinant $g < 0$:

$$*F_{\mu\nu} = (1/2)\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} g^{\rho\alpha} g^{\sigma\beta} F_{\alpha\beta} = (1/2)\varepsilon_{\mu\nu\rho\sigma} C^{\rho\alpha} C^{\sigma\beta} F_{\alpha\beta} \tag{1c}$$

with $C^{\mu\nu} = (-g)^{1/4} g^{\mu\nu}$.

The $C_{\mu\nu}$ defined by the condition

$$C_{\mu\rho} C^{\nu\rho} = \delta^{\nu}_{\mu}, \quad C_{\mu\nu} = (-g)^{-1/4} g_{\mu\nu}, \tag{1d}$$

are the components of the conformal metric which gives the angular metric associated to the Riemannian metric $g_{\mu\nu}$. The $C_{\mu\nu}$ are obviously invariant for the change of gauge $g_{\mu\nu} \to S g_{\mu\nu}$, S denoting an arbitrary scalar. The definition (1*c*) of $*F$ depends only on the conformal metric $C_{\mu\nu}$.

We found that the Riemannian metric $g_{\mu\nu}$ of the four-dimensional world-manifold is determined up to a scalar factor by the properties (1*a*) *and* (1*b*) *of the electromagnetic linear operator L, because they determine the Lorentz group at each point of the world-manifold.* The proof of this Fundamental Theorem will be given in Section 4. In particular, the minus sign in $L^2 = -1_{op}$ plays a central rôle and determines the signature of $g_{\mu\nu}$.

The Fundamental Theorem results immediately from the fact that the linear operator L with $L^2 = -1_{op}$ defines a complex structure in the six-dimensional vector space S_6 constituted by the antisymmetric tensors $A_{\mu\nu}$ at any point of the world-manifold, which allows to go over from the six-dimensional real space S_6 endowed with the indefinite metric $\varepsilon^{\kappa\lambda\mu\nu} A_{\kappa\lambda} A_{\mu\nu}$ to a three-dimensional complex Euclidean vector space. The $SO(3, C)$ of the latter vector space is isomorphic to the Lorentz group component containing the identity. *The Lorentz group at a point x determines $g_{\mu\nu}(x)$ up to a numerical factor $S(x)$.*

It is well known that the dimensionality $n = 4$ is exceptional with respect to its orthogonal groups. For $n > 2$, the groups $SO(n)$ and $SO(n-p, p)$ are all simple when $n \neq 4$. *But for $n = 4$ only $SO(3, 1)$, the Lorentz group, is simple.* This property is related to the equation $L^2 = -1_{op}$ and to the complex structure it defines in S_6, $L^2 = 1_{op}$ corresponds to both the definite metric and to the indefinite metric with signature O, whose orthogonal groups are not simple.

The above results, especially the Fundamental Theorem, show clearly that the properties of dielectricity and magnetic permeability of space can be used to give a physical basis to the construction of a conformal geometry of the world-manifold, which can only be associated to tensors $g_{\mu\nu}$ with the Minkowski signature (*normal hyperbolic metric*). *The aforementioned symmetry property of L cannot be introduced in manifolds of dimensionality $n \neq 4$, because it depends on the existence of the Ricci fourth order tensor-density $\varepsilon^{\kappa\lambda\mu\nu}$, which allows to get a symmetric bilinear form $\varepsilon(A, B)$ of antisymmetric covariant tensors of the second order A and B,*

$$\varepsilon(A, B) = (1/4)\varepsilon^{\kappa\lambda\mu\nu} A_{\kappa\lambda} B_{\mu\nu}. \tag{2a}$$

The symmetry condition (1b) is equivalent to

$$\varepsilon(A, LB) = \varepsilon(LA, B) \quad \textit{so that} \quad \varepsilon(LA, LB) = -\varepsilon(A, B). \tag{2b}$$

It will be shown in Section 3 that the possibility of the rotational form of the Maxwell equations for the two fields F and $*F$ is also related to the existence of $\varepsilon(A, B)$ and can be used as a condition to get $n = 4$, taking as a basic postulate that F and $*F$ are both of the second order.

The electromagnetic linear operator L determines only up to a sign a relative symmetric tensor $C_{\mu\nu}$, which defines a conformal metric. The introduction of the electromagnetic potentials leads to the imposition of the Lorentz condition, which cannot be written only in terms of $C_{\mu\nu}$. *Thus we get a basis for the introduction of a quantity Z of the kind of* $(-g)^{1/4}$, *such that the indetermination in the sign of the* $C_{\mu\nu}$, *also related to a power* 1/4, *be compensated and we get an electromagnetic symmetric tensor* $g_{\mu\nu}$ *with the Minkowski signature,*

$$g_{\mu\nu} = ZC_{\mu\nu} \quad \textit{so that} \quad g = -Z^4. \tag{3}$$

The availability of an electromagnetic symmetric tensor $g_{\mu\nu}$ with $g \neq 0$ allows to get the covariant differentiation of the physical quantities by means of the Christoffel symbols of $g_{\mu\nu}$, especially of the electromagnetic tensor of energy and momentum. *Thus we are able to get the Lorentz force as a covariant divergence of that tensor.*

The Maxwell equations associated to the algebraic equations (1a) and (1b) are not sufficient for the determination of the fields F and $*F$, because they do not allow the determination of the field $L^{\mu\nu}_{\kappa\lambda}(x)$. We can obtain $L(x)$ in terms of the $g_{\mu\nu}(x)$,

$$L^{\mu\nu}_{\kappa\lambda} = \varepsilon^{\mu\nu\rho\sigma} C_{\kappa\rho} C_{\lambda\sigma} = (-g)^{-1/2} \varepsilon^{\mu\nu\rho\sigma} g_{\kappa\rho} g_{\lambda\sigma}. \tag{4}$$

We are naturally led to choose, as equations for the determination of the $L(x)$, *the Einstein type of equations for the* $g_{\mu\nu}(x)$ *on covariance grounds.* We shall actually postulate those equations directly in another way.

The mathematical problem of finding the equations for the determination of the electromagnetic $g_{\mu\nu}$ equivalent to Z and the $L^{\mu\nu}_{\kappa\lambda}$ is the same as that of the determination of the gravitational potentials in General Relativity. We need a set of 10 covariant equations for the electromagnetic $g_{\mu\nu}$, related by 4 identities because the values of the $g_{\mu\nu}(x)$ cannot be totally determined without the fixation of the four arbitrary functions involved in the choice of the coordinates x^μ on the world-manifold. The equations must be of the form

$$R_{\mu\nu} - (1/2) g_{\mu\nu} R + \lambda g_{\mu\nu} = \theta_{\mu\nu} \textit{ with constant } \lambda \textit{ and } \theta_{\mu\nu} = \theta_{\nu\mu}, \tag{5}$$

because of a well known theorem of Cartan which shows that the only second order symmetric tensors built with the $g_{\mu\nu}$ and their derivatives up to order 2 having covariant divergence 0 are of the form of the tensor in the left-hand side of (5). Thus the covariant divergence of $\theta_{\mu\nu}$ is 0:

$$D_\rho(g^{\rho\mu}\theta_{\mu\nu}) = 0 \text{ with } D_\rho = x^\rho \text{ covariant derivative of the metric } g_{\mu\nu}. \quad (6)$$

In order to choose $\theta_{\mu\nu}$, the deflection of light rays by large stellar masses gives the fundamental clue: *we can take the electromagnetic tensor* $g_{\mu\nu}$ *as the metric and* $\theta_{\mu\nu} = m\kappa T_{\mu\nu}$, *m being a numerical constant and* $T_{\mu\nu}$ *the energy-momentum tensor of matter of all kinds, including the electromagnetic field itself, because the masses are seen to act on the electromagnetic waves and on the other hand* $D_\rho(g^{\rho\mu}T_{\mu\nu}) = 0$. $\kappa = G/c^4$, with G denoting the gravitational constant, must be introduced for dimensionality reasons.

The constants λ and m must be determined by experiment or observation. It is satisfactory to take

$$\lambda = 0 \text{ and } m = 8\pi, \text{ so that } \theta_{\mu\nu} = 8\pi\kappa T_{\mu\nu} \text{ and } \kappa = G/c^4. \quad (7)$$

The deflection of light rays by a large mass not only gives us the tensor but provides also a direct proof of the relations between the Riemannian metric and the properties of dielectricity and magnetic permeability of space. *It can be seen as resulting from a variation of the refractive index of space, which leads to curved light rays, or alternatively from a strong curvature of the world-manifold in the neighbourhood of a large mass.* It is well known that the Riemann-Christoffel tensor $R^\alpha_{\beta,\gamma\delta}$ describes precisely the curvature of the manifold endowed with the Riemannian metric $g_{\mu\nu}$. Cartan has shown that $R_{\mu\nu}-(1/2)g_{\mu\nu}R$ describes the curvature of the three-dimensional infinitesimal domains of a Riemannian manifold.

The red-shift of the light emitted from the stars gives the same clue to the choice of $\theta_{\mu\nu}$ *as the deflection of light rays, showing the influence of large masses on electromagnetic radiation.*

In order to find $\theta_{\mu\nu}$ we assumed the conservation laws $D_\rho T^\rho_\mu = 0$. But once the equations for the $g_{\mu\nu}$ were found, we may change our point of view and take $D_\rho T^\rho_\mu = 0$ as a corollary of the fundamental law of Physics given by the equation $R_{\mu\nu}-(1/2)g_{\mu\nu}R = 8\pi\kappa T_{\mu\nu}$. *The constant* κ *appears now as a link between the electromagnetic tensor* $g_{\mu\nu}$, *equivalent to Z and L, and mechanics: the fundamental mechanical equation* $D_\rho T^\rho_\mu = 0$ *can be obtained from* $R_{\mu\nu}-(1/2)g_{\mu\nu}R = 8\pi\kappa T_{\mu\nu}$ *because* $\kappa \neq 0$.

It is important to note that κ^{-1} *is a natural unit of force, both in General*

Relativity as in our electromagnetic theory, very large but finite. To neglect κ amounts to take the natural unit of force as infinite.

