ACADEMIA BRASILEIRA DE CIÊNCIAS

FORMAL SERIES AND DISTRIBUTIONS

Carmen Lys Ribeiro Braga and Mario Schönberg

SEPARATA DO VOL. 31 N.º 3 DOS "ANAIS DA ACADEMIA BRASILEIRA DE CIÊNCIAS"

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Formal Series and Distributions

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INTRODUCTION

1. It is well known that the concept of numerically valued function of n numerical variables is not broad enough to cover all the types of distribution of matter or probability. The mathematical physicists have been using in an unsystematic way generalized functions since the XVIIIth century. The development of the quantum mechanics led to the systematic use of generalized functions. The concept of function may be generalized in different ways. The best known generalization is given by the theory of distributions [1] and its extensions.

In the present paper we shall discuss a generalization of the concept of numerically valued functions of numerical variables by means of formal series of orthogonal functions, introduced by one of us [2, 3, 4]. In the theory of the Fourier series, the periodic integrable function $f(x)=f(x+2\pi)$ is associated to the formal series $\sum_{-\infty}^{+\infty} a_n e^{inx}$ with $2\pi a_n = \int_0^{2\pi} f(x)\, e^{-inx}\, dx$; even when the series is not convergent. *Our idea is to regard any formal series* $\sum_{-\infty}^{+\infty} b_n e^{inx}$ *as defining a generalized function and the series* $\sum_{-\infty}^{+\infty} inb_n e^{inx}$ *as the derivative of the generalized function* $\sum_{-\infty}^{+\infty} b_n e^{inx}$, *and other kinds of formal series of functions and generalized functions.*

The formal Fourier series are convenient to obtain a generalization of the periodic functions. Strictly speaking we get a generalization of the classes of periodic functions $\{f(x)\}$, $\{f(x)\}$ being constituted by all the functions having almost everywhere the same value as f(x). We shall be interested mainly in functions defined in the interval $(-\infty, +\infty)$, but not necessarily periodic. Thereby we shall use formal series of Hermite functions $\sum a_n h_n(x)$

$$h_n(x) = (2^n\, n!\, \sqrt{\pi})^{-1/2} \exp(-x^2/2)\, H_n(x),\quad H_n(x) = (-1)^n e^{x^2} D_n e^{-x^2} \tag{1}$$

or other complete orthonormal systems $\varphi_n(x)$ of the interval $(-\infty, +\infty)$, the $\varphi_n(x)$ being indefinitely differentiable. The functions F such that $\int_{-\infty}^{+\infty} |F(x)|^2 dx < \infty$ will be associated to the formal series $\sum a_n(F) \varphi_n(x)$ with $a_n(F) = \int_{-\infty}^{+\infty} \varphi_n(x)^* F(x) dx$. Since all the functions of the class $\{F(x)\}$ are associated to the same formal series, we shall write $\{F(x)\} \sim \Sigma a_n(F) \varphi_n(x)$. *We shall extend the association to any functions* f *for which the* $a_n(f)$ *exist, even when* f *does not belong to* $L^2(-\infty, +\infty)$.

It seems natural to define the derivative of the order p of the formal series $\Sigma a_n \varphi_n(x)$ as the series $\Sigma a_n D^p \varphi_n(x)$. With an arbitrary choice of the $\varphi_n(x)$ the formal series for the derivatives will in general not be of the type $\Sigma b_n \varphi_n(x)$, *unless the derivatives* $D \varphi_n(x)$ *be linear combinations of finite numbers of functions* $\varphi_r(x)$. In the case of the Hermite functions $h_n(x)$

$$D h_n(x) = \sqrt{n/2}\, h_{n-1}(x) - \sqrt{(n+1)/2}\, h_{n+1}(x) \tag{2}$$

Thereby all the derivatives of any formal series $\Sigma a_n h_n(x)$ *are formal series of the same kind* $\Sigma a_n^{(p)} h_n(x)$, p *denoting the order of the derivative*. The generalization of the functions by means of formal series of Hermite functions has therefore the same advantage as that given by the distributions with respect to the existence of derivatives of all orders.

Let Λ denote a linear functional whose domain of definition includes the basic functions $\varphi_n(x)$. *We shall define* $\Lambda[\Sigma a_n \varphi_n(x)]$ *as* $\Sigma a_n \Lambda[\varphi_n(x)]$, *when this series converges.* Let us consider in particular the functional $\Lambda_F[f] = \int_{-\infty}^{+\infty} F(x)^* f(x) dx$. We shall write $\Lambda_F[\Sigma a_n \varphi_n(x)] = \int_{-\infty}^{+\infty} F(x)^* \cdot \{\Sigma a_n \varphi_n(x)\} dx$. Thus we get a definition of the integral of the product of a function $F(x)^*$ by a formal series $S_a(x) = \Sigma a_n \varphi_n(x)$. We shall write more generally $\int_{-\infty}^{+\infty} S_a(x)^* S_b(x) dx = \Sigma a_n^* b_n$. It is important to note that a product $S_a(x)^* S_b(x)$ is not a formal series of the type $S_a(x)$.

The Dirac symbolic function $\delta(x - x')$ may be defined as the formal series $\Sigma \varphi_n(x')^* \varphi_n(x)$

$$\delta(x - x') = \Sigma \varphi_n(x')^* \varphi_n(x) \tag{3}$$

for

$$\int_{-\infty}^{+\infty} F(x)^* \delta(x - x') dx = \Sigma a_n(F)^* \varphi_n(x')^* \sim F(x')^* \tag{4}$$

with the above definition of the integral. When the series $\Sigma a_n(F) \varphi_n(x')$ converges and its sum is $F(x')$ we may write $\int_{-\infty}^{+\infty} F(x)^* \delta(x - x') dx = F(x')^*$.

Let us consider now a linear functional Λ^t depending on a parameter t and defined for the $\varphi_n(x)$, for all the values of t. We have $\Lambda^t[\varphi_n] = \psi_n(t)$. *We shall now define the functional* $\Lambda^t[S_a]$ *as the formal series* $\Sigma a_n \psi_n(t)$.

When this series does not converge, the present definition is more general than the above one. The most interesting Λ^t is the Fourier functional $\mathcal{F}_t$

$$\mathcal{F}_t[f] = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(x)\, e^{-itx}\, dx \qquad (5)$$

We shall now take $\varphi_n(x) = h_n(x)$ and use the well known formula $\mathcal{F}_t[h_n] = (-i)^n h_n(t)$.

We have

$$\mathcal{F}_t[S_a] = \Sigma\, (-i)^n\, a_n\, h_n(t) \qquad (6)$$

All the S_a *have Fourier transforms which are also formal series of Hermite functions.*

The coefficients $a_n(f)$ corresponding to formal series of Hermite functions exist even for functions diverging strongly at infinity such as e^x. *Our method allows to define a Fourier transform of* e^{x^α} *with* $|\alpha| < 2$. Those Fourier transforms are formal series of Hermite functions that cannot be associated to Schwarz distributions. We shall prove in section 9 that it is possible to associate a formal series of Hermite functions to any temperate distribution.

It follows from the above discussion that it is possible to give a meaning to many divergent Fourier integrals, such as $\mathcal{F}_t[x^p]$ with $p \geq 0$ or $\mathcal{F}_t[e^x]$, by means of the formal series of Hermite functions. *Those integrals may be regarded as formal integrals analogous to the formal Fourier series.*

The formal series $S(x)$ constitute a linear space with the following definition of the basic operations

$$S_a(x) + S_b(x) = \Sigma\, (a_n + b_n)\, \varphi_n(x)\ ,\quad cS_a(x) = \Sigma\, c\, a_n\, \varphi_n(x) \qquad (7)$$

This linear space is a Wiener differential space, an extension of the Hilbert space of the functions $F(x)$ with $\int_{-\infty}^{+\infty} |F(x)|^2\, dx < \infty$. WIENER [5] was led to the differential space in his work on the integration of functionals. The use of the differential space for the generalization of the concept of function is suggested by the quantum mechanics. Equation (3) is well known from the quantal formalism. The convenience of using series of Hermite functions is shown clearly by the discussion of the Heisenberg ring of the position and momentum operators of a quantal particle given by one of us in reference [2, 3, 4].

The linear space of the $S(x)$ is isomorphic to that of the linear functionals Λ_S, analogous to the above Λ_F

$$\Lambda_S[F] = \int_{-\infty}^{+\infty} S(x)^*\, F(x)\, dx = \{\Lambda_F[S]\}^*;\quad \Lambda_{\varphi_n}[F] = a_n(F) \qquad (8)$$

Any Λ_S *is defined on a part of the Hilbert space of the* F *which contains all the* $\varphi_n(x)$. Conversely, we can associate to each linear functional defined for the $\varphi_n(x)$, and such that $\Lambda\left[F\right] = \Sigma\, a_n(F)\, \Lambda\left[\varphi_n\right]$, the formal series $\Sigma\left\{\Lambda\left[\varphi_n\right]\right\}^* \varphi_n(x)$. *Our generalization of the concept of function by means of the formal series* $S(x)$ *amounts to the consideration of the linear functionals* Λ *of the Hilbert space of the* F*, defined for the functios* $\varphi_n(x)$ *of a complete orthonormal system, and such that for any* F *in the domain of definition of the functional the series* $\Sigma\, a_n(F)\, \Lambda\left[\varphi_n\right]$ *be convergent and equal to* $\Lambda\left[F\right]$ *and that all the* F *with this property belong to that domain.* In other words, the domain of definition of each Λ in the Hilbert space of the F is constituted by all the F for which the series $\Sigma\, a_n(F)\, \Lambda\left[\varphi_n\right]$ converges, $\Lambda\left[F\right]$ being the sum of the corresponding serie.

It is important to note that we get in general different linear spaces of generalized functions according to the choice of the system of the $\varphi_n(x)$. All those spaces of generalized functions contain however the Hilbert space of the F and some generalized functions such as $\delta\,(x - a)$. Let L be a linear operator defined for all the $F(x)$, we have $\Sigma\, \varphi_n(a)^*\, L\, \varphi_n(x) = \Sigma\, h_n(a)\, L\, h_n(x)$, for any orthonormal complete set of functions $\varphi_n(x)$; $\delta\,(x-a)$ corresponds to the unity operator 1_{op}.

The linear space associated to the formal series of Hermite functions has a special importance as a consequence of its relations with the Fourier transformation. The $h_n(x)$ *being integral analytic functions, we may consider the formal series* $\Sigma\, a_n\, h_n(z)$ *for all complex values of the variable z, and define the derivative with respect to z as in the case of a real variable.* Thus we get an interesting relation between generalized functions of a real variable and generalized functions of a complex variable, which may be regarded in some sense as analytic.

We may also use formal series of non orthogonal functions to obtain generalized functions. The formal power series are obviously of great interest and the formal series of Hermite polynomials too. Let the $\xi_n(x)$ be a complete set of linearly independent functions. We shall associate to them the linear functionals Λ_n satisfying the conditions $\Lambda_m\left[\sum_{n=0}^{\infty} A_n\, \xi_n(x)\right] = A_n$, in particular $\Lambda_m\left[\xi_n(x)\right] = \delta_{m,n}$. In the case of the complete orthonormal systems, we have $\Lambda_m\left[F(x)\right] = \int_{-\infty}^{+\infty} \varphi_m(x)^*\, F(x)\, dx = a_m(F)$. When there are functions $\zeta_n(x)$ constituting with the $\xi_n(x)$ a biorthogonal system $\int_{-\infty}^{+\infty} \zeta_m(x)\, \xi_n(x)\, dx = \delta_{m,n}$, we have $\Lambda_m\left[F(x)\right] = \int_{-\infty}^{+\infty} \zeta_m(x)\, F(x)\, dx$. We can now define $\delta\,(x - a)$ by the formal series $\Sigma\, \zeta_n(a)\, \xi_n(x)$.

The Hermite polynomials $H_n(x)$ can be taken as $\xi_n(x)$ and the associated functions $\zeta_n(x)$ will be $\zeta_n(x) = (\sqrt{\pi}\, 2^n\, n!)^{-1} \exp(-x^2)\, H_n(x)$. Thus we have a simple example of a biorthogonal system of functions and of the corresponding formal series $\Sigma\, A_n\, H_n(x)$. In the case of the formal Taylor series we can take $\xi_n(x) = x^n/(n!)$ and $\Lambda_n\left[f(x)\right] = f^{(n)}(0)$. In the present case there are no functions $\zeta_n(x)$ constituting a biorthogonal system with the $x^n/(n!)$. We define the derivative of the formal series $\Sigma A_n H_n(x)$ as $\Sigma 2n A_n H_{n-1}(x)$, since $DH_n = 2nH_{n-1}$. The derivative of the formal Taylor series $\Sigma A_n x^n/(n!)$ will be defined as the formal series $\Sigma A_n x^{n-1}/\{(n-1!)\}$.

The generalized functions obtained by means of the series of orthogonal and non orthogonal functions are not sufficient to deal with all the kinds of problems found in Physics. *It is necessary to introduce formal series of generalized functions of the above kind. Those series will be discussed in section 7. They lead to the introduction of a kind of generalized complex (or real) numbers.*

THE BASIC ALGEBRAIC DUALITY OF THE THEORY OF THE FORMAL SERIES

2. Let us consider the formal series $S_A = \Sigma A_n \xi_n$. The space of the S_A will be endowed with a structure of linear space by taking $S_A + S_B = \Sigma (A_n + B_n) \xi_n$, $c S_A = \Sigma c A_n \xi_n$, c denoting a complex number. *The linear space of the S_A is isomorphic to the linear space of the sequences of complex numbers $A_0, A_1, \dots, A_n, \dots$.* It is convenient to associate S_A to the one-column matrix whose elements are the coefficients A_n.

We shall denote by $\lambda_\alpha = \Sigma \alpha_n \Lambda_n$ the linear functional on the S_A defined as follows

$$\lambda_\alpha [S_A] = \Sigma \alpha_n A_n$$

the α_n denoting complex numbers. The domain of definition of any λ_α in the space of the S_A contains always the basic vectors ξ_n. We shall associate λ_α to the one-row matrix whose elements are the α_n. Thus $\lambda_\alpha [S_A]$ is equal to the product of the one-row matrix of λ_α by the one-column matrix of S_A.

The linear space of the λ_α is isomorphic to that of the S_A. *There is obviously an algebraic duality between those two isomorphic linear spaces.* This is the basic duality of the theory of the formal series. When the ζ_n and ξ_n constitute a bi-orthogonal system we have $\Lambda_n[F] = \int_{-\infty}^{+\infty} \zeta_n(x) \cdot F(x)\, dx$, so that we may consider the space of the formal series $\Sigma \alpha_n \zeta_n$ instead of that of the λ_α. Thus we get an algebraic duality between two linear spaces of formal series.

Let L be a linear operator on the F(x) whose domain of definition contains all the $\xi_n(x)$ and such that the $L \xi_n = \sum_m L_{mn} \xi_m$. The coefficients L_{mn} are the elements of the matrix of L with respect to the basic functions ξ_n. It is interesting to note that $L_{mn} = \Lambda_m [L \xi_n]$. In the case of a bi-orthogonal system we have $L_{mn} = \int_{-\infty}^{+\infty} \zeta_m(x)\, L\, \xi_n(x)\, dx$. *When the series $\sum_n L_{mn} A_n$ converge for all the m, we shall define $L S_A$ as the formal series whose coefficients are those series*

$$L S_A = \sum_m \left\{ \sum_n L_{mn} A_n \right\} \xi_m \tag{9}$$

The one-column matrix of the formal series $L\ S_A$ is simply the product (L) (A) of the matrix of L by the one-column matrix of S_A. When the matrix of L has only a finite number of non-null elements in each row, $L\ S_A$ exists for all the S_A.

When the series $\sum_m L_{mn}\ \alpha_m$ converge for all the n, we shall define the linear functional $\lambda_\alpha\ L$ as follows

$$\lambda_\alpha\ L = \sum_n \left\{ \sum_m \alpha_m\ L_{mn} \right\} \Lambda_n \tag{9a}$$

The one-row matrix of $\lambda_\alpha\ L$ is the product (α) (L) of the one-row matrix of λ_α by that of L. When there is only a finite number of non-null elements in each column of (L), $\lambda_\alpha\ L$ exists for any λ_α.

We can always define the linear functional $\delta_u = \Sigma\ \xi_n(u)\ \Lambda_n$, closely related to the operator unity $1_{op} = \Sigma\ \xi_n\ \Lambda_n$. δ_u is analogous to the functional represented by the same notation in the distribution theory, since $\delta_u \left[S_A \right]$ gives the value $S_A(u)$ of the function corresponding to a S_A with a finite number of terms. δ_u is particularly interesting in the case of the formal Taylor series, in which $\xi_n(x) = x^n/(n!)$, $\Lambda_n \left[S_A \right] = D^n\ S_A(x)_{x=0}$

$$\delta_u \left[S_A \right] = \Sigma \left\{ D^n\ S_A(x)_{x=0} \right\} u^n/(n!) \tag{10}$$

The true importance of δ_u will become clear in section 8.

In the case of a bi-orthogonal system we may introduce the formal series $\Sigma\ \zeta_n(u)\ \xi_n$ and $\Sigma\ \xi_n(u)\ \zeta_n$, the latter being associated to the functional δ_u. and the former to the Dirac generalized function $\delta(x-u)$. *In our theory we have therefore both* δ_u *and* $\delta(x-u)$, *whereas in the theory of the distributions there is only* δ_u.

In the theory of the distributions are used the derivatives of linear functionals. We shall define the derivative of order p of λ_α as $(-1)^p\ \lambda_\alpha\ D^p$, in order to obtain results analogous to those of the theory of distributions. It is interesting to note that the derivation of formal series S_A and linear functionals λ_α are very similar operations in the present formalism, as shown by equations (9) and (9a). The derivation of S_A corresponds to the product (D) (A) and the derivation of λ_α to the product $-$ (α) (D) of matrices. *The matrices of the* D^p *with respect to* $x^n/(n!)$, $h_n(x)$ *and the* $H_n(x)$ *have only a finite number of elements in each row and in each column, since* $D\ x^n = n\ x^{n-1}$, $D\ H_n = 2n\ H_{n-1}$ *and* $D\ h_n = \sqrt{n/2}\ h_{n-1} - \sqrt{(n+1)/2}\ h_{n+1}$. *Hence in those three cases all the* S_A *and* λ_α *have derivatives of all orders.*

The algebraic duality of the spaces of the S_A and λ_α is associated to the affine geometry of vector spaces of infinite dimensionality. It corresponds to the duality of the contravariant and covariant vectors of a finite-dimensional affine space. The use of a complete system of orthonormed functions φ_n is associated to the unitary geometry of a space of infinite dimensionality. The linear space of the $\Sigma\ \alpha_n\ \zeta_n$ is now that of the $\Sigma\ a_n^*\ \varphi_n^*$ and we can associate to each S_a the linear functional $\Lambda_{S_a} = \Sigma\ a_n^*\ \Lambda_n$. In the unitary geometry it is possible to introduce a unitary Fourier operator $\mathcal{F}_{op}$. *The space of the*

S_a *corresponding to the Hermite functions* h_n *admits the Fourier transformation as an automorphism* $S_a = \Sigma\, a_n h_n \longrightarrow \mathcal{F}_{op} S_a = \Sigma\, a_n (-i)^n h_n$, *for* $\mathcal{F}_{op} h_n = (-i)^n h_n$. The matrix of $\mathcal{F}_{op}$ with respect to the basic functions h_n is diagonal.

The representation of the generalized functions by one-column matrices shows that the classical theory of the functions of real variables can be replaced, to a large extent, by an algebraic formalism that treats in the same way ordinary and generalized functions, the derivatives of all orders and the Fourier transformation being applicable to all the ordinary and generalized functions included in the formalism.

The theory of the distributions and its generalizations are based on the properties of the topological vector spaces. Our generalization of the functions is based on the algebraic theory of the linear space of the sequences. There is however a natural topology in the linear space of the sequences. Let us consider a sequence of sequences $S(p)$, $p = 0, 1, 2, \ldots$, the elements of the sequence $S(p)$ being the $a_n(p)$. *We shall say that the sequence of the* $S(p)$ *tends to the sequence* S *whose elements are the* a_n, *when the sequences* $a_n(p)$, *with* n *fixed and* p *variable, are convergent and* $\lim_{p \to \infty} a_n(p) = a_n$.