The two conditions for the operator L, $L^2 = -1_{op}$ and $\varepsilon(A, LB) = \varepsilon(B, LA)$, are also satisfied by $(-L)$. *The sign of L is however essential because it determines the sign of the energy-density of the electromagnetic field*:

$$T^{(e)}(F)^{\mu}_{\nu} = (1/8)(-g)^{-1/2}\,\varepsilon^{\mu\alpha\beta\gamma}((LF)_{\alpha\beta}F_{\gamma\nu} - (LF)_{\gamma\nu}F_{\alpha\beta}) \tag{8}$$

Equations (8) shows that we can obtain from F and L the tensor-density of electromagnetic energy-momentum $\mathcal{U}^{(e)}(F)^{\mu}_{\nu} = (1/8)\,\varepsilon^{\mu\alpha\beta\gamma}((LF)_{\alpha\beta}F_{\gamma\nu} - (LF)_{\gamma\nu}F_{\alpha\beta})$, but that we need Z^2 to get $T^{(e)}(F)$. *We can therefore use the energy-momentum tensor* $T^{(e)}(F)$ *as the basis for the introduction of* Z^2, *instead of the Lorentz condition for the potentials.*

We shall prove in a following Section that the conditions for L can be obtained by imposing suitable conditions on the energy-momentum tensor-density $\mathcal{U}^{(e)}(F)^{\mu}_{\nu}$, without making use of the Maxwell equations. *Thus the four-dimensionality of the world-manifold and the normal hyperbolic type of its Riemannian metric can be associated to rather simple properties of its energy-momentum distribution.*

2. The Fundamental Postulates

We shall now give a more systematic development of the idea that electromagnetism plays a central rôle in Physics, following the general line of thought outlined in Section 1, but in a more radical way. The basic importance of electromagnetism for the physical construction of geometry was already shown in Section 1, without using the more radical approach of this Section.

Besides the Basic Postulate of the existence of the two fields F and $*F$, described by antisymmetric covariant tensors of the second order of the world-manifold, we shall also assume the Maxwell equations and the algebraic electromagnetic equations for L, the Lorentz condition for the potentials, as well as the restriction on L necessary to render the energy-density of the electromagnetic field non negative. These matters were already discussed in Section 1 and will be further analysed in Section 3.

In this Section we shall introduce four postulates of a somewhat different kind, which will be called the Fundamental Postulates:

I. *The Fundamental Mechanical Postulate.*

II. *The Fundamental Geometrical Postulate.*

III. *The Existence of a Natural Unit of Electromagnetic Field-Intensity.*

IV. *The Fundamental Postulate on the Parallel Displacement.*

The Postulate III can be put in different forms. It is essentially a postulate on the existence of *one natural unit of electromagnetic nature*: unit of electric charge, unit of magnetic mass, unit of field-intensity. Postulate I introduces the unit of force κ^{-1} and there is of course the velocity of light c. Thereby the existence of one electromagnetic natural unit implies that of the others. We preferred to postulate the unit of field-intensity in order not to close the question on the priorities of the units of electric charge or magnetic mass. From a purely experimental point of view, the most natural thing is to postulate the existence of a natural unit of electric charge, namely the charge e of the known elementary particles. But this may perhaps not be the most satisfactory form of our Postulate.

Postulate I. The tensor of energy and momentum distribution of matter $T_{\mu\nu}$ is proportional to the tensor $R_{\mu\nu}-(1/2)g_{\mu\nu}R$ obtained with the electromagnetic tensor $g_{\mu\nu}$ and its Riemann-Christoffel tensor $R^{\alpha}_{\beta,\gamma\delta}$, the proportionality factor being $(8\pi G/c^4)^{-1}$, G denoting the gravitational constant (The Fundamental Mechanical Postulate):

$$8\pi\kappa T_{\mu\nu} = R_{\mu\nu}-(1/2)g_{\mu\nu}R \quad \textit{with} \quad \kappa = G/c^4. \tag{1}$$

Equation (1) is obviously a reinterpretation of the Einstein gravitational equations. In our theory Equation (1) completes the set of the Maxwell and algebraic electromagnetic equations, giving the differential equations for the tensors $L^{\mu\nu}_{\kappa\lambda}$ and Z.

Equation (1) *is the essential link between electromagnetism and mechanics, whose basic equation is obtained by taking the covariant divergence of both sides of* (1):

$$D_\mu T^\mu_\nu = 0 \quad \textit{with} \quad D_\mu = (x^\mu \textit{ covariant derivative}) \textit{ and } T^\mu_\nu = g^{\mu\rho} T_{\rho\nu}. \tag{2}$$

The gravitational constant appears now as the bridge between electromagnetism and mechanics. κ^{-1} is actually a natural unit of force.

$T_{\mu\nu}$ includes also the electromagnetic energy-momentum tensor $T^{(e)}_{\mu\nu}$, whose divergence $D_\mu T^{(e)\mu}_{\nu}$ introduces in (2) *the Lorentz force on the charges.*

Postulate II. *The measurements of space and time in Physics are based on the Riemannian geometry of the four-dimensional world-manifold defined by the electromagnetic tensor $g_{\mu\nu}$ obtained from L and Z. (The Fundamental Geometrical Postulate).*

The two Fundamental Postulates give a precise formulation of the idea that electromagnetism plays a central rôle in physical measurements and observations. In particular the Postulate II bases the Riemannian geometry of the world-manifold on electromagnetic quantities. We shall see that the Postulate II is closely related to the existence at each point of the world-manifold of the electromagnetic Lie algebra $\mathscr{E}$ isomorphic to the Lie algebra of the Lorentz group.

We were able to put together the Maxwell and Einstein equations as fundamental differential equations of the electromagnetic theory. Thus we have the two kinds of basic fields $F_{\mu\nu}$ and $g_{\mu\gamma}$, with $*F$ expressed in terms of them. *We gained a new insight on the nature of the field $g_{\mu\nu}$, which now synthesizes the linear operator L and the weighted scalar Z. The signature of $g_{\mu\nu} dx^\mu dx^\nu$ is seen to follow from the fact that L determines a complex structure in the vector space S_6 of the antisymmetric tensors $A_{\mu\nu}$.*

The mathematics of our theory of electromagnetism and geometry is essentially based on the exceptional rôle played by the vector space of the $A_{\mu\nu}$ in the case of a four-dimensional manifold. *This mathematical fact gives a strong support to the idea that electromagnetism and also other fields described by tensors $A_{\mu\nu}$ must have a distinguished part in Physics.*

The vector space S_6 is intrinsically endowed with a bilinear form $\varepsilon(A, B)$, whose importance we have already stressed. Since $\varepsilon(A, B)$ is a scalar-density, we need a scalar-density field $\omega(x)$ to obtain a scalar metric $\omega^{-1}\varepsilon(A, A)$. It seems natural to take this metric independent of that given by the $g_{\mu\nu}$, since $\varepsilon(A, A)$ does not depend on the $g_{\mu\nu}$. *We are therefore led to assume that ω does not coincide with $\sqrt{-g}$:*

$$\omega(x) = S(x)\sqrt{-g(x)}, \quad S(x) \quad \textit{being a scalar field.} \tag{3}$$

In a four-dimensional manifold it is natural to assume the existence of a scalar field $S(x)$ besides the Riemannian metric $g_{\mu\nu}(x)$, in order to have the independent metric $\omega^{-1}\varepsilon(A, A)$ for the $A_{\mu\nu}$. This suggests a scalar-tensor theory of geometry and gravitation, more general than that given in this paper.

The tensors of S_6 seem to be of special interest to Physics. The S_6 vectors are the $A_{\mu\nu}$, in particular the $F_{\mu\nu}$. The antisymmetric tensors of the second order of S_6 are the $U_{\alpha\beta,\gamma\delta}$ antisymmetric with respect to α and β and also in γ, δ with

$$U_{\alpha\beta,\gamma\delta} + U_{\gamma\delta,\alpha\beta} = 0. \tag{4}$$

The electromagnetic energy-momentum tensor $T^{(e)}$ is equivalent to the $U^{(e)} = {}^*F \wedge F$,

$$T^{(e)\mu}_{\nu} = (1/4)(-g)^{-1/2}\,\varepsilon^{\mu\alpha\beta\gamma}\,U^{(e)}_{\alpha\beta,\gamma\nu} \quad \textit{with} \quad U^{(e)}_{\alpha\beta,\gamma\delta} = {}^*F_{\alpha\beta}F_{\gamma\delta} - {}^*F_{\gamma\delta}F_{\alpha\beta}. \tag{5}$$

The U can be used to describe non polarized light.

The Riemann-Christoffel curvature tensor $R_{\alpha\beta,\gamma\delta}$ is a symmetric second order tensor of S_6 satisfying the condition $\varepsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta,\gamma\delta} = 0$. $R_{\alpha\beta,\gamma\delta}$ is indeed antisymmetric with respect to the two indices of each pair and $R_{\alpha\beta,\gamma\delta} = R_{\gamma\delta,\alpha\beta}$. It is interesting to note that the various symmetries of the indices of $R_{\alpha\beta,\gamma\delta}$ mean simply that it is a symmetric tensor of the second order of S_6, with the trace 0 for the intrinsic metric of S_6.

The antisymmetric tensors of the third order of S_6 are the $W_{\alpha\beta,\gamma\delta,\kappa\lambda}$, antisymmetric with respect to the three pairs of indices and the indices of each pair. There is a very simple relation between W and the symmetric tensors $S_{\mu\nu}$:

$$\omega^{-1}\varepsilon^{\alpha\beta\gamma\delta}\,W_{\mu\alpha,\beta\gamma,\delta\nu} = 2S_{\mu\nu}. \tag{6}$$

We can raise the indices of W by means of the metric tensor $\omega^{-1}\varepsilon^{\alpha\beta\gamma\delta}$ and obtain $W^{\alpha\beta,\gamma\delta,\kappa\lambda}$ from which we can extract a symmetric tensor $2\hat{S}^{\mu\gamma} = \omega\varepsilon_{\alpha\beta\gamma\delta}\,W^{\mu\alpha,\beta\gamma,\delta\nu}$. *The vector space of the W is the direct sum of two ten-dimensional vector spaces, equivalent to those of the $S_{\mu\nu}$ and $\hat{S}^{\mu\nu}$, when a scalar-density ω is given.*

By means of three linearly independent $A^{(a)}_{\mu\nu}$, we can build a $W = A^{(1)} \wedge A^{(2)} \wedge A^{(3)}$ by outer products of S_6. These W are the simple trivectors of S_6. It is possible to get from the real simple trivectors the $S_{\alpha\beta}$ with non Minkowskian signatures. *The $g_{\mu\nu}$ of the world-manifold can be obtained from the complex outer product of three linearly independent eigenvectors of the same eigenvalue of L, by means of formula (6).*

The vector space S_6 can be transformed into a richer algebraic structure, namely a Lie algebra, by the introduction of a vector product $A_1 \times A_2$, which is again an A, by means of the structure constants $C^{\alpha\beta,\gamma\delta}_{\mu\nu}$ of the Lorentz group of $g_{\mu\nu}$:

$$(A_1 \times A_2)_{\mu\nu} = (1/4)\, C^{\alpha\beta,\gamma\delta}_{\mu\nu} A_{1;\alpha\beta} A_{2;\gamma\delta} \quad \text{with} \quad C^{\alpha\beta,\gamma\delta}_{\mu\nu} = g^{\rho\sigma}(\delta^{\alpha\beta}_{\mu\rho}\delta^{\gamma\delta}_{\sigma\nu} - \delta^{\alpha\beta}_{\nu\rho}\delta^{\gamma\delta}_{\alpha\mu}). \quad (7)$$

Thus S_6 becomes the Lie algebra of the Lorentz group of the metric $g_{\mu\nu}$.

Here comes in another peculiarity of the dimensionality $n = 4$: there is essentially only one definition of the vector product of S_6 giving a simple Lie algebra, namely that corresponding to a $g_{\mu\nu}$ with a Minkowskian signature. This results from the fact that the Lorentz group is the only simple Lie group with six parameters, up to isomorphisms.

The dimensionality $n = 4$ of the world-manifold and the Minkowskian signature of its Riemannian metric are determined by the condition that the Lie algebra structure of the vector space of the A at any point x of the world-manifold determined by the orthogonal group of $g_{\mu\nu}$ be simple and the only possible type of simple Lie algebra for a vector space of dimensionality n. This theorem results from the fact that for $n \neq 4$ and larger than 2, the groups $0(n-p, p)$ are all simple, so that there are several types of simple Lie algebras.

The above discussion shows that the Lie algebra structure of S_6 is really the most essential mathematical feature of the geometry of the world-manifold. It must therefore correspond to an essential algebraic structure $\mathscr{E}$ of the electromagnetic theory. In order to be able to define a Lie product $F_1 \times F_2$ of two fields, of the same nature as a field F, we need a new Fundamental Postulate:

Postulate III. *There is a natural unit ϕ of electromagnetic field intensity.*

By means of ϕ we can give a satisfactory definition of the Lie product $F_1 \times F_2$ of two fields corresponding to the above Lie algebra structure of the $A_{\mu\nu}$ because $\phi^{-1} F_{\mu\nu}$ is a tensor with dimensionless components. Thus we get the product of $\mathscr{E}$

$$(F_1 \times F_2)_{\mu\nu} = (1/4)\phi^{-1} C^{\alpha\beta,\gamma\delta}_{\mu\nu} F_{1;\alpha\beta} F_{2;\gamma\delta} = \phi^{-1} g^{\rho\sigma}(F_1 \wedge F_2)_{\mu\rho,\sigma\nu}. \quad (8)$$

The Lie algebra of the Lorentz group is not changed when we replace $g_{\mu\nu}$ by $Sg_{\mu\nu}$, S being a scalar, because the group is the same for both $g_{\mu\nu}$ and $Sg_{\mu\nu}$. The definition of the Lie algebra of S_6 shows that the substitution

$g_{\mu\nu} \to Sg_{\mu\nu}$ corresponds to an automorphism not affecting the vector-space structure of S_6. *In the case of the electromagnetic Lie algebra with the product* (8) *we must rule out the automorphism induced by* $g_{\mu\nu} \to Sg_{\mu\nu}$ *because it amounts to a change of* ϕ. This is related to a passage from the conformal geometry determined by the Lorentz groups at the different points of the manifold, or by the corresponding linear operators L, to the Riemannian geometry given by the $g_{\mu\nu}$.

The introduction of ϕ *gives us a fundamental length* $(\phi\sqrt{\kappa})^{-1}$, *which is probably related to the Planck length* $(h c \kappa)^{1/2}$. *The theory contains also a constant* $(\phi\kappa)^{-1}$ *which can be a fundamental electric charge or a fundamental magnetic mass.* It is interesting to note that by taking for $(\phi\kappa)^{-1}$ the Dirac value for the elementary magnetic mass $(1/2)\alpha^{-1}\ e$, we get for $(\phi\sqrt{\kappa})^{-1}$ the value $(8\pi\alpha)^{-1/2}(hc\kappa)^{1/2}$, very nearly the Planck length, with $\alpha = e^2/\hbar c$.

If follows from the above considerations that there are other postulates equivalent to Postulate III, because of the existence of the natural unit of force κ^{-1} introduced by Postulate I. We may assume, instead of Postulate III, any one of the following postulates:

Postulate IIIa. There is a natural unit e_0 *of electric charge.*

Postulate IIIb. There is a natural unit m_0 *of magnetic mass.*

Postulate IIIc. There is a natural unit of length Λ_0.

Postulate IIIa is of course the most obvious, because of the experimental fact of the existence of the charge e of the known elementary particles. We discussed already the interest of having the Dirac value of the magnetic mass as a constant in our theory, in order to get from it the Planck length. This can be done directly assuming the Postulate IIIb, with m_0 taken as the Dirac magnetic mass.

In the case of Postulate IIIc, there would be the nice feature of having a natural unit of length, from the beginning, in the physical construction of geometry. The Planck length would eventually be a good choice for Λ_0. *Thus the Planck constant h could be obtained from the elementary length* Λ_0, G *and* c.

There are of course other possible choices of a basic natural unit different from those discussed in the above four forms of the Postulate III. A particularly interesting one is that of a natural unit of angular momentum or action, which corresponds to the following form of our Postulate:

Postulate IIId. There is a natural unit $\hbar$ of angular momentum.

The angular momentum is a physical quantity particularly related to the rotations and the Lorentz group. The introduction of a natural unit of angular momentum allows us to get a Lie algebra of the angular momentum tensors $M_{\mu\nu}$ directly from that of the Lorentz group, with the multiplication rule $(M_1 \times M_2)_{\mu\nu} = \hbar^{-1} C_{\mu\nu}^{\alpha\beta,\gamma\delta} M_{1;\alpha\beta} M_{2;\gamma\delta}$, similar to the electromagnetic Lie algebra $\mathscr{E}$. *Thus the Planck constant h would come in in a non quantized theory as a natural unit of angular momentum.* The passage to the relativistic quantum theory of the angular momentum would correspond to the introduction of the representations of the Lie algebra of the $M_{\mu\nu}$ by Lie algebras of operators of Hilbert spaces, in which the M are associated to linear operators whose commutators correspond to the Lie products of the Lie algebra. *In the case of the angular momentum, as well as in other cases, it is possible to define Lie algebras for physical quantities, involving the Planck constant h as a dimensional constant, before the quantization, which is associated to a representation of those Lie algebras by linear operators of Hilbert spaces with finite or infinite dimensionality.* The Lie products give rise to commutators in the quantized formalism.

The electromagnetic Lie algebra $\mathscr{E}$ does not depend on the Maxwell equations. It is an algebraic structure of the electromagnetic theory underlying at each point x the geometric Lie algebra of the vector space S_6. Thus it gives the foundation of the Postulate II, which bases the geometry of the world-manifold on the electromagnetic $g_{\mu\nu}$. We shall see in Postulate IV how $\mathscr{E}$ underlies the definition of the parallel displacement on the world-manifold.

$\mathscr{E}$ shows that the $g_{\mu\nu}$ have a definite algebraic rôle in electromagnetism: they and the constant ϕ give the structure constants $\phi^{-1} C_{\mu\nu}^{\alpha\beta,\gamma\delta}$ of the electromagnetic Lie algebra $\mathscr{E}$.

We shall now see that the Lie product $F_1 \times F_2$ is very closely related to the Lorentz force on the charged particles. This is due to the fact that the Lie product $F_1 \times F_2$ is a special case of a kind of product $F \times T$ which exists for any covariant tensor $T_{\mu_1,\ldots,\mu_s}$:

$$(F \times T)_{\mu_1,\cdots,\mu_s} = \phi^{-1} g^{\rho\sigma}(F_{\mu_1\rho} T_{\sigma,\mu_2,\cdots,\mu_s} + \ldots + F_{\mu_s\rho} T_{\mu_1,\cdots,\mu_{s-1},\sigma}). \quad (9)$$

In particular, when T is a vector P_μ

$$(F \times P)_\mu = \phi^{-1} g^{\rho\sigma} F_{\mu\rho} P_\sigma = \phi^{-1} F_{\mu\rho} V^\rho \quad \textit{with} \quad V^\rho = g^{\rho\sigma} P_\sigma. \tag{9a}$$

Hence

$$(F \times P)_\mu = \kappa e_0 F_{\mu\rho} V^\rho \quad \textit{with} \quad e_0 = (\kappa\phi)^{-1} \tag{10}$$

$F \times P$ is therefore the Lorentz force on a particle of charge e_0 with the velocity vector V, measured with the natural unit of force κ^{-1}.

$\eta(F \times T)$, with η infinitesimal, is the change of T for the infinitesimal Lorentz transformation which changes P into $P + \eta(F \times P)$. Along the element ds of the path of charged particle of mass m_0 and charge e_0, the field F generates an infinitesimal Lorentz transformation changing the momentum $m_0 V$ by $e_0(F \times P)ds$.

$\mathscr{E}$ is actually a subalgebra of the Lie algebra $\mathscr{E}_P$ of the fields $F_{\mu\nu}$ and currents J_μ, with the multiplication rules (8), (8*a*) *and* (8*b*):

$$(F \times J)_\mu = \phi^{-1} g^{\rho\sigma} F_{\mu\rho} J_\sigma \textit{ and } (J \times F)_\mu = \phi^{-1} g^{\rho\sigma} J_\rho F_{\sigma\mu} = -(F \times J)_\mu, \tag{8a}$$

$$J^{(1)} \times J^{(2)} = 0 \tag{8b}$$

The rule (8*b*) follows from the fact that $J^{(1)} \times J^{(2)}$ is the scalar $g^{\rho\sigma}(J^{(1)}_\rho J^{(2)}_\sigma - J^{(1)}_\sigma J^{(2)}_\rho) = 0$.

The rôle of the Lorentz force is seen explicitly in (8*a*). Moreover from the Jacobi condition

$$(F_1 \times F_2) \times J + (J \times F_1) \times F_2 + (F_2 \times J) \times F_1 = 0 \tag{11}$$

and (8*a*), we get (8).

In the Minkowski space-time, the group of the displacements is the well known Poincaré group generated by the Lorentz group and the Abelian group of translations. In the case of a curved world-manifold there is no Abelian group of translations with four independent parameters, so that there is no true analog of the Poincaré group of the flat space-time. It is nevertheless possible to introduce at each point x of the world-manifold a Lie algebra of the $A_{\mu\nu}$ and P_μ with the multiplication rules $(A_1 \times A_2)_{\mu\nu} =$

$$= g^{\rho\sigma}(A_{1;\mu\rho} A_{2;\sigma\nu} - A_{1;\nu\rho} A_{2;\sigma\mu});\ (A \times P)_\mu = g^{\rho\sigma} A_{\mu\rho} P_\sigma;\ P^{(1)} \times P^{(2)} = 0 \tag{12}$$

which is essentially the Lie algebra of the Poincaré group of the flat space-time tangent at the point x to the world-manifold.

The electromagnetic Lie algebra $\mathscr{E}_P(x)$ is isomorphic to the Lie algebra of the flat Minkowskian tangent space at the point x. The $\mathscr{E}_P$ at all the points of the world-manifold are therefore isomorphic Lie algebras. The $\mathscr{E}$ at all the points of the world-manifold are all isomorphic to the Lie algebra of 0(3,1) and thereby isomorphic to each other.