We shall discuss the topology based on this convergence in section 7.

* * *

GENERALIZED AND ORDINARY DERIVATIVES OF FUNCTIONS

3. Let $f(x)$ be a function for which the $a_n(f) = \int_{-\infty}^{+\infty} f(x)\, h_n(x)\, dx$ exist. It follows from equation (2) that the coefficients of the series of Hermite functions obtained by applying to that of $f(x)$ the definition of derivative given in section 1 are $a_n^{(1)}(f)$

$$a_n^{(1)}(f) = \sqrt{(n+1)/2}\; a_{n+1}(f) - \sqrt{n/2}\; a_{n-1}(f) = -\int_{-\infty}^{+\infty} f(x)\,\{D\, h_n(x)\}\, dx \quad (11)$$

It may happen that the derivative $D\, f(x)$ *exists but cannot be associated to a formal series of Hermite functions, because some of the integrals* $\int_{-\infty}^{+\infty} h_n(x)\,\{Df(x)\}\, dx$ *are divergent. In such a case the generalized derivative defined by the derivation of the formal series for* $f(x)$ *is not equivalent to the ordinary derivative* $D\, f(x)$. This situation occurs for instance with $x^{-1/3}$, since the $a_{2n}(x^{-4/3})$ are divergent integrals.

The coefficients of the formal series of Hermite functions obtained by p successive derivations of that for $f(x)$ will be denoted by $a_n^{(p)}(f)$. It is eaily seen that

$$a_n^{(p)}(f) = (-1)^p \int_{-\infty}^{+\infty} f(x)\,\{D^p\, h_n(x)\}\, dx = Fp\left(\int_{-\infty}^{+\infty} h_n(x)\,\{D^p\, f(x)\}\, dx\right) \quad (12)$$

Fp denoting the Hadamard finite part of the integral. The finite part coincides of course with the Lebesgue integral, whenever the integral exists(*).

The operator D *involved in the theory of the Hilbert space of the* F(x) *is not the ordinary derivative but our generalized one.* Indeed; in that theory D is characterized by its matriz (D_{mn}) and the product of (D_{mn}) by the one-column matrix corresponding to F(x) is equivalent to our generalized derivative of the formal series $\Sigma\, a_n(F)\, (h_n(x)$. Even when F(x) is everywhere differentiable and F′(x) belongs to the Hilbert space, we can transform F into a function discontinuous everywhere, by modifying its values at the points of a set of measure zero. The modified function corresponds to the same vector of the Hilbert space as F(x) and the vector of comsponent $\sum_n D_{mn}\, a_n(F)$ may be considered as the derivative of the modified discontinuous function. The passage from the classical definition of the derivative to ours is, in the present case, related to the identification of functions that differ only at the points of a set of measure zero.

The necessity of distinguishing in some cases the ordinary derivative of a function from a generalized derivative appears also in the theory of the distributions, for instance when the derivative of a locally integrable differentiable function is not locally integrable. The finite parts of the integrals come in in the differentiation of distributions in a way similar as they do in our theory.

We have introduced the formal series $\Sigma\, a_n\, h_n(x)$ for real values of the variable x. *We may extend any* $S_a(x)$ *to complex values of* x *by taking the Hermite functions for such values. The definition of the derivative of a formal series of Hermite functions of a complex variable is the same as in the real case. Any such series has derivatives of all orders, and may be regarded as a kind of generalized analytic function.* We shall not discuss this point in the present paper, it will be treated in detail elsewhere.

Our kind of generalized functions and their derivatives may be conveniently used in the theory of differential equations, as has been done with the Schwarz distributions. Our theory seems to be naturally indicated when the theory of the Hilbert space is applied to differential equations, since our theory is an extension of that of the Hilbert space of the F(x). In many cases, formal series of Laguerre functions, formal series of spherical functions, formal series of Fourier functions etc. are more adequate than the formal series of Hermite functions.

The introduction of the generalized derivative is extremely simple in the case of a formal Fourier series $\sum_{-\infty}^{+\infty} a_n\, e^{inx}$: the generalized derivative

*) The authors are indebted to PROF. HANS JOOS for the introduction of the finite parts in connection with the derivation of formal series of Hermite functions.

of this series is $\sum_{-\infty}^{+\infty}$ in $a_n\, e^{inx}$. The part of $\delta\,(x - y)$ is now played by the series $\sum_{+\infty}^{-\infty}$ $e^{in(x-y)}/2\,\pi$. This kind of expansion comes in naturally when x is an angle.

* * *

3a. *The above discussion shows that there may be a formal series of Hermite functions associated to a function f when some or all the integrals* $\int_{-\infty}^{+\infty} f(x)\, h_n(x)\, dx$ *diverge, but the Hadamard finite parts* $Fp\left\{ \int_{-\infty}^{+\infty} f(x)\, h_n(x)\, dx \right\} = b_n(f)$ *exist, namely the series* $\Sigma\, b_n(f)\, h_n(x)$. *We shall denote the formal series* $\Sigma\, b_n(f)\, h_n(x)$ *by* $Fp\,\{f(x)\}$ *and by* $PV\,\{f(x)\}$ *when the finite parts involved in the definition of the* $b_n(f)$ *are simply Cauchy principal values of integrals.* Thus we get the $Fp\,\{x^{-2p}\}$ and the $PV\,\{x^{-(2p+1)}\}$. It is easily seen that those formal series are obtained by successive derivations of the formal series of Hermite functions of $\log |x|$.

Let us consider now the Heaviside discontinuous function $\varepsilon\,(x) = 0$ for $x < 0$ and $\varepsilon\,(x) = 1$ for $x \geq 0$. It follows from (11) that

$$a_n^{(1)}\,(\varepsilon\,(x - a)) = h_n(a) \tag{13}$$

the formal series obtained by the derivation of that for $\varepsilon\,(x - a)$ is therefore $\delta\,(x - a) = \Sigma\, h_n(a)\, h_n(x)$, as should be expected.

It is convenient to introduce a special symbol Dg for the generalized derivative, in order to distinguish it from the ordinary one. It follows from the above discussion that

$$D_g \log |x| = PV\,\{x^{-1}\}\,,\quad D_g\, Fp\,\{x^{-n}\} = -\,n\; Fp\,\{x^{-(n+1)}\} \tag{14}$$

$$D_g\, \varepsilon\,(x - a) = \delta\,(x - a)\,,\quad D_g\, \delta^{(p)}\,(x - a) = \delta^{(p+1)}\,(x - a) \tag{15}$$

It follows from the definition of $\int_{-\infty}^{+\infty} S_a(x)^*\, S_b(x)\, dx$ given in section 1 that

$$\int_{-\infty}^{+\infty} \delta^{(p)}\,(y - x)\; S_a(y)\; dy = (-\,1)^p\; D_g^p\; S_a(x) \tag{16}$$

* * *

GENERALIZED FUNCTIONS OF THE OPERATOR x_{op}

4. We shall denote by x_{op} the linear operator defined by the equation

$$x_{op}\; F(x) = x\; F(x) \tag{17}$$

$f(x_{op})$ is the linear operator whose matrix elements with respect to the Hermite functions are

$$\{f\,(x_{op})\}_{mn} = \int_{-\infty}^{+\infty} h_m(x)\; h_n(x)\; f(x)\; dx \tag{18}$$

We shall define $S_a(x_{op})$ as the linear operator whose matrix elements with respect to the $h_n(x)$ are

$$\{ S_a(x_{op}) \}_{mn} = \int_{-\infty}^{+\infty} \{ h_m(x)\, h_n(x) \}\, S_a(x)\, dx \tag{19}$$

provided those generalized integrals exist. Thus we get

$$\{ \delta(x_{op} - y) \}_{nm} = h_m(y)\, h_n(y)\,,\ \{ \delta^{(p)}(x_{op} - y) \}_{nm} = (-1)^p\, D_y^p\, (h_m(y)\, h_n(y)) \tag{20}$$

It is important to note that the $S_a(x_{op})$ *are true, not symbolic, linear operators.* We shall write

$$h_m(x)\, h_n(x) = \sum_p h_{mn,p}\, h_p(x)\ ,\ h_{mn,p} = \int_{-\infty}^{+\infty} h_m(x)\, h_n(x)\, h_p(x)\, dx \tag{21}$$

$$h_{mn,p} = \{ h_p\, (x_{op}) \}_{mn} \tag{22}$$

Hence

$$\{ S_a(x_{op}) \}_{mn} = \sum_p h_{mn,p}\, a_p = \sum_p a_p \{ h_p\, (x_{op}) \}_{mn} \tag{23}$$

We shall denote by $[L, M]$ the commutator $L\,M - M\,L$ of the linear operators L and M. It is readily seen that

$$[x_{op}, S_a(x_{op})] = 0\ ,\ [D, S_a(x_{op})] = D_g\, S_a(x_{op}) \tag{24}$$

by means of the matrix multiplication. *Since the* $S_a\,(x_{op})$ *are ordinary operators of the Hilbert space of the* F(x), *the second equation* (24) *shows again that* D_g *is the kind of derivative involved in the theory of the Hilbert space.*

The Fourier operator $\mathcal{F}_{op}$ of section 2 has a diagonal matrix in the basis of the h: $\{ \mathcal{F}_{op} \}_{mn} = (-i)^m\, \delta_{m,n}$. The unitary transformation $\mathcal{F}_{op}^{-1}\, L\, \mathcal{F}_{op}$ is defined for all the L having a matrix (L_{mn}) with respect to the $h_n(x)$. Since

$$\mathcal{F}_{op}^{-1}\, x_{op}\, \mathcal{F}_{op} = -iD \tag{25}$$

we may define $S_a(-iD)$, whenever $S_a(x_{op})$ exists, as follows

$$S_a(-iD) = \mathcal{F}_{op}^{-1}\, S_a(x_{op})\, \mathcal{F}_{op} \tag{26}$$

$$\{ S_a(-iD) \}_{mn} = i^{m-n}\, \{ S_a(x_{op}) \}_{mn} = \sum_p a_p\, \{ h_p(-iD) \}_{mn} \tag{27}$$