We shall now consider the determination of the affine connection $\Gamma^{\rho}_{\kappa\lambda}$ of the world-manifold in the electromagnetic construction of geometry. In the ordinary Riemannian geometry the affine connection $\Gamma^{\rho}_{\kappa\gamma}$ is assumed to be symmetric

$$\Gamma^{\rho}_{\kappa\lambda} = \Gamma^{\rho}_{\lambda\kappa} \tag{13}$$

so that the torsion tensor $T^{\rho}_{\kappa\lambda} = \Gamma^{\rho}_{\kappa\lambda} - \Gamma^{\rho}_{\lambda\kappa} = 0$ everywhere. The symmetric $\Gamma^{\rho}_{\kappa\lambda}$ are obtained from the Riemannian metric tensor $g_{\mu\nu}$ by imposing the condition of invariance of the length of the vectors V^{μ} by parallel displacement:

$$g_{\mu\nu}(x)\,V^{\mu}V^{\nu} = g_{\mu\nu}(x+dx)\,(V^{\mu}+\delta V^{\mu})\,(V^{\nu}+\delta V^{\nu}) \text{ with } \delta V^{\rho} = -\Gamma^{\rho}_{\kappa\lambda}V^{\kappa}dx^{\lambda}. \tag{14}$$

We get from (14)

$$\partial_{\rho}g_{\mu\nu} - \Gamma^{\alpha}_{\rho\mu}g_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu}g_{\mu\alpha} = 0 \quad \text{so that} \quad D_{\rho}g_{\mu\nu} = 0. \tag{15}$$

It follows from equations (15) and (13) that the components of the affine connection Γ are the Christoffel symbols

$$\Gamma^{\rho}_{\kappa\lambda} = (1/2)g^{\rho\alpha}(\partial_{\kappa}g_{\lambda\alpha} + \partial_{\lambda}g_{\kappa\alpha} - \partial_{\alpha}g_{\kappa\lambda}) \quad \text{so that} \quad \Gamma^{\rho}_{\kappa\lambda} = \left\{ {\rho \atop \kappa\lambda} \right\}. \tag{16}$$

We want to keep equations (13) and (16) in our electromagnetic approach, with the $g_{\mu\nu}$ taken as the electromagnetic ones, equivalent to Z and L, in order that the theory of the parallel displacement be in agreement with the Postulate II. The condition (14) will be replaced by another involving the Lie algebra $\mathscr{E}$, through the following Fundamental Postulate of the Parallel Displacement:

Postulate IV. *The parallel displacement is defined by a symmetric affine connection $\Gamma^{\rho}_{\kappa\lambda}$ and induces an isomorphic correspondence between $\mathscr{E}(x)$ and $\mathscr{E}(x + dx)$.*

The Postulate IV requires that the Lie product of the parallel displaced tensors $F_{1,\mu\nu} + \delta F_{1,\mu\nu}$ and $F_{2,\mu\nu} + \delta F_{2,\mu\nu}$ at the point $x + dx$, involving the $g_{\mu\nu}(x + dx)$, is the tensor $(F_1 \times F_2)_{\mu\nu} + \delta(F_1 \times F_2)_{\mu\nu}$ with $F_1 \times F_2$ involving the $g_{\mu\nu}(x)$. A straightforward computation shows that as a consequence of the symmetry of $\Gamma^{\rho}_{\kappa\lambda}$, we get again the equations (15) and (16). *Now equation* (14) *is a corollary of* (15), *so that the invariance of* $g_{\mu\nu} V^{\mu} V^{\nu}$ *by parallel displacement is a consequence of the Postulate* IV, *the* $g_{\mu\nu}$ *being of course the electromagnetic ones.*

It follows from the validity of equations (16) for the electromagnetic $g_{\mu\nu}$ that the Riemann-Christoffel tensor $R^{\alpha}_{\beta,}{}^{\gamma\delta}$ of those $g_{\mu\nu}$ is actually the curvature tensor of the world-manifold, corresponding to the affine connection $\Gamma^{\rho}_{\kappa\lambda}$. *Thus the equations* (1) *have the same geometric content as the Einstein gravitational equations, as a consequence of the Postulate* IV.

3. The Maxwell Equations

We shall now discuss the form of the Maxwell equations in a differentiable world manifold not endowed with either affine, conformal or metric properties. *This requires a two-field formalism.* We shall be mainly interested in showing how the structure of the Maxwell equations in the two-field formalism requires that the dimensionality of the world-manifold be $n = 4$, when the fields $*F$ and F are assumed to be both described by covariant antisymmetric tensors of the second order.

Let us firstly give the definitions of the fundamental differential operators *Rot* and *Div*. The operation *Rot* can be applied to the antisymmetric covariant tensors A of order $p < n$, in a n-dimensional differentiable manifold, the tensor *Rot A* being antisymmetric covariant of order $p + 1$:

$$(Rot\,A)_{\mu_1,\cdots,\mu_{p+1}} = (p\,!)^{-1}\,\delta^{\rho_1,\cdots,\rho_{p+1}}_{\mu_1,\cdots,\mu_{p+1}}\,\partial_{\rho_1} A_{\rho_2,\cdots,\rho_{p+1}} \quad \text{with } \partial_\rho = x^\rho \text{ derivative.} \tag{1}$$

The operation *Div* can be applied to the antisymmetric contravariant tensor-densities $\mathscr{B}$ of order $p > 0$ in a n-dimensional differentiable manifold, $Div\,\mathscr{B}$ being an antisymmetric contravariant tensor-density of order $p-1$:

$$(Div\,\mathscr{B})^{\mu_2,\cdots,\mu_p} = \partial_{\mu_1}\,\mathscr{B}^{\mu_1,\cdots,\mu_p} \tag{2}$$

The covariant differential operations Rot and Div involve only ordinary derivatives of the components of the antisymmetric tensors A and tensor-densities $\mathscr{B}$. *Rot* corresponds to the Cartan differential of an external differential form. *Div* can be expressed in terms of *Rot* as we shall now see.

The antisymmetric contravariant tensor-density of order p, $\mathscr{B}$, is equivalent to an antisymmetric covariant tensor B of order $n-p$, because of the existence of the Ricci relative tensor $\varepsilon_{\mu_1,\cdots,\mu_n}$:

$$B_{\mu_1,\cdots,\mu_{n-p}} = (p\,!)^{-1}\,\varepsilon_{\mu_1,\cdots,\mu_n}\,\mathscr{B}^{\mu_{n-p+1},\cdots,\mu_n} \tag{3}$$

This B, $\mathscr{B}$ duality associates to $Rot\,B$ the tensor-density $(-1)^{n-p}\,Div\,\mathscr{B}$:

$$(Rot\,B)_{\mu_1,\cdots,\mu_{n-p+1}} = ((p-1)\,!)^{-1}(-1)^{n-p}\,\varepsilon_{\mu_1,\cdots,\mu_n}(Div\,\mathscr{B})^{\mu_{n-p+2},\cdots,\mu_n} \tag{4}$$

The homogeneous Maxwell equation involves only the field F associated to the Lorentz force-density $F_{\mu\nu}\,\mathscr{J}^{\nu}$ on the charge-current distribution. F must be a covariant antisymmetric tensor of the second order, because of the contravariant nature of the tensor-density $\mathscr{J}^{\mu}$ associated to particle velocities v^{μ}. The homogeneous Maxwell equation

$$Rot\,F = 0 \tag{5a}$$

can obviously be written for any world dimensionality $n > 3$.

The non homogeneous Maxwell equation will be taken firstly in a divergencial form

$$Div\,{}^*\mathscr{F} = \mathscr{J} \tag{5b'}$$

$^*\mathscr{F}$ must be an antisymmetric contravariant tensor-density of order 2, because $\mathscr{J}$ is a vector-density. The equation (5b′) can be written for $n > 2$.

In the n-dimensional case $^*F$ is an antisymmetric covariant tensor of order $n-2$ as shown by equation (3). For any value of n compatible with (5b′), $^*F$ has the same number of components as F. *But only for $n = 4$ is $^*F$ a tensor of the same nature as F.* This is due to the fact that the Ricci relative tensor has four indices for $n = 4$: $^*F_{\kappa\lambda} = (1/2)\varepsilon_{\kappa\lambda\mu\nu}\,{}^*\mathscr{F}^{\mu\nu}$.

The non homogeneous Maxwell equation can be written in a rotational form for $n > 3$:

$$Rot\,{}^*F = (-1)^{n-2}\,J. \tag{5b}$$

J is now an antisymmetric covariant tensor of order $n-1$.

The non homogenous Maxwell equation is closely related to the conservation of the electric charge expressed by the equation

$$Div\,\mathscr{J} = 0 \quad \text{or} \quad Rot\,J = 0 \tag{6}$$

It is well known that, with suitable conditions imposed to the world-manifold, $Div\ \mathcal{J} = 0$ implies the existence of fields $^*\mathcal{F}$ satisfying the equation $Div\ ^*\mathcal{F} = \mathcal{J}$, $^*\mathcal{F}$ may be seen as a kind of potential tensor for the charge-current density $\mathcal{J}$, from a purely mathematical point of view.

We shall assume the following Basic Postulate of the electromagnetic theory, which renders $^*F$ an antisymmetric covariant tensor of the second order, as a consequence of F having the same property:

*BASIC POSTULATE: The two basic tensors F and *F are both antisymmetric covariant of the same order.*

It follows from the Basic Postulate that the dimensionality of the world-manifold is $n = 4$, since the discussion of the Maxwell equations showed that the order of $^*F$ is $n-2$. *Thus we see that the four-dimensionality of the world-manifold is a consequence of the structure of the Maxwell equations* (5a) *and* (5b) *which involve fields F and *F of the same tensorial nature.* The Basic Postulate gives to the two Maxwell equations the same structure in vacuum and allows to replace them by a single equation for a complex field F^+ (or F^-):

$$Rot\ F^+ = -iJ \quad and \quad Rot\ F^- = iJ, \quad with \quad F^+ = F - i^*F \quad and \quad F^- = F + i^*F \tag{5}$$

Until now we did not assume any relation between F and $^*F$. *The four-dimensionality of the world-manifold is associated to the differential electromagnetic equations, without any consideration of L and the algebraic electromagnetic equations.* It is interesting to note that the introduction of the linear operator L presupposes the Basic Postulate.

The introduction of L and the algebraic electromagnetic equations corresponds to a second stage of the physical construction of geometry. The F^+ are now eigenvectors of L corresponding to the eignevalue i and the F^- eigenvectors of L of the eigenvalue $(-i)$:

$$LF^+ = iF^+ \quad and \quad LF^- = -iF^- \quad since \quad ^*F = LF. \tag{7}$$

L being a real operator of square -1_{op} has only two eigenvalues i and $(-i)$, both with the multiplicity 3. The complex vector spaces of the F^+ and F^- must thereby be three-dimensional. *The introduction of L leads therefore to a description of the electromagnetic field by a complex three-*

dimensional vector F^+ *or* $F^- = (F^+)^*$. *Those three-dimensional vectors correspond to the two kinds of second order van der Waerden spinors, built with the two-component Weyl spinors.*

The bilinear form $\varepsilon(A, B)$ exists already in the first stage of the construction of geometry, but $\varepsilon(A, LB)$ belongs to the second stage. In the first stage F and $*F$ are still independent, so that F^+ and F^- are vectors of a six-dimensional complex Euclidean space $S_6(C)$ with the metric form $\varepsilon(A, A)$. In the second stage F^+ is restricted to the three-dimensional sub-space S_3^+ of $S_6(C)$ defined by $LF^+ = iF^+$ and F^- to the sub-space S_3^- defined by the equation $LF^- = -iF^-$.

We shall see in Section 4 that the linear operator L with $L^2 = -1_{op}$ allows to render the six-dimensional vector space S_6 of the real F into a complex three-dimensional Euclidean space equivalent to S_3^+ or to S_3^-.

3a. The Maxwell equations appear as a special case of a more general type of equations in which there is a field A described by an antisymmetric covariant tensor A of order $p < n$ and another field $*A$ described by an antisymmetric covariant tensor of order $n-p$. The two fields have the same number of components C_n^p and satisfy equations of the form

$$Rot\, A = B, \qquad Rot\, {*A} = B', \tag{1}$$

B and B' being antisymmetric covariant tensors of order $p + 1$ and $n-p+1$, respectively.

The case of $p = 2$ is particularly interesting when $B = 0$ because it follows from a well known general theorem of Poincaré that under certain conditions there is a covariant vector P such that

$$A = Rot\, P, \qquad (B = 0,\ p = 2) \tag{2}$$

so that A can be described by a vector field P. *In the electromagnetic case* P *is the vector potential. It is not completely determined by the field* A, *there being the possibility of the gauge transformation*

$$P \to P + Rot\, S, \quad \textit{with } S \textit{ an arbitrary differentiable scalar field.} \tag{3}$$

In the case of $p = 2$, B' *describes a conservative current, since it follows from the second equation* (1) *that*

$$Rot\, B' = 0 \quad \textit{so that} \quad \mathscr{B}' = 0 \quad \textit{for} \quad p = 2. \tag{4}$$

In the electromagnetic case equation (4) *is the law of conservation of the electric charge.*

The introduction of a Riemannian metric in the n-dimensional differentiable manifold allows to associate to the field A a field $*A$ by the operation of tensor duality for all the possible values of p. This follows from the possibility of associating to A an antisymmetric contravariant tensor $A^{\mu_1,\cdots,\mu_p} = g^{\mu_1\rho_1}\ldots g^{\mu_p\rho_p} A_{\rho_1,\cdots,\rho_p}$ by means of the metric $g_{\mu\nu}$ and to build a contravariant tensor-density of order p, $*\mathscr{A}$:

$$*\mathscr{A}^{\mu_1,\cdots,\mu_p} = -(\theta g)^{1/2} A^{\mu_1,\cdots,\mu_p} \quad \textit{with} \quad \theta^2 = 1, \quad \theta g > 0. \tag{5}$$

From the tensor-density $*\mathscr{A}$ of order p we get the covariant antisymmetric of order $n-p$, $\overset{*}{A}$. *The metric *duality allows the definition of the divergence Div* $*\mathscr{A}$, *equivalent to Rot* $\overset{*}{A}$. *Thus with the present choice of* $*A$ *the system of equations* (1) *appears as a direct generalization of the system rot* $\mathbf{A} = \mathbf{B}$, *div* $\mathbf{A} = \mathbf{S}$ *for a vector field* $\mathbf{A}$ *in the three-dimensional Euclidean vector analysis.*

In the case of $p = 2$ the introduction of the metric g and the corresponding *duality allows to define the dual $*P$ of the vector P, which is an antisymmetric covariant tensor of the order $n-1$. The arbitrariness of the scalar S in equation (3) can now be restricted by the generalized Lorentz condition

$$\mathit{Rot}\, *P = 0 \quad \text{equivalent to} \quad \mathit{Div}\, *\mathscr{P} = 0. \tag{2a}$$

It is important to note that in the case of $n = 4$ *and* $p = 2$ *the definition of* $*A$ *can be given in terms of the conformal metric* $C_{\mu\nu} = (\theta g)^{-1/4} g_{\mu\nu}$, *but the definition of* $*P$ *requires the Riemannian metric* $g_{\mu\nu}$. *Thereby the Lorentz condition for the electromagnetic potentials involves the Riemannian metric of the world-manifold.* On the other hand it is well known that in general relativity the Maxwell equations involve only the conformal metric. This follows from the fact that in General Relativity the field $*F$ is taken as the *dual of F.

The Equations (2) and (2a) are a system of the type (1) for the vector P. By taking the *dual of both sides of (2) we get $*(\mathit{Rot}\, P) = *F$ and by making use of the inhomogeneous Maxwell equation we obtain the relation between P and J

$$\mathit{Rot}\,(*(\mathit{Rot}\, P)) = J, \quad (p = 2). \tag{6}$$

It is not necessary to use the explicit definition of the *dual of a second order antisymmetric tensor to get the equation corresponding to (6) in the electromagnetic case in which we have a linear operator L such that $*A = LA$ for $p = 2$. Equation (6) can be written as

$$Rot\,(L\,(Rot\,P)) = J, \qquad \text{for} \qquad n = 4, \; p = 2. \tag{7}$$

3b. The duality B, $\mathscr{B}$ corresponds to the existence of a bilinear form $\varepsilon_n(A, B)$ of the antisymmetric covariant tensors A of order p and B of order $n-p$

$$\varepsilon_n(A, B) = (p\,!(n-p)\,!)^{-1}\,\varepsilon^{\rho_1,\cdots,\rho_n} A_{\rho_1,\cdots,\rho_p} B_{\rho_{p+1},\cdots,\rho_n}, \tag{1}$$

$$\varepsilon_n(A, B) = (-1)^{p(n-p)}\,\varepsilon_n(B, A). \tag{2}$$

For even values of n, $n = 2r$, the B of order r have $\mathscr{B}$ of the same order r. Now $\varepsilon_n(A, B)$ gives a bilinear form for the tensors of order r. It follows from (2) that it is either symmetric or antisymmetric according to r:

$$\varepsilon_{2r}(A, B) = (-1)^r\,\varepsilon_{2r}(B, A), \qquad (A \text{ and } B \text{ of order } r). \tag{3}$$

For $n = 2r$ the *dual $*A$ of an A of order r is also of order r, so that we can define a linear operator L such that

$$*A = LA \qquad \text{for the } A \text{ of order } r. \tag{4}$$

Since we have for any $p < n$

$$*(*A) = (-1)^{p(n-p)}\,\theta A \qquad (p = \text{order of } A), \tag{5}$$

$$L^2 = (-1)^r\,\theta 1_{op} \qquad \text{with} \qquad 1_{op} = \text{unit operator on the } A \text{ of order } r. \tag{4a}$$

It follows from (3) that

$$\varepsilon_{2r}(A, LB) = \varepsilon_{2r}(B, LA). \tag{4b}$$

The conditions imposed in Section 1 *to the electromagnetic linear operator* L *are precisely of the type* (4), (4a), (4b) with $\theta = -1$ *and* $r = 2$, *corresponding to the dimensionality* $n = 4$ *and the normal hyperbolic type of the metric.*

It is easily seen that in the case of $n = 2r$ the *duality for the A of order r depends only on the conformal metric

$$*A_{\mu_1,\cdots,\mu_r} = (r\,!)^{-1}\,\varepsilon_{\mu_1,\cdots,\mu_r,\rho_1,\cdots,\rho_r} C^{\mu_1\rho_1} \ldots C^{\mu_r\rho_r} A_{\rho_1,\cdots,\rho_r}, \tag{6}$$

$$C^{\mu\nu} = (\theta g)^{1/2r} g^{\mu\nu}. \tag{7}$$

Equation (4*b*) shows that, for even values of n, L allows always to build a symmetric bilinear form of the A of order $r = n/2$:

$$L(A, B) = L(B, A) = \varepsilon_{2r}(A, LB). \tag{8}$$

For odd values of r, the bilinear form $\varepsilon_{2r}(A, B)$ is antisymmetric.

We shall prove in Section 4*c* that when $n = 4$ and $\theta = -1$ the conditions (4), (4a) and (4b), determine the conformal metric $C_{\mu\nu}$ up to a minus sign, *but not for $n = 4$ and $\theta = 1$, when there are different possibilities for the signature of the Riemannian metric.*

It is easily seen that, both for the definite Riemannian metric and for the indefinite Riemannian metric of a four-dimensional manifold, the determinant g is positive so that $\theta = 1$ in both cases and the duality operators L satisfy the same conditions $L^2 = 1_{op}$ and $\varepsilon(A, LB) = \varepsilon(B, LA)$.

3c. The non conformal nature of the Lorentz condition $Div\ {}^*\mathscr{P} = 0$ follows from the fact that ${}^*\mathscr{P}^\mu$ cannot be expressed in terms of P_μ and the $C^{\mu\nu}$ only, a Z is also needed:

$${}^*\mathscr{P}^\mu = -Z^3 C^{\mu\nu} P_\nu \quad \textit{with} \quad Z = (\theta g)^{1/4}, \quad {}^*P_{\kappa\lambda\mu} = (\theta g)^{3/4} \varepsilon_{\kappa\lambda\mu\nu} C^{\nu\rho} P_\rho, \tag{1}$$

as a consequence of Eq. $(2a-5)$. *The Lorentz condition gives therefore the electromagnetic basis for the introduction of Z and, together with the linear operator L, of the tensor $g_{\mu\nu} = ZC_{\mu\nu}$, since L determines $C_{\mu\nu}$ up to the sign.*

It is not surprising that the Lorentz condition should give something beyond the Maxwell equations, since it is a restriction on the potentials not required by the Maxwell equations. The remarkable fact is that it leads to the introduction of the measure of the hypervolumes and the orientation of the world-manifold by the scalar density $Z^2 = (\theta g)^{-1/2}$.

In fact, it is more correct to say that the introduction of Z is necessary in order to build the scalar-density $Div\ {}^*\mathscr{P}$ by means of the vector field P. The importance of the scalar-density $Div\ {}^*\mathscr{P}$ was clearly shown by the Hamiltonian formalism of the electromagnetic field and even more by the quantum electrodynamics, in which the Lorentz condition is closely related to the Coulomb force. Since the Coulomb force is basic for the existence of solid bodies, it must also be fundamental for the physical construction of geometry. *It seems therefore plausible to think that the basic role of $Div\ {}^*\mathscr{P}$ in the electromagnetic foundation of geometry is related to the Coulomb force.*

The passage from the relative tensor $C_{\mu\nu}$ to the absolute tensor $g_{\mu\nu}$ has far reaching consequences, because the field $g_{\mu\nu}(x)$ allows to define the covariant derivatives D_μ of the tensors and thus to give a meaning to the absolute variation of the physical quantities from a point x to a neighbouring point $x + dx$ of the world-manifold.