We shall write $L = \sum_{n=0}^{\infty} L_p$ when $L_{mn} = \sum_p \{ L_p \}_{mn}$. Thus we get

$$S_a(x_{op}) = \Sigma\, a_n\, h_n(x_{op})\ ,\ S_a(-iD) = \Sigma\, a_n\, h_n(-iD) \tag{28}$$

We shall define the integral $\int_\sigma L(\alpha)\, d\alpha$ of the linear operator $L(\alpha)$, depending on the α, on the set σ of the $\alpha-$ space as the linear operator such that

$$\left\{ \int_\sigma L(\alpha)\, d\alpha \right\}_{mn} = \int_\sigma \{ L(\alpha) \}_{mn}\, d\alpha \tag{29}$$

It follows from equation (20) that

$$x_{op} = \int_{-\infty}^{+\infty} y\ \delta(x_{op} - y)\ dy\ ,\ f(x_{op}) = \int_{-\infty}^{+\infty} f(y)\ \delta(x_{op} - y)\ dy \tag{30}$$

The first equation (30) *gives the spectral decomposition of* x_{op}. Indeed, let $f_\sigma(x)$ denote the characteristic function of the measurable set σ on the x-axis, we have

$$f_\sigma(x_{op})\ f_\tau(x_{op}) = f_{\sigma\tau}(x_{op})\ ,\ f_\sigma(x_{op}) = \int_\sigma \delta(x_{op} - y)\ dy \tag{31}$$

and $\int_{-\infty}^{+\infty} \delta(x_{op} - y)\ dy$ is the operator unity 1_{op}. In a similar way the spectral decomposition of $-iD$ is

$$-iD = \int_{-\infty}^{+\infty} y\ \delta(-iD - y)\ dy \tag{32}$$

It is readily seen that

$$\delta(-iD - y)\ S_a(x) = (2\pi)^{-1/2}\ e^{ixy}\ \mathcal{F}_{op}\ S_a(y) \tag{33}$$

Hence

$$\int_{-\infty}^{+\infty} S_a(t)\ dt = 2\pi\ \delta(-iD)\ S_a(x) \tag{34}$$

This equation gives a remarkable expression of the integral in terms of a function of the derivative.

* * *

THE FORMAL DOUBLE SERIES OF A LINEAR OPERATOR

5. We shall now associate to the matrix (L_{mn}) of the linear operator L, with respect to the complete orthonormal set of the $\varphi_n(x)$, the formal double series L(x,y)

$$L(x,y) = \sum_{m,n} L_{mn}\ \varphi_m(x)\ \varphi_n(y)^* = \sum_n \varphi_n(y)^*\ L\ \varphi_n(x) \tag{35}$$

$\delta(x - y)$ *is the formal double series corresponding to the operator unity* 1_{op}. Now we regard $\Sigma\ \varphi_n(x)\ \varphi_n(y)^*$ as a formal series in two variables x and y, instead of a formal series in the variable x whose coefficients depend on a parameter y. The formal double series L(x,y) may be regarded as a series in x with coefficients $\sum_n L_{mn}\ \varphi_n(y)^*$, when the latter series converge for all the values of m. *We may define* L(x,y) *by means of a bi-orthogonal system:* $L(x,y) = \Sigma\ L_{mn}\ \xi_m(x)\ \zeta_n(y)$.

We shall define the formal integral $\int_{-\infty}^{+\infty} L(x,y)\ S_a(y)\ dy$ as follows

$$\int_{-\infty}^{+\infty} L(x,y)\ S_a(y)\ dy = \sum_n \Big\{ \sum_n L_{mn}\ a_n \Big\}\ \varphi_m(x) \tag{36}$$

when all the series $\sum_n L_{mn} a_n$ are convergent, i.e. when $S_a(x)$ belongs to the domain of definition of the linear operator L. It follows from (36) that

$$\int_{-\infty}^{+\infty} L(x,y)\, S_a(y)\, dy = L\, S_a(x) \tag{37}$$

L (x,y) *may therefore be called the kernel of the operator* L.

The formal integral $\int_{-\infty}^{+\infty} S_a(x)^* L(x,y)\, dx$ will be defined as follows

$$\int_{-\infty}^{+\infty} S_a(x)^* L(x,y)\, dx = \sum_n \left\{ \sum_m a_n^* L_{mn} \right\} \varphi_n(y)^* \tag{38}$$

when the series $\sum_m a_m^* L_{mn}$ are convergent for all the values of n, i.e. when the product of the one-row matrix whose elements are the a_m^* by the matrix (L_{mn}) exists.

We need a third kind of formal integral $\int_{-\infty}^{+\infty} L(x,y)\, M(y,z)\, dy$ involving the kernels of two linear operators L and M. When the product L M exists we take

$$\int_{-\infty}^{+\infty} L(x,y)\, M(y,z)\, dy = \sum_{m,n} \left\{ \sum_p L_{mp} M_{pn} \right\} \varphi_m(x)\, \varphi_n(z)^* \tag{39}$$

The formal double series in the right-hand side of (39) is the kernel K(x,z) of the linear operator K = L M.

We can associate to L another kind of double formal series

$$L(x,x) = \sum_{m,n} L_{mn}\, \varphi_m(x)\, \varphi_n(x)^* \tag{40}$$

and the formal integral

$$\int_{-\infty}^{+\infty} L(x,x)\, dx = \sum_n L_{nn} = \text{Trace of } L \tag{41}$$

When $L_{mn} = a_m b_n^*$ we shall write $L(x,y) = S_a(x)\, S_b(y)^*$ and $L(x,x) = S_a(x) \cdot S_b(x)^*$. The trace of L coincides now with the formal integral $\int_{-\infty}^{+\infty} S_b(x)^* S_a(x)\, dx$ defined in section 1.

There is a linear operator $S_{a,op}$ associated to each $S_a(x)$

$$(S_{a,op})_{mn} = a_m a_n^* \tag{42}$$

When $S_a(x)$ is a normalized function F(x), $S_{a,op}$ is the projector associated to F(x) in the theory of the Hilbert space. We have $S_{a,op}(x,y) = S_a(x)\, S_a(y)^*$.

It follows from equations (3) and (9) that

$$L(x,y) = L\, \delta(x-y) \tag{43}$$

The operator L *is therefore determined by the formal series* $L\,\delta(x-y)$ *taken as a double series of type* (35). When $L = x^p_{op}$, $L(x,y) = x^p\,\delta(x-y)$. The double series of the operator D^p is $\delta^{(p)}(x-y)$.

* * *

THE PRODUCT OF FORMAL SERIES $S_a(x)$

6. We shall introduce the following definition of the product $S_a(x)\,S_b(x)$

$$S_a(x)\,S_b(x) = \sum_p \left\{\sum_m a_m \left\{\sum_n b_n\,\varphi_{mn,p}\right\}\right\}\varphi_p(x) \tag{44}$$

$$\varphi_{mn,p} = \int_{-\infty}^{+\infty} \varphi_p(x)^*\,\varphi_m(x)\,\varphi_n(x)\,dx \tag{45}$$

When the series $\sum_{m,n} a_m\,b_n\,\varphi_{mn,p}$ *are absolutely convergent,* $S_a\,S_b$ *and* $S_b\,S_a$ *exist both and* $S_a\,S_b = S_b\,S_a$. *Since* $\{S_a(x_{op})\}_{pn} = \sum_m a_m\,\varphi_{mn,p}$, *when those conditions of absolute convergence are satisfied we have*

$$S_a(x)\,S_b(x) = S_a(x_{op})\,S_b(x) = S_b(x_{op})\,S_a(x) \tag{46}$$

In the definition (44) the $\sum_m a_m \left\{\sum_n b_n\,\varphi_{mn,p}\right\}$ are taken as convergent in order that $S_a(x)\,S_b(x)$ be a $S_c(x)$. *It is possible to get a more general definition by dropping that requirement, and allowing* $S_c(x)$ *to be a generalized formal series whose coefficients* c_n *are formal series of some parameters involved in the* a_n *and* b_n. Thus

$$\delta(x-a)\,S_b(x) = S_b(a)\,\delta(x-a) \tag{47}$$

the coefficients c_n are now formal series: $c_n = \varphi_n(a)^*\,S_b(a)$. An important case of (47) is given by the well known formula

$$\delta(x-u)\,\delta(x-v) = \delta(u-v)\,\delta(x-u) \tag{48}$$

From a purely formal point of view, we should take for the product $S_a\,S_b$ the series $\sum_m a_m \left\{\sum_n b_n\,\varphi_m\,\varphi_n\right\}$. Since $\varphi_m\,\varphi_n \sim \sum_p \varphi_{mn,p}\,\varphi_p$, it is natural to take $S_a\,S_b = \sum_m a_m\left\{\sum_n b_n\left\{\sum_p \varphi_{mn,p}\,\varphi_p\right\}\right\}$. $\sum_n \varphi_{mn,p}\,\varphi_p$ is a generalized function represented by a formal orthogonal series and $\sum_n b_n\left\{\sum_p \varphi_{mn,p}\,\varphi_p\right\}$ a formal series of such generalized functions, hence a mathematical entity of a more complicated kind than the above generalized functions. *Thus* $S_a\,S_b$ *appears as a series whose terms are series of generalized functions. The product of two formal orthogonal series appears therefore in general as a mathematical entity more complicated than a series of generalized functions* (such entities will be discussed in section 7b). In some special cases those mathematical entities may be identified with generalized functions represented by formal orthogonal series and we are then led to the particular definition given by equation (44).

We have seen that it is sometimes possible to define a linear operator $S_a(x_{op})$ associated to the generalized function $S_a(x)$. The product $S_a(x_{op}) \cdot S_b(x_{op})$ of two linear operators is determined by the product of the matrices of the two operators. It is readily seen that $\{S_a(x_{op})\ S_b(x_{op})\}_{mn} = \sum_r \left\{ \sum_{s,t} a_s\, b_t\, \varphi_{sr,m} \varphi_{tn,r} \right\}$. The matrix elements of the product operator are given by a series whose terms are themselves double series. The matrix elements of the product may sometimes be expressed in terms of generalized functions. Thus we have

$$\{\delta(x_{op} - a)\ \delta(x_{op} - b)\}_{mn} = \delta(a - b)\ \varphi^*_m(a)\ \varphi_n(b) \tag{49}$$

When the φ_p are Hermite functions we get easily

$$\{\delta(x_{op} - a)\, \delta(x_{op} - b)\}_{mn} = \delta(a - b)\, h_m(a)\, h_n(a) = \delta(a - b)\, \{\delta(x_{op} - a)\}_{mn} \tag{50}$$

We may therefore write formally

$$\delta(x_{op} - a)\ \delta(x_{op} - b) = \delta(a - b)\ \delta(x_{op} - a) \tag{51}$$

This operator equation corresponds obviously to (48). $\delta(x_{op} - a)\ \delta(x_{op} - b)$ is a kind of generalized operator whose matrix elements are not ordinary numbers. We shall introduce in section 8 a kind of generalized numbers which allow to deal satisfactorily with the present situation.

SERIES OF GENERALIZED FUNCTIONS

7. Let us consider a sequence of generalized functions $S_{A(N)} = \sum_n A(N)_n\, \xi_n$, N taking the values 0, 1, 2... The sequence will be called convergent when the $A(N)_n$ tend to limits A_n as $N \longrightarrow \infty$. The generalized function $S_A = \sum A_n\, \xi_n$ will be called the limit of the sequence of the $S_{A(N)}$. *The linear space of the S_A is obviously closed with respect to the present definition of the convergence. Any generalized function S_A can be obtained as a limit of series $S_{A(N)}$ with a finite number of terms, which therefore correspond to ordinary functions; we may for instance take* $A(N)_n = A_n\, \varepsilon(N - n)$, *ε being the Heaviside function.* The formal orthogonal series $\sum a_n\, \varphi_n$ are therefore limits of functions of the Hilbert space $L^2(-\infty, +\infty)$ of the F. *It is important to note that the definitions of convergence corresponding to different complete orthogonal systems are not equivalent, so that we get different kinds of closure of $L^2(-\infty, +\infty)$.*

The linear space of the formal Taylor series $\sum c_n\, x^n$ corresponds to the $\xi_n(x) = x^n$. It may be considered as a closure of the linear space of the polynomials of x. The space of the formal series of Hermite polynomials is also a closure of the space of the polynomials of x, since any of these polynomials is a finite combination of Hermite polynomials.

We may extend the above definition of convergence to formal series $S_{A(u)}$, whose coefficients $A(u)_n$ depend on a continuous parameter u and tend to limits A_n as $u \to u_o$: $\lim_{u \to u_o} S_{A(u)} = S_A$. Thus by means of the formal Hermite series we get

$$\lim_{u \to o} (\sqrt{2\pi}\,|u|)^{-1} \exp(-x^2/2u^2) = \delta(x) \tag{52}$$

since

$$(\sqrt{2\pi}\,|u|)^{-1} \int_{-\infty}^{+\infty} \exp(-x^2/2u^2)\, h_{2n}(x)\, dx = h_{2n}(0)\,(1-u^2)^n/(1+u^2)^{n+1/2} \tag{53}$$

It is readily seen that the derivatives $\delta^{(p)}(x)$ are also the limits of the corresponding derivatives of $(\sqrt{2\pi}\,|u|)^{-1} \exp(-x^2/2u^2)$ for $u \to 0$.

The introduction of a definition of convergence in the space of the S_A allows us to use the series of generalized functions $\sum_N S_{A(N)}$, which will be called convergent when the sequence of the partial sums $\sum_N S_{A(N)}\, \varepsilon(p-N)$ convergences for $p \to \infty$, the limit of this sequence being the formal ξ-series taken as the sum of the above series of generalized functions. A necessary and sufficient condition of convergence of $\sum_N S_{A(N)}$ is the convergence of the numerical series $s_n = \sum_N A(N)_n$. We have actually $\sum_N S_{A(N)} = \sum_n s_n\, \xi_n$. The $S_{A(N)}$ may in particular be associated to ordinary functions. *Thus we may interpret a formal series* $\Sigma A_n \xi_n$ *as a series of generalized functions, from the point of view of the Hermite series, by introducing into* $\Sigma A_n \xi_n$ *the expansions of the* ξ_n *in terms of the* h_n.

The series of generalized functions obtained from $\Sigma A_n \xi_n$ by replacing the ξ_n by the corresponding Hermite series will in general not converge, because the numerical series $s_p = \sum_n A_n \int_{-\infty}^{+\infty} \xi_n(x)\, h_p(x)\, dx$ may be divergent. *In order to be able to deal simultaneously with different kinds of formal series, we need an extension of the linear space of the formal Hermite series* $S_a = \Sigma a_n h_n$ *including the non-convergent series of generalized functions* $\sum_N S_{a(N)}$. This extension is obviously analogous to that of the space of the convergent ξ-series by the formal ξ-series.

* * *

7a. The existence of different extensions of the Hilbert space $L^2(-\infty, +\infty)$ associated to different kinds of formal orthogonal series is an unpleasant feature of the theory of the formal orthogonal series, at the first level of development. We shall now pass to a higher level, by the introduction of a linear space constituted by all the series of generalized functions $\sum_N S_{a(N)} =$

$= \sum_N \left\{ \sum_n a(N)_n h_n \right\}$, with the rules $\sum_N S_{a(N)} + \sum_N S_{b(N)} = \sum_N \{S_{a(N)} + S_{b(N)}\}$ and $c \sum_N S_{a(N)} = \sum_N c\, S_{a(N)}$. The formal series $\Sigma\, b_n \varphi_n$ will be identified to $\sum_N \left\{ b_N \sum_n a_n(\varphi_N) h_n \right\}$, with $a_n(\varphi_N) = \int_{-\infty}^{+\infty} \varphi_N(x)\, h_n(x)\, dx$. *The passage from the linear space of the* S_a *to that of the general series* $\sum_N S_{a(N)}$ *allows us to deal with all kinds of formal orthogonal series in a more unified way, but it allows to introduce new kinds of mathematical entities such as the series of derivatives of the Dirac generalized function* $\Sigma\, c_n\, \delta^{(n)}(x)$, *with arbitrary coefficients* c_n, *and the formal series* $\Sigma\, A_n\, \xi_n$ *of functions* ξ_n *for which the* $a_n(\xi_N)$ *are finite.* The ξ_n need not to tend to zero as $|x| \longrightarrow \infty$, we may take for instance $\xi_n(x) = x^n$. The possibility of identifying the formal Taylor series with series $\sum_N S_{a(N)}$ is particularly interesting, for it shows that we can now deal with functions increasing at infinity as fast as any integral analytic function.

The exponentials $\exp(-\lambda x)$, with any complex λ, have finite Hermite coefficients. We may therefore associate to any formal series $\Sigma\, c_n \exp(-\lambda_n x)$ a series $\sum_N S_{a(N)}$. *The non-separable Hilbert space of the almost periodic functions of a real variable may therefore be identified with a sub-space of the* $\sum_N S_{a(N)}$.

We shall define the derivative of order p $D^p \sum_N S_{a(N)}$ of $\sum_N S_{a(N)}$ as $\sum_N D^p S_{a(N)}$. *All the* $\sum_N S_{a(N)}$ *have thus derivatives of all orders. There is also a natural definition of the Fourier transform* $\mathcal{F} \sum_N S_{a(N)} = \sum_N \mathcal{F} S_{a(N)}$ *for any* $\sum_N S_{a(N)}$. The Fourier transformation defines an automorphism of the linear space of the series $\sum_N S_{a(N)}$. Any linear operator L of $L^2(-\infty, \infty)$, with a row-finite matrix with respect to the basis of the h_n, can be extended to all the $\sum_N S_{a(N)}$, in the same way as D^p or $\mathcal{F}$, for $L\, S_{a(N)}$ is a Hermite series.

Let $\Sigma\, b_n \varphi_n$ and $\Sigma\, c_n \psi_n$ be two different orthogonal expansions of the same F. They correspond to different series of generalized functions, which must be regarded as equivalent. *We shall therefore divide the series* $\sum_N S_{a(N)}$ *into classes of equivalent series, all the series of the same class describing the same mathematical entity.* The series $\sum_N S_{a(N)}$ and $\sum_N S_{\mathit{a}(N)}$ will be taken as equivalent when $\sum_N \{S_{a(N)} - S_{\mathit{a}(N)}\}$ is a convergent series of generalized functions of sum 0, with the definition of convergence corresponding to the space of the formal Hermite series. The criterion of equivalence is therefore that all the numerical series $\sum_N (a(N)_n - \mathit{a}(N)_n)$ be convergent and have sum zero. This criterion is satisfactory in the above case of the two different orthogonal expansions of a function F. It is easily seen that the derivatives and Fourier transforms of equivalent series of generalized funtions are also equivalent.

* * *

7b. *The Hermite series and the series of generalized functions are the two simplest types of the series* $\sum_{p_1}\left\{\sum_{p_2}\left\{\ldots\left\{\sum_{p_r}\left\{\sum_n a(p)_n h_n\right\}\right.\right.\right.$, *corresponding to* $r=0$ *and* $r=1$, *respectively. We are led to the consideration of series with* $r=2$ *by the product of two Hermite series, and to series with higher values of r by the multiplication of several Hermite series. The same procedure applied to the series of generalized functions allows to define derivatives of all orders and Fourier transforms for the series with* $r>1$. *All the above series are special cases of the following type of series* $\sum_r\left\{\sum_{p_1}\left\{\ldots\sum_{p_r}\left\{\sum_n a(p_1,\ldots,p_r)_n h_n\right\}\right.\right.$. The space of the latter series can be endowed with a structure of linear space by taking the sum of the series with the coefficients $a(p_1,\ldots,p_r)_n$ and $\mathit{a}(p_1,\ldots,p_r)_n$ as the series with the coefficients $a+\mathit{a}$ and the product of the series of coefficients a by the number c as the series with the coefficients c a. We shall define the product $\left(\sum_r\left\{\sum_p\left\{\sum_n a(p)_n h_n\right\}\right\}\right)\times\left(\sum_r\left\{\sum_p\left\{\sum_n \mathit{a}(p)_n h_n\right\}\right\}\right)$ as the series of coefficients $b(p)_n$ taken as follows

$$b(p_1,\ldots,p_r)_n=\sum_s \varepsilon(r-s-2)\, a(p_1,\ldots,p_s)_{p_{r-1}}\, \mathit{a}(p_{s+1},\ldots,p_{r-2})_{p_r}\, h_{p_{r-1}\,p_r,\,n}$$

This choice of the b is a consequence of the equation $h_m(x)\,h_n(x)=\sum h_{mn,\,k}\,h_k(x)$. *The series* $\sum_r\left\{\sum_p\left\{\sum_n\left\{a(p)_n h_n\right\}\right.\right.$ *constitute a non-associative and non-commutative algebra, with the above definitions of the sums and the products.*

At the presente level of development of the theory of the formal series it is not always convenient to start from the formal Hermite series. It may be preferable to start from the formal Taylor series $\sum c_n x^n$ and to go over to the general series $\sum_r\left\{\sum_p\left\{\sum_n c(p_1,\ldots,p_r)_n x^n\right\}\right.$, since the Hermite functions can be expressed as Taylor series. Let us write $h_m(x)=\sum_n h_{m,n}\,x^n$. We can take $\sum_r\left\{\sum_p\left\{\sum_m a(p)_m h_m(x)\right\}=\sum_s\left\{\sum_q\left\{\sum_n c(q)_n x^n\right\}\right.\right.\right.$ with $c(p_1,\ldots,p_r)_n=a(p_1,\ldots,p_{r-1})_{p_r}\,h_{p_r,n}$.