4. The Fundamental Theorem and the Lorentz Group

We shall now give the first proof of the Fundamental Theorem on the determination of the conformal geometry of the world-manifold at the point x by its operator L. In Sections 4b and 4c we shall give a second proof and show how $g_{\mu\nu}$ can be built by means of the antisymmetric tensors of the third order $W_{\kappa\lambda,\mu\nu,\rho\sigma}$ of the vector space S_6. The first proof is based on the discussion of the complex structure defined in S_6 by the linear operator L.

It is well known that a real vector space of even dimensionality $2r$ can be rendered into a complex r-dimensional vector space by means of a real operator I with $I^2 = -1_{2r}$, 1_{2r} denoting the unit operator of the $2r$-dimensional real space. It suffices to give a definition of iA in terms of I in one of the following ways:

$$iA = IA \quad \text{or} \quad iA = -IA \quad (A = \text{vector of the } 2r\text{-dimensional space}) \tag{1}$$

This requires a restriction of the linear operators of the 2r-dimensional real vector space: only the linear operators commutable with I give linear operators of the r-dimensional complex vector space, because in the latter I must be treated as a number.

The above method can be applied to S_6 with the linear operator L. S_6 is rendered a complex three-dimensional vector space S_3^+ by taking $iA = LA$. By taking $iA = -LA$ we get the vector space $S_3^- = S_3^{+*}$, the complex conjugate space of S_3^+. The algebra of the linear operators of S_3^+ and S_3^- is constituted by the linear operators K of S_6 commutable with L:

$$[K, L] = 0. \tag{2}$$

We must now take into account that S_6 is endowed with the symmetric bilinear form $\varepsilon(A, B)$, which distinguishes the group of linear operators N of S_6 such that

$$\varepsilon(NA, NB) = \varepsilon(A, B). \tag{3}$$

The group of the N is isomorphic to 0(3,3). Only the operators $\hat{N}$ commutable with L are linear operators of S_3^+ and S_3^-. It follows from (3) and $[L, \hat{N}] = 0$ that

$$\varepsilon(\hat{N}A,\ \hat{N}B) = \varepsilon(A,\ B) \quad \text{and} \quad \varepsilon(\hat{N}A,\ L\hat{N}B) = \varepsilon(A,\ LB). \tag{4}$$

The conditions (4) *characterize the* $\hat{N}$ *because they give* $[L, \hat{N}] = 0$. It follows indeed, from the first equation (4), that $\varepsilon(A, LB) = \varepsilon(\hat{N}A, \hat{N}LB)$ and since for any A' there is an A with $A' = \hat{N}A$ because of the existence of $\hat{N}^{-1}$, for arbitrary A' and B we have $\varepsilon(A', \hat{N}LB) = \varepsilon(A', L\hat{N}B)$, *so that* $\hat{N}L = L\hat{N}$.

Since the $\hat{N}$ are real linear operators, we can replace the equations (4) by

$$E^+(\hat{N}A,\ \hat{N}B) = E^+(A,\ B) \text{ with } 2E^+(A,\ B) = \varepsilon(A,\ B) - i\varepsilon(A,\ LB), \tag{5$^+$}$$

or

$$E^-(\hat{N}A,\ \hat{N}B) = E^-(A,\ B) \text{ with } 2E^-(A,\ B) = \varepsilon(A,\ B) + i\varepsilon(A,\ LB). \tag{5$^-$}$$

$E^+(A, B)$ *and* $E^-(A, B)$ *are symmetric bilinear forms of the complex vector spaces* S_3^+ and S_3^-, respectively, because $E^+(LA, B) = E^+(A, LB) = iE^+(A, B)$ and $E^-(LA, B) = E^-(A, LB) = -iE^-(A, B)$, so that the multiplication of $E^+(A, B)$ by i is equivalent to replace either A by LA or B by LB and the multiplication of $E^-(A, B)$ by $(-i)$ is equivalent to replace A by LA or B by LB. *Equations* (5$^+$) *and* (5$^-$) *show therefore that the group constituted by the* $\hat{N}$ *is the orthogonal group of the complex Euclidean vector spaces* S_3^+ *and* S_3^-, *with the metric quadratic forms* $E^+(A, A)$ *and* $E^-(A, A)$, *respectively.*

We can now use the well known isomorphism of $SO(3, C)$ and the continuous Lorentz group $SO_0(3, 1)$ to obtain at each point of the world-manifold a linear representation of the Lorentz group by the linear operators $\hat{N}$ on the A. *The* $\hat{N}$ *give the Ad representation of the Lorentz group of the corresponding point x of the world-manifold, the fundamental representation associated to the Lie algebra of the Lorentz group.*

The above discussion shows that the operator L is related to one of the Casimir operators of the Lorentz group, because of its commutability with all the $\hat{N}$. *It is easily seen that it is actually related to the Casimir operator* C_{op},

$$C_{op} = (1/8)\,\varepsilon_{\alpha\beta\gamma\delta}\,\delta_{op}^{(\alpha\beta)}\,\delta_{op}^{(\gamma\delta)}, \tag{6}$$

the $\delta^{(\kappa\lambda)}$ *being the basic tensors A of the coordinate system, of components*

$\delta^{(\kappa\lambda)}_{\mu\nu} = \delta^{\kappa\lambda}_{\mu\nu}$, *and the* $\delta^{(\kappa\lambda)}_{op}$ *the corresponding linear operators in a representation of the Lie algebra.*

Thus we get the following form of the Fundamental Theorem:

THEOREM. The linear operator $L(x)$ *of dielectricity and magnetic permeability of space at the point* x *of the world-manifold determines its Lorentz group and the conformal geometry at* x *corresponding to a Minkowskian signature of the Riemannian metric, which is determined only up to a scalar factor* $s(x)$ by $L(x)$.

It is well known[3] that a Lie algebra $\mathscr{L}$ over the real numbers is said to have a complex structure when the vector space constituted by the elements X of $\mathscr{L}$ has a complex structure defined by a linear operator I with $I^2 = -1_{op}$, $1_{op}X = X$, such that for any elements X and Y of $\mathscr{L}$,

$$X \times (IY) = I(X \times Y) \quad \text{with} \quad X \times Y = \text{Lie product of } \mathscr{L}. \tag{7}$$

The Lie algebra of the Lorentz group of the quadratic form $g_{\mu\nu} V^\mu V^\nu$ *has a complex structure in the four-dimensional case with* $g < 0$, *defined by the linear operator* L *on the* A *corresponding to* $g_{\mu\nu}$:

$$(LA)_{\mu\nu} = (1/2)\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} g^{\rho\alpha} g^{\sigma\beta} A_{\alpha\beta} \quad \text{with} \quad L^2 = -1_{op}. \tag{8}$$

It follows indeed from the definition (2-7) that

$$L(A \times B) = A \times (LB). \tag{9}$$

The discussion of the complex structure of S_6 defined by the electromagnetic linear operator L shows that the complex structure of the Lie algebra of the Lorentz group at a point x of the world-manifold follows from the commutability of L with the linear operators $Ad\,A$ of the infinitesimal Lorentz transformations on the vectors of S_6. Equation (9) can indeed be written as

$$L(Ad\,A)B = (Ad\,A)(LB). \tag{10}$$

Thus we see that the electromagnetic linear operator L *defines directly the complex structure of the Lie algebra of the Lorentz group at the corresponding point of the world-manifold.* L *defines also the complex structure of the electromagnetic Lie algebra* $\mathscr{E}$.

Equation (9) is a fundamental property of the Lie algebra of the Lorentz group. It leads to several interesting relations

$$A \times {}^*B = -B \times {}^*A \quad \textit{so that} \quad A \times C = -{}^*A \times {}^*C \quad \textit{and} \quad {}^*A \times A = 0. \tag{11}$$

The second equation (11) shows the effect of the tensor-duality on the Lie algebra. *The third equation* (11) *shows that the infinitesimal Lorentz transformations corresponding to A and $^*A$ are commutable.*

4a. Let us consider now the vector space $S_6(C)$ of the complex antisymmetric tensors $A_{\mu\nu}$, in order to be able to introduce the eigenvectors A^+ and A^- of L corresponding to the eigenvalues i and $(-i)$ respectively,

$$LA^+ = iA^+, \quad LA^- = -iA^-. \tag{1}$$

Since L is a real operator, its eigenvalues i and $(-i)$ are triple and the A^+, A^- constitute two three-dimensional vector spaces of $S_6(C)$. *The A^+ and A^- vector spaces are equivalent to the S_3^+ and S_3^- vector spaces of Section* 3, *respectively*. Let us write $2A^+ = A + iB$ with A and B real. The condition $L(A + iB) = i(A + iB)$ gives $LA = -B$ and $LB = A$, so that $2A^+ = A - iLA$. Similarly we see that the A^- are of the form $2A^- = A + iLA$ with A real. *The complex conjugate of a A^+ is a A^- corresponding to the same real tensor A.*

For the sake of simplicity we shall denote by S_3^+ the vector space of the A^+ and by S_3^- the vector space of the A^-. *$S_6(C)$ is the direct sum of S_3^+ and S_3^-, since any complex A is the sum of a A^+ and a A^-,*

$$A = A^+ + A^- \quad \textit{with} \quad A^+ = (1/2)(1_{op} - iL)A, \; A^- = (1/2)(1_{op} + iL)A. \tag{2}$$

The linear operators L^+ and L^-,

$$L^+ = (1/2)(1_{op} - iL), \quad L^- = (1/2)(1_{op} + iL), \tag{3}$$

are determined by the vector spaces S_3^+ and S_3^- and conversely determine them completely. We have

$$(L^+)^2 = L^+, (L^-)^2 = L^-, L^+L^- = L^-L^+ = 0, L^+ = (L^-)^*, L^+ + L^- = 1_{op}, \tag{4}$$

$$\varepsilon(A, L^+B) = \varepsilon(B, L^+A) \quad \textit{and} \quad \varepsilon(A, L^-B) = \varepsilon(B, L^-A). \tag{5}$$

It follows immediately from (5) *that* A^+ *and* B^- *are orthogonal with respect to the two basic symmetric bilinear forms of* $S_6(C)$,

$$\varepsilon(A^+, B^-) = 0 \quad \textit{and} \quad \varepsilon(A^+, LB^-) = 0. \tag{6}$$

The symmetric bilinear form $\varepsilon(A, B)$ *defines a complex Euclidean metric in* $S_6(C)$, *which induces in* S_3^+ *and* S_3^- *the three-dimensional complex Euclidean metrics defined by the symmetric bilinear forms* E^+ *and* E^- *of equations* (4-5) *and* (4-5*):

$$E^+(A, B) = \varepsilon(L^+ A, L^+ B) \quad \textit{and} \quad E^-(A, B) = \varepsilon(L^- A, L^- B). \tag{7}$$

Equations (7) show in a very clear way why E^+ and E^- are three-dimensional complex bilinear forms of S_3^+ and S_3^-.