The advantage of the Taylor series lies in the simplicity of the multiplication rule $x^m\,x^n=x^{m+n}$ and the derivation rule $D\,x^n=n\,x^{n-1}$. On the other hand the Fourier transformation is much more complicated in the case of the Taylor series, since $\mathcal{F}_{op}\,x^n=\sqrt{2\pi}\;i^n\,\delta^{(n)}(x)$ and the derivatives of $\delta(x)$ are expressed by series of the type $\sum_m\left\{\sum_n c_{m,n}\,x^n\right\}$.

We shall now introduce a criterion of equivalence of the generalized series of Hermite functions: *the series of coefficients* a *and* â *are taken as equivalent when* $\sum_r \sum_p \left\{ a(p)_n - \hat{a}(p)_n \right\} = 0$ *for any value of* n, *i.e. when all those numerical multiple series are convergent and their sums are equal to zero.* This criterion contains as a special case that given for the series of generalized functions in section 7a. *There is obviously a serious difficulty in connection with the above definition of product, for the product of two generalized Hermite series is not always equivalent to that of two generalized series respectively equivalent to the former two.* The linear space of the generalized Hermite series is now divided into classes of equivalent series, each class defining a new mathematical entity, which is a new kind of generalized function. The natural definition of the product of two generalized Hermite series does not lead to a definition of the product of the associated generalized functions. The set of the present kind of generalized functions is naturally endowed with a structure of linear space by the operations of addition of generalized Hermite series and multiplication of such series by numbers.

Instead of the above criterion of equivalence of two generalized Hermite series, we may use the following criterion: *the generalized Hermite series of coefficients* a *and* â *are taken as equivalent when* $\sum_r \left\{ \sum_{p_1} \left\{ \dots \left\{ \sum_{p_r} a(p)_n - \hat{a}(p)_n \right\} \right. \right.$ $= 0$ *for all the values of n.* The present criteion is also equivalent to that of section 7a when applied to the series of generalized functions $\sum_N S_a(N)$. The new criterion of equivalence leads also to serious difficulties with the product of generalized Hermite series, analogous to those discussed in connection with the first criterion.

In the case of the generalized formal Taylor series, a criterion of equivalence of two series is also required. We get a criterion for the equivalence of the generalized Taylor series by replacing the coefficients a of the generalized Hermite series by the corresponding ones of the generalized Taylor series in the above conditions of equivalence of the generalized Hermite series.

FORMAL SERIES AND GENERALIZED NUMBERS

8. The complex valued function f defines a correspondence between the real numbers x and the complex numbers. The notation f(x) for the function f is not quite satisfactory, for it actually indicates the complex number associated to the real number x by the function f. In the present section we shall distinguish in the above way the function f and the number f(x), and we shall denote the formal series by $\sum a_n \varphi_n$, instead of $\sum a_n \varphi_n(x)$.

We shall now introduce a special kind of generalized complex numbers, in order that the generalized functions $\sum a_n \varphi_n$ be associated to correspondences between real numbers x and those generalized complex numbers. When the series Σc_n of complex terms converges, it defines a complex number, namely its sum. The convergent series Σc_n and $\Sigma c_n'$ define the same complex numbers when and only when the sum of the series $\Sigma (c_n - c_n')$ is zero. *It is thus seen that the ordinary complex numbers can be identified with the classes of sequences of complex numbers c_n for which Σc_n converges, each class being constituted by the sequences whose series Σc_n have the same sum. We shall associate to any sequence of complex numbers c_n a generalized complex number, which is the symbol of the class of all the sequences of numbers such that the sum of the series $\Sigma (c_n - c_n')$ is zero.* We shall say that the generalized complex number is the sum of its numerical series Σc_n and we shall denote it by Σc_n. The generalized numbers associated to the convergent series can be obviously identified to the ordinary complex numbers.

We shall define the sum of the generalized numbers Σc_n and $\Sigma c_n'$ as the generalized number $\Sigma (c_n + c_n')$. The product of the ordinary number a by the generalized number Σc_n will be defined as the generalized number $\Sigma a c_n$. *Thus the set of the generalized complex numbers is endowed with the structure of a linear space.* The product of two ordinary complex numbers appears here as a special case of the product of an ordinary complex number by a generalized one. There are difficulties in the definition of the product of two proper generalized numbers. Such a product should be associated to a product of two non-convergent numerical series, which is a non-convergent double series. The transformation of a double series into a single series presents in general an arbitrariness, which leads to difficulties even in the case of the product of two convergent series, as well known.

We are now in condition to use both the formal series $\Sigma A_n \xi_n$ and the numerical series $\Sigma A_n \xi_n(x)$ for the different values of the independent variable. *We shall call the generalized number $\Sigma A_n \xi_n(x)$ the value of the formal series $\Sigma A_n \xi_n$ for the value x of the independent variable. The introduction of the values of the formal series allows us to obtain a criterion of equivalence of series of different kinds: we may regard the formal series $\Sigma A_n \xi_n$ and $\Sigma B_n \eta_n$ as equivalent when their values coincide for all the values x of the independent variable, i.e. when the generalized numbers $\Sigma A_n \xi_n(x)$ and $\Sigma B_n \eta_n(x)$ are equivalent for all the values of x.*

The concept of formal series can be extended by the introduction of series $\Sigma \{A_n\} \xi_n$ whose coefficients $\{A_n\}$ are generalized numbers. The series of generalized functions $\Sigma S_{a(N)}$ with $S_{a(N)} = \Sigma a_n(N) h_n$ may be identified to the Hermite series $\Sigma \{a_n\} h_n$ with $\{a_n\} = \sum_N a_n(N)$. *The criterion of equivalence of the above series of generalized functions introduced in section 7 means simply that the generalized Hermite series associated to the* *two series of generalized functions are identical, the coefficients of any* h_n *being equivalent generalized numbers.*

The generalized numbers can be conveniently used to extend the domain of definition of the linear functionals on the vectors of a linear space, by allowing the value of the linear functional to be a generalized number. Thus in the case of the linear space of the formal series $\Sigma A_n \xi_n$, any linear functional Λ, whose ordinary domain of definition includes all the basic functions ξ_n, can be extended to the whole linear space by taking $\Lambda[\Sigma A_n \xi_n]$ as the generalized number $\Sigma A_n \Lambda[\xi_n]$.

The value $\Sigma A_n \xi_n(x)$ *of the formal series* $\Sigma A_n \xi_n$ *for the value* x *of the independent variable is obtained by the application to the formal series of the valuation functional* δ_x *defined in section 2:* $\delta_x[\Sigma A_n \xi_n] = \Sigma A_n \xi_n(x)$. The basic meaning of δ_x appears now in a very clear way for any series $\Sigma A_n \xi_n$. The distribution δ_u corresponds to the Dirac generalized function $\delta(x-u)$. Our theory distinguishes the valuation functional δ_u and the generalized function $\delta(x-u)$, which play both important rôles.

Divergent integrals may be interpreted as generalized numbers. Let f(x) be a locally summable function for which the Hermite coefficients are finite. When $\int_{-\infty}^{+\infty} f(x)\,dx$ diverges, we can take $\int_{-\infty}^{+\infty} f(x)\,dx = \Sigma a_n(f) \cdot \int_{-\infty}^{+\infty} h_n(x)\,dx$, the left-hand side being interpreted as a generalized number. Thus we get the formula

$$\int_{-\infty}^{+\infty} f(x)\,dx = \sqrt{2\pi}\ \Sigma(-1)^n a_{2n}(f)\, h_{2n}(0) \tag{54}$$

In connection with the present definition of $\int_{-\infty}^{+\infty} f(x)\,dx$ as a generalized number, it is important to note that we have

$$\int_{-\infty}^{+\infty} S_a(x)\,dx = \Sigma a_n \int_{-\infty}^{+\infty} h_n(x)\,dx = \sqrt{2\pi}\ \Sigma(-1)^n a_{2n}\, h_{2n}(0) \tag{55}$$

so that we have taken

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} S_{a(f)}(x)\,dx \tag{56}$$

The above definition of $\int_{-\infty}^{+\infty} f(x)\,dx$ as a generalized number cannot be applied to any locally summable function. *It is however possible to define generalized numbers by means of integrals* $\int_{-\infty}^{+\infty} f(x)\,dx$ *of locally summable functions, instead of numerical series.* Such a theory was developed by one of us (Schönberg, unpublished work) in connection with the divergences of the quantum theory of fields. It will be discussed in a forthcoming publication. The symbol $\int_{-\infty}^{+\infty} f(x)\, dx$ is identified to an ordinary number when the integral converges, and taken as a generalized number when it diverges, the ordinary number being the value of the Lebesgue integral. The symbols $\int_{-\infty}^{+\infty} f(x)\, dx$ and $\int_{-\infty}^{+\infty} g(x)\, dx$ are taken as the same generalized number when the integral $\int_{-\infty}^{+\infty} [f(x) - g(x)]\, dx$ converges and has the value 0. These generalized numbers constitute a linear space with $\int_{-\infty}^{+\infty} f(x)\, dx + \int_{-\infty}^{+\infty} g(x)\, dx = \int_{-\infty}^{+\infty} [f(x) + g(x)]\, dx$ and $c \int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{+\infty} c\, f(x)\, dx$., c denoting an ordinary complex number. *Equation* (54) *may now be used to express the old kind of generalized numbers in terms of the present ones, in some cases.*

We may now introduce series $\Sigma [a_n(f)] h_n$ associated to any locally summable functions f(x), the coefficients $[a_n(f)]$ being the generalized numbers $[a_n(f)] = \int_{-\infty}^{+\infty} f(x)\, h_n(x)\, dx$ of the new kind. The generalized numbers defined by symbolic integrals are convenient for the theory of the formal series of orthogonal functions, but not yet sufficiently general to define the "value" of a general series $\Sigma [a_n] h_n(x)$. We need a theory of "numbers" of the type $\sum_0^\infty c_n [a_n]$, for any values of the ordinary complex numbers c_n. This matter will be discussed in the section 8a.