The linear operators K commutable with L are those which transform an A^+ into a B^+ and a A^- into a C^-. This explains why the K define linear operators K^+ in S_3^+ and K^- in S_3^-:

$$K^+ = KL^+,\ K^- = KL^- \quad \text{and also} \quad K^+ = L^+ K,\ K^- = L^- K, \tag{8}$$

because K is commutable with L^+ and L^-. It follows from (8) that

$$K = K^+ + K^- \quad \text{with} \quad K^+ K^- = 0. \tag{9}$$

4b. The discussion of Section 4a showed that the introduction of a real linear operator L satisfying the conditions $L^2 = -1_{op}$ and $\varepsilon(A, LB) = \varepsilon(B, LA)$ is associated to the splitting of $S_6(C)$ into the direct sum of two complex conjugate three-dimensional vector spaces S_3^+ and S_3^-, orthogonal with respect to the complex six-dimensional metric $\varepsilon(A, A)$. L is in fact expressed in terms of the two projectors L^+ and L^- of S_3^+ and S_3^-,

$$L = i(L^+ - L^-). \tag{1}$$

We shall now see that any pair of complex conjugate three-dimensional vector spaces S, S^* with $S_6(C) = S \oplus S^*$ determines a real linear operator L_S with $L_S^2 = -1_{op}$ and $\varepsilon(A, L_S B) = \varepsilon(B, L_S A)$, provided the vectors A_S be orthogonal to the B_{S*} with respect to the metric defined by $\varepsilon(A, B)$: $\varepsilon(A_S, B_{S*}) = 0$. We can define L_S by the conditions $L_S A_S = iA_S$ and $L_S A_{S*} = -iA_{S*}$, since any $A = A_S + A_{S*} : L_S A = i(A_S - A_{S*})$.

When A is real, $A_{S*} = A_S^*$ and $L_S A$ is also real, so that L_S is a real linear operator and for, any A, $L_S^2 A = i(L_S A_S - L_S A_{S*}) = -A$, so that $L_S^2 = -1_{op}$.

We have $\varepsilon(A, L_S B) = i\varepsilon(A_S + A_{S*}, B_S - B_{S*}) = i\varepsilon(A_S, B_S) - i\varepsilon(A_{S*}, B_{S*}) = = \varepsilon(B, L_S A)$, because of the ε-orthogonality of S and S^*.

A three-dimensional vector space S of $S_6(C)$ can be described by a Grassmann outer product of three linearly independent $A_S^{(a)}$: $W_S = A_S^{(1)} \wedge \wedge A_S^{(2)} \wedge A_S^{(3)}$. W_S is an antisymmetric tensor of the third order of $S_6(C)$ of components $W_{S;\kappa\lambda;\mu\nu,\rho\sigma}$, determined by S up to a numerical factor. From W_S we can get a symmetric tensor-density $G_{S,\alpha\beta}$ of determinant $G_S \neq 0$:

$$6G_{S;\alpha\beta} = (1/2)\varepsilon^{\kappa\lambda\mu\nu} W_{S;\alpha\kappa,\lambda\mu,\nu\beta}. \tag{2}$$

We shall see in Section 4*c* that $G_{S;\alpha\beta} = \sigma(-g_S)^{1/2} g_{S;\alpha\beta}$, σ being an arbitrary complex number depending on the choice of the $A_S^{(a)}$, and $g_{S;\alpha\beta}$ a real tensor with $g_S < 0$.

The above discussion gives a new way to obtain the relation between the electromagnetic linear operator $L(x)$ and the symmetric tensor $g_{\mu\nu}(x)$ at the point x of the world-manifold, $g_{\mu\nu}(x)$ being expressed in terms of three linearly independent eigenvectors $A^{+(a)}(x)$, $a = 1,2,3$, of $L(x)$ and of an arbitrary scalar field $\sigma(x)$. *Thus we obtain a new proof of the Fundamental Theorem relating the conformal geometry of the world-manifold to the properties of dielectricity and magnetic permeability of space described by the linear operator L.*

4c. We shall now apply to $S_6(C)$ the theory of the tensor-duality, taking the metric of $S_6(C)$ as defined by the quadratic form $\varepsilon(A, A)$. We shall consider especially the antisymmetric tensors $W_{\kappa\lambda,\mu\nu,\rho\sigma}$ of the third order of $S_6(C)$, for which there is a linear operator of the type L, which will be denoted by L_6. It follows from the general discussion of Section 3b that $L_6^2 = 1_{6,op}$, $1_{6,op}$ denoting the unit operator on the W, because now $n = = 2r = 6$ and $\varepsilon(A, A)$ has signature 0 for real A, so that $\theta = -1$. L_6 has the eigenvalues 1 and -1, and there are real eigenvectors. We shall denote by X and Y the eigenvectors of L_6:

$$L_6 X = X \quad \textit{and} \quad L_6 Y = -Y. \tag{1}$$

The X are self-dual and the Y anti-self-dual.

The simple p-vectors of $S_6(C)$ are its antisymmetric tensors of order p obtainable as Grassmann outer products of p linearly independent vectors of $S_6(C)$. They describe the p-dimensional vector spaces of $S_6(C)$. *The* *tensor duality of a n-dimensional metric vector space transforms its p-dimensional non null vector spaces into their orthogonal $n-p$ dimensional vector spaces, through the associate simple p and $n-p$ vectors.*

The tensor duality of $S_6(C)$ corresponding to the metric $\varepsilon(A, A)$ exchanges the conjugate orthogonal vector spaces S and S^*, and in particular S_3^+ and S_3^-. *It allows us to determine in a very convenient way the structure of the simple W_S describing S.* $L_6 W_S$ is a simple trivector W_{S*} of S^* and must therefore be of the form kW_S^*, k denoting a numerical factor, because S and S^* are complex conjugate vector spaces. We have $L_6^2 W_S = kL_6 W_S^* = k(L_6 W_S)^*$ because L_6 is real, so that $kk^* = 1$ and $k = e^{iu}$ with u real. Since $L_6(e^{-iu/2} W_S) = (e^{-iu/2} W_S)^*$,

$$W_S = e^{iu/2}(X_S + iY_S) \quad \textit{with real } X_S \textit{ and } Y_S, \tag{2}$$

$$L_S X_S = X_S, \; L_S Y_S = -Y_S \quad \textit{with real } X_S \textit{ and } Y_S. \tag{3}$$

S determines the associated pair X_S, Y_S up to a real numerical factor. X_S and Y_S for a given W_S are determined up to a common sign.

A simple direct calculation shows that a necessary and sufficient condition for a Y is

$$\varepsilon^{\kappa\lambda\mu\nu} Y_{\alpha\kappa,\,\lambda\mu,\,\nu\beta} = 0. \tag{4}$$

We have therefore $\varepsilon^{\kappa\lambda\mu\nu} X_{\alpha\kappa,\,\lambda\mu,\,\nu\beta} \neq 0$ for $X \neq 0$. Thus only the X_S part of W_S contributes to the symmetric tensor-density $G_{S;\,\alpha\beta}$ defined by Equation (4b-2):

$$6G_{S;\,\alpha\beta} = (1/2)e^{iu/2}\varepsilon^{\kappa\lambda\mu\nu} X_{S;\,\alpha\kappa,\,\lambda\mu,\,\nu\beta}, \; (e^{-iu/2} G_{S;\,\alpha\beta} \text{ is real}). \tag{5}$$

Since $S_6(C) = S \oplus S^*$, $S \neq S^*$ and $X_S \neq 0$, $Y_S \neq 0$. $e^{-iu/2} G_{S;\,\alpha\beta}$ is therefore a real nonzero symmetric tensor density, for any value of u. *In particular the $G^{(o)}_{S;\alpha\beta}$ corresponding to $u = 0$ are real and nonzero.* Any $G_{S;\,\alpha\beta}$ is of the form $re^{iu/2} G^{(o)}_{S;\alpha\beta}$ with a given $G^{(o)}_{S;\alpha\beta}$ and real values of r.

We can define by means of an equation of the form (4b-2) a $G_{S;\,\alpha\beta}$ for any three-dimensional vector space S of S_6 or $S_6(C)$ by means of a W_S which is the outer product of three linearly independent tensors $A^{(a)}$ of S. In particular S can be taken complex, as in Section 4b, or real. $G_{S;\,\alpha\beta}$ will be 0 in case W_S is a Y. We shall assume that there are in S non null $A_S^{(a)}$ satisfying the conditions of orthogonality and normalization

$$\varepsilon(A_S^{(a)}, A_S^{(b)}) = 2\omega_a \delta_{ab} \quad (\omega_a \neq 0). \tag{6}$$

The determinant D_S of the elements $(1/2)\varepsilon(A_S^{(a)}, A_S^{(b)})$ is the analog of the Gram determinant of the three vectors $A_S^{(a)}$ for the metric $(1/2)\varepsilon(A, A)$. D_S is therefore nonzero for any set of three linearly independent $A_S^{(a)}$. With the choice (6), we get $D_S = \omega_1 \omega_2 \omega_3$. *We shall see that the determinant G_S of the $G_{S;\alpha\beta}$ is D_S^2, hence nonzero.* With the choice (6),

$$G_S = (\omega_1 \omega_2 \omega_3)^2. \tag{7}$$

It follows from the definition of the Grassmann outer product $A \wedge B$ in S_6:

$$(A \wedge B)_{\kappa\lambda,\,\mu\nu} = A_{\kappa\lambda} B_{\mu\nu} - A_{\mu\nu} B_{\kappa\lambda}, \tag{8}$$

and from the identity

$$(1/2)\varepsilon^{\kappa\lambda\mu\nu}(A_{\alpha\kappa} B_{\lambda\mu} + A_{\lambda\mu} B_{\alpha\kappa}) = -\varepsilon(A, B)\delta_\alpha^\nu \tag{9}$$

that

$$G_{S;\alpha\beta} = (1/2)\varepsilon^{\kappa\lambda\mu\nu} A^{(1)}_{S;\alpha\kappa} A^{(2)}_{S;\lambda\mu} A^{(3)}_{S;\nu\beta}. \tag{10}$$

It follows from equation (10) that the matrix of the $G_{S;\alpha\beta}$ is the product of the three matrices of elements $A^{(1)}_{S;\alpha\beta}$, $(1/2)\varepsilon^{\alpha\beta\gamma\delta} A^{(2)}_{S;\gamma\delta}$ and $A^{(3)}_{S;\alpha\beta}$, whose determinants are the corresponding ω_a^2. *Since G_S is the product of those determinants, we get equation* (7).