When the matrix elements $L_{mn} = \int_{-\infty}^{+\infty} \zeta_m(x)\, L\, \xi_n(x)\, dx$ are divergent integrals which can be interpreted as generalized numbers, we may associate to L a matrix whose elements are such numbers. Generalized numbers can also be applied to the extension of the definition of the $L\, S_A$: *we shall define* $L\, S_A$ *for any* S_A *by equation* (9), *when the* L_{mn} *are ordinary numbers, the right-hand side of equation* (9) *being taken as a formal series* $\Sigma \{B_n\} \xi_n$ *whose coefficients are generalized numbers* $\{B_m\} = \sum_n L_{mn} A_n$.

* * *

8a. We shall say that the sequence of generalized numbers $\{A(N)\} = \sum_n A(N)_n$ tends to the generalized number $\{A\}$ as $N \to \infty$, when the $\{A\} - \{A(N)\}$ are equivalent to ordinary numbers for sufficiently large values of N and the sequence of these numbers converges to zero. We shall write $\{A\} = \lim_{N \to \infty} \{A(N)\}$. A similar definition can be given in the case of the sequences of generalized numbers $\left[f_N\right] = \int_{-\infty}^{+\infty} f_N(x)\,dx$: the sequence of the $\left[f_N\right]$ tends to the generalized number $\left[f\right] = \int_{-\infty}^{+\infty} f(x)\,dx$ when the integrals $\int_{-\infty}^{+\infty} (f(x) - f_N(x))\,dx$ are convergent for sufficiently large N and the sequence of their values converges to zero.

The series $\sum_N \{A(N)\}$ will be said to converge to the generalized number $\{B\}$, when the sequence of the partial sums $\sum_N \varepsilon(n - N)\{A(N)\}$ tends to $\{B\}$ as $n \longrightarrow \infty$. $\{B\}$ will be called the sum of the series of generalized numbers. *The consideration of the non-convergent series of genera lized numbers $\Sigma\{A(N)\}$ leads to an extension of the concept of generalized numbers $\{A\}$, which allows to define the value $\sum_N S_{A(N)}(x)$ of the series of generalized functions $\Sigma\, S_{A(N)}$ for the value x of the independent variable.* The value of $S_{A(N)}$ for the value x of the independent variable is the generalized number $\sum_n A(N)_n\, \xi_n(x) = S_{A(N)}(x)$; when the series of generalized numbers $\sum_N S_{A(N)}(x)$ converges, its sum is the value of $\Sigma\, S_{A(N)}$ for the value x of the independent variable. In order to define $\Sigma\, S_{A(N)}(x)$ for any $S_{A(N)}$ and any value of x, we need to interpret the non-convergent series of generalized numbers of the kind $\{A\}$ as a new kind of generalized numbers. We shall say that $\Sigma\{A(N)\}$ and $\Sigma\{A'(N)\}$ are equivalent when the series $\Sigma(\{A(N)\} - \{A'(N)\})$ has the sum 0. *The new kind of generalized numbers are the symbols of the classes of equivalent series $\Sigma\{A(N)\}$. When the sum of $\Sigma\{A(N)\}$ is $\{B\}$, we identify the symbol of the class of the series equivalent to $\Sigma\{A(N)\}$ with the generalized number $\{B\}$ of the old type.*

We may use the series of generalized numbers $\Sigma\left[f_N\right]$ to extend the space of the $\left[f\right]$, in the same way as we used the series $\Sigma\{A(N)\}$ to extend the space of the $\{A\}$. Thus we can build the theory of the series $\Sigma\, c_n\left[a_n\right]$ mentioned at the end of section 8. The series of the $\left[f_N\right]$ corresponding to the convergent integrals $\int_{-\infty}^{+\infty} f_N(x)\,dx$ may be identified with those of the type $\{A\}$.

* * *

8b. The two kinds of generalized numbers introduced in sections 8 and 8a correspond to the formal series of functions and to the series whose terms are formal series of functions. It is clearly necessary to extend still more the concept of generalized number in order to deal with the generalized kind of formal series of functions discussed in section 7b.

We shall now consider numerical series of the type $\sum_r \Big\{ \sum_{p_1} \Big\{ \ldots \ldots \Big\{ \sum_{p_r} a(p_1, \ldots, p_r) \Big\} \Big\}$. The series of coefficients a and â will be regarded as equivalent when the series of coefficients $a - \hat{a}$ converges and has sum 0. *The classes of equivalent series will be associated to the generalized numbers.* The generalized numbers of section 7 are those associated to the classes of series which contain only terms corresponding to $r = 1$; the generalized numbers of section 7a correspond to the classes of series containing only terms with $r = 2$.

The generalized Hermite series $\sum_r \left\{ \sum_p \left\{ \sum_n a(p)_n h_n \right\} \right.$ may be regarded as Hermite series $\sum \{a\}_n h_n$ whose coefficients $\{a\}_n$ are the generalized numbers corresponding to the series $\sum_r \left\{ \sum_p a(p)_n \right\}$. The second criterion of equivalence of generalized Hermite series introduced in section 7b means that two such series are taken as equivalent when the numerical series corresponding to the same h_n are equivalent, with the above definition of equivalence, i. e. that they correspond to the same generalized number. *The generalized number corresponding to* $\sum_r \left\{ \sum_p \left\{ \sum_n a(p)_n h_n(x) \right\} \right.$ *may be regarded as the "value" of the generalized Hermite series for the value* x *of the independent variable.*

The set of the present generalized numbers may be endowed with the structure of a linear space by taking as sum of the generalized numbers corresponding to the series of coefficients a and a' that associated to the series of coefficients $a + a'$ and taking as the product of the ordinary complex number c by the genralized number corresponding to $\sum_r \left\{ \sum_p a(p) \right\}$ that corresponding to $\sum_r \left\{ \sum_p ca(p) \right\}$.

FORMAL SERIES AS CONTINUOUS LINEAR FUNCTIONALS

9. Let us consider the formal series $\sum A_n \xi_n$ in the case of a bi-orthogonal system ξ_n, ζ_n. We shall now introduce the linear space E_ξ of the finite linear combinations ω_α of the ζ_n

$$\omega_\alpha = \sum_0^N \alpha_n \zeta_n \tag{57}$$

The ω_α correspond to the linear functionals $\sum \alpha_n \Lambda_n$ on the S_A which are defined for all the S_A. E_ξ *is obviously a linear space of ordinary functions.*

The S_A will now be identified to linear functionals $S_A [\omega_\alpha]$ on the functions ω_α

$$S_A [\omega_\alpha] = \sum_0^N \alpha_n A_n \tag{58}$$

The domain of definition of S_A *is the whole space* E_ξ, *since the right-hand side of* (58) *is a finite sum.* (We are now using only the ordinary numbers and the ordinary domain of definition of a linear functional).

There is a natural definition of convergence in E_ξ: *the sequence of vectors* $\omega_{\alpha(r)}$, $r = 0,1, \ldots,$ *converges to 0 when there is an integer* $\mathcal{N}$, *independent of r, such that* $\alpha(r)_n = 0$ *for all the values of r when* $n > \mathcal{N}$ *and the* $\alpha(r)_n$ *tend to 0 as* $r \longrightarrow \infty$. *The linear functionals* S_A *are continuous with respect to the above definition of the convergence in* E_ξ. *Any linear functional* λ *defined for all the* ω_α *is a* $S = \Sigma \lambda \left[\zeta_n\right] \xi_n$.

The space of the S_A is the dual of E_ξ with the above definition of the convergence in E_ξ. This is a topological duality, whereas the duality of the S_A and the λ_α of section 2 is algebraic. We considered only S_A corresponding to bi-orthogonal systems of functions ξ_n, ζ_n. It is possible to extend the above topological duality to any complete system of linearly independent ξ_n by using the linear space of the finite linear combinations $\mu_\alpha = \sum_0^N \alpha_n \Lambda_n$ of the Λ_n, instead of E_ξ. The S_A may be interpreted as linear functionals on the μ_α

$$S_A \left[\mu_\alpha\right] = \sum_0^N \alpha_n A_n \tag{59}$$

We can introduce in the space of the μ_α a convergence analogous to that of E_ξ and render the S_A into continuous linear functionals on the vectors μ_α.

In the case of a complete orthonormal system of basic functions φ_n, the ω_α are the finite linear combinations of the conjugate functions $\overset{*}{\varphi}_n$. Thereby the formal series $\Sigma\, a_n \varphi_n$ may be interpreted as the continuous linear functionals on the $\sum_0^N b_n^* \overset{*}{\varphi}_n$. In particular, the formal Hermite series correspond to the continuous linear functionals on the finite Hermite series.

The above discussion shows clearly that our algebraic method of defining generalized functions by means of formal series of functions is equivalent to a topological method analogous to that of the theory of the distributions, in which the functions are generalized by means of continuous linear functionals on function spaces.