We can associate to S the relative tensor $G_S^{-1/4} G_{S;\alpha\beta}$ and the absolute tensor

$$\mathscr{L}_{S;\kappa\lambda}^{\ \ \mu\gamma} = G_S^{-1/2}\, \varepsilon^{\mu\nu\rho\sigma}\, G_{S;\kappa\rho} G_{S;\lambda\sigma}, \tag{11}$$

which do not depend on the choice of the $A_S^{(a)}$. *The $\mathscr{L}_{S;\alpha\beta}$ define a linear operator $\mathscr{L}_{\mathscr{S}}$ on the $A: (\mathscr{L}_{\mathscr{S}} A)_{\mu\nu} = \mathscr{L}^{\rho\sigma}_{S;\,\mu\nu} A_{\rho\sigma;}$*

Let $\mathscr{S}$ denote the three-dimensional vector space of S_6 or $S_6(C)$ orthogonal to S with respect to the metric $\varepsilon(A, A)$. Since $\varepsilon(A_{\mathscr{S}}, A_S^{(a)}) = 0$, it follows from (10) and (9) that the vectors $A_{\mathscr{S}}$ of $\mathscr{S}$ are eigenvectors of the eigenvalue (-1) of $\mathscr{L}_S$ and the A_S eigenvectors of the eigenvalue 1 of $\mathscr{L}_S$:

$$\mathscr{L}_S A_S = A_S,\ \mathscr{L}_S A_{\mathscr{S}} = -A_{\mathscr{S}} \quad \textit{so that} \quad \mathscr{L}_S^2 = 1_{op}. \tag{12}$$

When S is real, the $A^{(a)}$ can be taken as real tensors. D_S is also real and $G_S = D^2 > 0$. Thus $G_{S;\alpha\beta}$ and $G_S^{-1/4} G_{S;\alpha\beta}$ are real. We can associate to the real S any tensor $g_{S;\alpha\beta}$ with determinant $g_S > 0$ such that

$$g_S^{-1/4} g_{S;\alpha\beta} = G_S^{-1/4} G_{S;\alpha\beta} \quad \textit{so that} \quad g_{S;\alpha\beta} = g_S^{1/4} G_S^{-1/4} G_{S;\alpha\beta}, \quad (g_S > 0). \tag{13}$$

In the case of a real S the $g_{S;\alpha\beta}$ cannot have the Minkowskian signature (−2) because $g_S > 0$. The real metric induced by $\varepsilon(A, A)$ in S can be either definite or indefinite, giving rise to a definite Riemannian metric $g_{S;\alpha\beta}$ and to an indefinite one of signature 0, respectively. Since in both cases the real operator $\mathscr{L}_S$ satisfies the same conditions $\mathscr{L}_S^2 = 1_{op}$ and $\varepsilon(A, \mathscr{L}_S B) = \varepsilon(B, \mathscr{L}_S A)$, *those conditions are not sufficient to determine uniquely the signature of the Riemannian metric.*

Let us consider now the case of two complex conjugate ε-orthogonal S, S^*. We need only to consider $G_S^{(o)-1/4} G_{S;\alpha\beta}^{(o)}$ in the calculation of $\mathscr{L}_S$, which would be real for $G_S^{(o)} > 0$ and imaginary for $G_S^{(o)} < 0$. Now A_S^* is a $A_{\mathscr{S}}$ and must correspond to the eigenvalue (−1) of $\mathscr{L}_S$, which cannot therefore be a real operator. $G_S^{(o)}$ *is therefore negative and we have now the tensors $g_{S;\alpha\beta}$ with $g_S < 0$:*

$$g_{S;\alpha\beta} = (-g_S)^{1/4}(-G_S)^{-1/4} G_{S;\alpha\beta} \quad \textit{with} \quad g_S < 0 \textit{ arbitrary}. \tag{14}$$

Now the Riemannian metrics must be normal hyperbolic with the Minkowskian signature (−2). The linear operator $L_S = i\mathscr{L}_S$ is real and satisfies the conditions

$$L_S^2 = -1_{op}, \quad \varepsilon(A, L_S B) = \varepsilon(B, L_S A), \tag{15}$$

which are sufficient to determine the signature of the metric.

In the case of electromagnetism, we can apply directly the results of the above discussion with $S = S_3^+$ and $S^* = S_3^-$, the two vector spaces associated to the eigenvalues i and $(-i)$ of the electromagnetic linear operator L, which is now $L = L_S$. Equation (14) shows that L determines $g_{S^+;\alpha\beta}$ up to the factor $\pm(-g_{S^+})^{1/4}$. In fact S does not determine the sign of $(-G_{S^+})^{-1/4}$ which comes in the conformal metric $(-G_{S^+})^{-1/4} G_{S^+;\alpha\beta}$.

Thus we have obtained a proof of the Fundamental Theorem based on the direct construction of the conformal metric from eigenvectors $A^{(a)+}$ of the linear operator L of dielectricity and magnetic permeability of space. This method allows also an easy parallel discussion of the non Minkowskian signatures, which are not related to complex structures.

We have excluded the three-dimensional S constituted by null vectors A_S with $\varepsilon(A_S, A_S) = 0$. Let us consider now a pair of such real vector spaces S, S_* with $S_6 = S \oplus S_*$, the $A_S^{(a)}$ and $A_{S*}^{(a)}$ satisfying the conditions:

$$\varepsilon(A_S^{(a)}, A_S^{(b)}) = 0; \quad \varepsilon(A_{S*}^{(a)}, A_{S*}^{(b)}) = 0; \quad (A_S^{(a)}, A_{S*}^{(b)}) = \omega\delta_{ab}. \tag{16}$$

(ω = *scalar-density and* $a, b = 1,2,3$)

We can define a linear operator L_S by the conditions

$$L_S A_S^{(a)} = A_{S*}^{(a)} \quad \text{and} \quad L_S A_{S*}^{(a)} = - A_S^{(a)}. \tag{17}$$

The $A_S^{(a)}$ and $A_{S*}^{(b)}$ constitute a basis for S_6 and thereby we have

$$L_S^2 = -1_{op} \quad \text{and} \quad \varepsilon(A, L_S B) = \varepsilon(B, L_S A) \tag{18}$$

The definition (17) of L_S renders the $A_S^{(a)}$ and $A_{S*}^{(b)}$ an orthonormal basis of S_6 for the metric $\omega^{-1}\varepsilon(A, L_S A)$ with signature 0

$$\varepsilon(L_S A_S^{(a)}, A_S^{(b)}) = \omega\delta_{ab}; \; \varepsilon(L_S A_S^{(a)}, A_{S*}^{(b)}) = 0; \; \varepsilon(L_S A_{S*}^{(a)}, A_{S*}^{(b)}) = -\omega\delta_{ab}. \tag{19}$$

4d. We shall now show how to obtain at each point x bases of orthogonal vectors $P^{(j)}$ with $j = 0,1,2,3$ from complex bases $A^{(a)+}$ with $a = 1,2,3$ in S_3^+. We take

$$\varepsilon(A^{(a)+}, A^{(b)+}) = -(i/2)\omega\delta_{ab} \textit{ with } \omega = \textit{scalar-density and } A^{(a)+} = (1/2)(A^{(a)} - i\, {}^*\!A^{(a)}) \tag{1}$$

The six real tensors $A^{(a)}$ and ${}^*A^{(a)} = LA^{(a)}$ constitute an orthonormal basis in S_6 for the metric $\omega^{-1}\varepsilon(A, LA)$ because of the equations (2) which follow from (1):

$$\varepsilon(A^{(a)}, A^{(b)}) = 0; \; \varepsilon({}^*A^{(a)}, {}^*A^{(b)}) = 0; \; \varepsilon(A^{(a)}, {}^*A^{(b)}) = \omega\delta_{ab}, \tag{2}$$

$$\omega^{-1}\varepsilon(A, LB) = A_a B_a - {}^*A_a {}^*B_a \quad \textit{with} \quad A = A_a A^{(a)} + {}^*A_a {}^*A^{(a)}. \tag{3}$$

It is well known that a necessary and sufficient condition for A to be a simple bivector $A = P \wedge P'$ is that $\varepsilon(A, A) = 0$, the symbol $\wedge$ denoting the outer product. A necessary and sufficient condition for two simple linearly independent bivectors A and $\bar{A}$ to be expressed in terms of three linearly independent covariant vectors P, $\bar{P}$ and P' as $A = P \wedge P'$, $\bar{A} = \bar{P} \wedge P'$ is that $\varepsilon(A, \bar{A}) = 0$. P' is determined up to a numerical factor as belonging to the intersection of the two-dimensional vector spaces associated to A and $\bar{A}$.

It follows from the equations (2) *that the six tensors* $A^{(a)}$ *and* ${}^*A^{(a)}$ *are all simple bivectors and can be expressed as outer products of vectors* P. By taking into account the three equations (2) it is seen that four linearly independent $P^{(j)}$ are sufficient and we have, either

$$A^{(a)} = P^{(o)} \wedge P^{(a)} \quad \textit{and} \quad *A^{(a)} = (1/2)\,\varepsilon_{abc} P^{(b)} \wedge P^{(c)} \quad \text{with } a,b,c = 1,2,3 \quad (4a)$$

or

$$*A^{(a)} = P^{(o)} \wedge P^{(a)} \quad \textit{and} \quad A^{(a)} = (1/2)\,\varepsilon_{abc} P^{(b)} \wedge P^{(c)} \quad (4b)$$

$$\varepsilon^{\alpha\beta\gamma\delta} P^{(o)}_{\alpha} P^{(1)}_{\beta} P^{(2)}_{\gamma} P^{(3)}_{\delta} = \omega \quad \textit{because} \quad \varepsilon(A^{(a)}, *A^{(a)}) = \omega \quad (5)$$

The $P^{(j)}$ can not be introduced with L alone, a ω is necessary. The $g_{\mu\nu}$ corresponding to L and $Z = \omega^{1/2}$ are given in terms of the $P^{(j)}$ by (6):

$$g_{\mu\nu} = \sum_j s_j P^{(j)}_{\mu} P^{(j)}_{\nu} \quad \text{with} \quad s_0 = 1 \quad \text{and} \quad s_1 = s_2 = s_3 = -1. \quad (6)$$

The $P^{(j)}$ give an orthonormal basis of the metric $g_{\mu\nu}$:

$$g^{\mu\nu} P^{(j)}_{\mu} P^{(k)}_{\nu} = s_j \delta_{jk}. \quad (7)$$

The $P^{(j)}$ are determined by the $A^{(a)}$ and $*A^{(a)}$ up to a simultaneous change of all their signs.

Thus we get a third proof of the Fundamental Theorem. The $g_{\mu\nu}$ are not entirely determined by L because $g = -\omega^2$, as a consequence of Equation (5).

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