It is intersting to note that even formal series of generalized functions may in some cases be identified to continuous linear functionals on convenient linear spaces of functions, by a procedure similar to that applied to the formal series of ordinary functions. Let us consider the series $\mathcal{S}_c = \Sigma\, c_n\, \delta^{(n)}$ of derivatives of the Dirac generalized function $\delta(x)$. Those series correspond to linear functionals on the space E_δ of the finite linear combinations of the functions $\zeta_n(x) = (-1)^n (n!)^{-1} x^n$

$$S_c \left[\omega_\alpha\right] = \sum_0^N \alpha_n c_n \tag{60}$$

with $\omega_\alpha = \sum_0^N \alpha_n \zeta_n$. The choice of the ζ_n is based on the well known formula $\int_{-\infty}^{+\infty} \delta^{(m)}(x)\, x^n\, dx = (-1)^m m!\, \delta_{m,n}$. E_δ is simply the linear space of the polynomials of one variable. We can introduce in E_δ a definition of convergence similar to that of the E_ξ, in order to render the $\mathcal{S}_c$ continuous linear functionals on the polynomials ω_α. We have proven that the series of derivatives of δ correspond to linear functionals defined on the whole space of the polynomials. Conversely any such functional λ corresponds to the series $\Sigma\ \lambda\left[\zeta_n\right]\delta^{(n)}$.

* * *

FORMAL HERMITE SERIES AND TEMPERATE DISTRIBUTIONS

10. The most important distributions are the temperate ones. The temperate distributions are those whose Fourier transforms are also distributions, which turn out to be also temperate distributions. The possibility of associating the temperate distributions on the real axis to Hermite series was already discussed by SCHAWRZ in his book [6].

The temperate distributions are the continuous linear functionals on the space S of the indefinitely differentiable functions ψ which tend to 0, together with their derivatives of all orders, for $|x| \longrightarrow \infty$ more rapidly than any negative power of $|x|$. The continuity of the linear functionals $T[\psi]$ is defined by the condition that the numerical sequence $T[\psi_n]$ converges to zero as $n \longrightarrow \infty$ whenever the sequence of the ψ_n converges to 0 in the sense of the convergence in S: the sequence of the functions ψ_n converges to zero as $n \longrightarrow \infty$ when the functions $P(x)\ Q(D)\ \psi_n(x)$ tend to zero uniformly over the real axis for any choice of the polynomials P and Q, the coefficients of Q being constants. The functions ψ of S are called rapidly decreasing functions.

The Hermite functions h_n belong obviously to S, so that it is possible to associate to any temperate distribution T the Hermite series $\Sigma\ a_n(T)\ h_n$ with $a_n(T) = T[h_n]$. It was proven by SCHWARZ [6] that a necessary and sufficient condition for a sequence of complex numbers a_n to be the $a_n(T)$ of a temperate distribution is that the sequence of the a_n be weakly increasing

$$\lim_{n \longrightarrow \infty} (1 + n^2)^{-p}\ |a_n| \longrightarrow 0 \text{ for a sufficiently large p} \tag{61}$$

The functionals $h_n[\psi] = \int_{-\infty}^{+\infty} h_n(x)\ \psi(x)\ dx$ are temperate distributions and the series of distributions $\Sigma\ a_n\ h_n$ converges in the space of the temperate distributions when the sequence of the a_n is weakly increasing, the temperate distribution $T = \Sigma\ a_n\ h_n$ having the a_n as its $a_n(T)$. *The formal Hermite series whose coefficients* a_n *constitute weakly increasing sequences of numbers may therefore be identified to the temperate distributions.*

It is easily seen that the application of our rules of computing the derivatives and the Fourier transforms of the formal Hermite series to those whose coefficients constitute weakly increasing sequences leads to the same results as those of the theory of the temperate distributions. *Our theory of the formal Hermite series is therefore a natural extension of the theory of the temperate distributions.*

The linear space E_h of the finite linear combinations $\sum_0^N \alpha_n h_n = \omega_\alpha$ is included in S and the convergence in E_h, introduced in section 9, leads to the convergence in S. The dual E'_h of E_h contains the dual S' of S, which is constituted by the temperate distributions T. S' is an extension of the Hilbert space $L^2(-\infty, +\infty)$ of the F contained in the space E'_h of all the Hermite series, the F being taken as functionals $F[\psi] = \int_{-\infty}^{+\infty} F(x)\,\psi(x)\,dx$.

The rapidly decreasing functions ψ are the functions F whose Hermite coefficients a_n constitute rapidly decreasing sequences of numbers [6].

$$\lim_{n \to \infty} n^p |a_n| = 0 \text{ for all the positive integers } p \qquad (62)$$

The formal series of exponentials $\sum_{-\infty}^{+\infty} b_s\, e^{isx}$ are related to the periodic distributions of period 2π, in a way similar to that of the Hermite series and the temperate distributions. It is easily seen that those periodic distributions can be identified to the formal Fourier series whose coefficients b_s constitute slowly increasing sequences of numbers. *The theory of the formal Fourier series is therefore an extension of the theory of the periodic distributions.*

GENERAL DISTRIBUTIONS AND SERIES OF ORTHOGONAL FUNCTIONS

11. Let S(a,b) denote the linear space of the functions which are indefinitely differentiable at all the points of the closed finite interval (a, b) and vanish, as well as their derivatives of all orders, at the end-points a, b and outside (a, b). In the theory of the distributions, the continuous linear functionals on (a, b) are defined as the linear functionals λ on $S_{(a,b)}$ such that $\lim \lambda[\psi_r] = \lambda[\psi]$, whenever the ψ_r and ψ belong to $S_{(a,b)}$ and the ψ_r converge uniformly to ψ and the derivatives $\psi_r^{(n)}$ converge uniformly to the $\psi^{(n)}$ for each n. A fundamental theorem of the theory of the distributions states that a continuous linear functional on (a, b) can be identified either to a Lebesgue summable function on (a,b) or to a derivative of finite order of such a function, regarded as a distribution.

Any Lebesgue summable function f(x) defined on a finite and closed interval (a,b) can be associated to a Hermite series, the coefficients $a_n(f)$ being taken as the integrals $\int_a^b f(x)\, h_n(x)\, dx$. The derivatives of the summable function regarded as a distribution can be associated to the derivatives of its Hermite series. *It follows from the above theorem that any continuous linear functional* λ *on* (a,b) *can be associated to a formal Hermite series, namely the series corresponding to the summable function* f(x), *when* λ *may be identified to this function, or to the series corresponding to* $f^{(r)}(x)$, *when* λ *may be identified to* $f^{(r)}(x)$. *with* $r > 0$, *but not to a Lebesgue summable function.*

A distribution on an open interval I is defined as a linear functional whose domain contains the $S_{(a,b)}$ of all the closed finite intervals (a,b) within I and which becomes a continuous linear functional on (a,b) when restricted to $S_{(a,b)}$. We may say that any distribution behaves locally as a continuous linear functional on a closed finite interval. It follows from the above discussion that any distribution may be identified locally with a formal Hermite series. The temperate distributions can be identified globally with formal Hermite series, but not all the distributions. For instance, the distributions corresponding to continuous functions f(x), for which some of the integrals $\int_{-\infty}^{+\infty} f(x)\, h_n(x)\, dx$ diverge, cannot be identified to Hermite series. On the other hand, there are Hermite series that do not correspond to distributions, as was already shown in section 1.

Let us consider the formal series $\Sigma\, c_n\, \xi_n(x)$ of locally integrable functions defined in the interval $-\infty, +\infty$. We may identify ξ_n with the distribution $\xi_n[\psi] = \int_{-\infty}^{+\infty} \xi_n(x)\, \psi(x)\, dx$, $\psi(x)$ denoting as usual any indefinitely differentiable function vanishing together with all its derivatives outside some finite open interval a,b. *A necessary and sufficient condition for the identification of our formal series with a distribution on the interval* $-\infty, +\infty$ *is the convergence of the series of distributions* $\Sigma\, c_n\, \xi_n$, *i.e. the convergence of the numerical series* $\Sigma\, c_n\, \xi_n[\psi]$ *for any* ψ *satisfying the above conditions.* When the $\xi_n(x)$ constitute a complete system, there is always some ψ for which an inifinite number of the $\xi_n[\psi]$ are $\neq 0$ and it is possible to choose the arbitrary coefficients c_n in such a way that the numerical series $\Sigma_n c_n\, \xi_n[\psi]$ be divergent. Thus we get a formal series $\Sigma\, c_n\, \xi_n[x]$ which cannot be identified to a distribution on the interval $-\infty, +\infty$.

Schwartz showed [7] that any distribution T may be obtained as a series of derivatives of continuous functions G_n, $T = \Sigma\, D^{p_n}\, G_n$, each $G_n(x)$ vanishing outside some finite interval a_n, b_n. *Since each distribution* $D^{p_n}\, G_n$ *can be associated to a formal series of Hermite functions, it follows from the above Schwarz theorem that any distribution in the n-dimensional space* R^n *can be associated to a series of generalized functions of n variables, each of those generalized functions being associated to a formal Hermite series of* n *variables.* The theory of the formal Hermite series is a generalization of the theory of the temperate distributions, not including all the distributions, but the theory of the series of generalized functions given by such formal series is a generalization of the theory of the general distributions. It is interesting to note that this generalization allows to obtain a Fourier transform for any distribution, since any series of such generalized functions has a Fourier transform, which is also a series of the same kind, as was pointed in section 7.

SUMMARY

The generalization of the concept of function by means of formal series of functions is discussed. The formal series of Hermite functions and other formal orthogonal series are examined. The formal Hermite series are shown to be of special importance, because of their relations with the derivation and the Fourier transformation; this allows to define the derivatives of all orders and the Fourier transforms of the generalized functions defined by those formal series. It is shown that generalized functions of the derivative are in some cases ordinary operators of $L^2(-\infty, +\infty)$. Series of generalized functions are discussed. It is shown that the formal series of Hermite functions give an extension of the temperate distributions and that any distribution can be associated to a series of generalized functions defined by formal Hermite series.

Formal Taylor series and more general formal series of non-orthogonal functions are also discussed. It is shown that a satisfactory theory of formal series of functions requires the introduction of mathematical entities of a higher kind, defined by a special type of multiple series. The discussion of such formal series of a higher kind leads to the consideration of a generalization of the ordinary complex numbers, the generalized numbers being associated to divergent series.

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