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QUANTUM THEORY AND GEOMETRY

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ABSTRACT

The mathematical formalisms of the quantum mechanics and the quantum field theory are interpreted as special kinds of geometric algebras closely related to the vector calculus and to the calculus of points. The algebras of operators of the quantized fields appear as geometric calculi of general three-dimensional spaces. It is shown that the operators and states of the quantal systems and fields have rather simple geometrical meanings. The present relations between the quantal physics and the geometry of the space-time are essentially different from those involved in the general theory of relativity and the classical unified field theories, for they depend on the finite value of the Planck constant.

The geometrization of the quantal physics requires a further development of the ordinary theory of space and the introduction of new kinds of geometric objects corresponding to the quantal states. The operators describing the physical quantities are, however, related to simple intuitive geometric objects, such as points, vectors, trihedrals and spheres. The present geometrical approach leads to a fusion of the Heisenberg and Schrödinger pictures of the quantal motion, the states being described by elements of the geometric algebras and the observables too.

The kinematical aspects of the quantum theory have already been extensively discussed along the present lines, but the treatment of the dynamic ones is still rudimentary, especially with respect to the interactions between fields. The relations between the present geometric theory and the general relativity are as yet not known.

I

Geometrization of the Energy-Momentum and the Spin. The relations between the quantum physics and the theory of the space-time properties are not yet well understood. It seems very likely that a deeper analysis of those relations will lead to important modifications of our geometrical ideas and will play a considerable part in the future development of the quantum theory, both in the clarification of its fundamental principles as in the construction of the theory of the elementary particles.

It is rather surprising that the relations between the quantum theory and the analysis of the space-time structure should not have attracted the interest of the physicists and mathematicians, especially after the intense interest raised by the general theory of relativity and its connections with the Riemannian geometry. The only branch of the geometrical theory whose significance for the quantum physics has been satisfactorily appreciated is the theory of the linear representations of the groups of displacements and reflections of the three-dimensional Euclidean space and the four-dimensional Minkowskian space-time. In connection with the representation theory some geometric algebras have been used to a considerable extent, the rôle of the Clifford algebra of the space-time being particularly important.

Recently there has been a renewal of the interest in the discussion of the relations between physics and geometry. The quantization of the gravitational field is being studied, as well as the inclusion of the quantum field theory in the frame of the RIEMANNian geometry. The fundamental results of LEE and YANG on the non conservation of the parity have given a strong incentive to the discussion of the geometrical problems of the quantum field theory.

The existence of a fundamental relation between the quantum theory and geometry is implicit in the DE BROGLIE formula $\lambda = h/p$, which allows to measure the momentum in cm^{-1}. *Thus the basic dynamical variable p is geometrized.* The geometrical meaning of the linear momentum is clearly shown by its association to the differential operators for the infinitesimal spatial translations. In a similar way the orbital angular momentum components are essentially the differential operators for the infinitesimal rotations.

It is well known that the CLIFFORD algebra C_4 is a kind of vector calculus for the metric geometry of a four-dimensional space. *The existence of such an algebra in the relativistic quantum theory of the particles with spin $1/2$ gives another basic connection between the quantum physics and the geometry of the space-time, which is of a simpler kind than that given by the linear and orbital angular momenta.* The anticommutativity of the operators γ_μ expresses the geometrical relation of orthogonality of the basic vectors of the cartesian reference frame of the space-time, which are described by the γ_μ in the CLIFFORD vector calculus.

The above interpretation of the anticommutation rules $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\, g_{\mu\nu} \cdot 1$ suggests that the commutation rules of the quantum theory express always geometrical relations. This idea led us in 1955 to look for a geometrical interpretation of the HEISENBERG commutation rules. We found a very simple interpretation by going over to the affine geometry of the EUCLIDean three-dimensional space [1]: *the p_j are the symbols of the basic contravariant vectors and the q^j those of the basic covariant vectors in an affine vector calculus that gives an extension of the commutative GRASSMANN algebra of the vectors.* In the p, q calculus the contravariant vector of components V^j and the covariant vector of components U_j are described by the symbols $\{V\} = V^j p_j$ and $\{U\} = U_j q^j$. *The geometrical relation between the vectors $\vec{U}$ and $\vec{V}$ is expressed by the commutation rule* $\{U\}\{V\} - \{V\}\{U\} = i\, U_j V^j \cdot 1$. The symbols $\{U\}$ and $\{U'\}$ of any two covariant vectors $\vec{U}$ and $\vec{U}'$ are commutative and those $\{V\}, \{V'\}$ of any two contravariant vectors $\vec{V}$ and $\vec{V}'$ are also commutative. It is clear that this commutativity is related to the impossibility of building an invariant with the U_j, U'_k or with the V^j, V'^k.

In 1956 we extended the above kind of affine vector calculus [2] by the introduction of a new generator $\{P\}$, which we interpreted later [3] as the symbol of the origin of the cartesian coordinates. *This calculus of points and vectors allows to deal not only with the observables of a spinless non relativistic particle but also with its states.* The present algebra gives a synthesis of the HEISENBERG and SCHRÖDINGER representations, the states being represented by the elements of a basic left-ideal. This geometric algebra gives an interesting extension of the separable HILBERT space, which allows to deal with plane waves and also with symbolic functions such as the DIRAC δ and its derivatives of all orders. *Our geometric algebra gives a geometric theory* *of the temperate distributions and more general mathematical entities, which are not* SCHWARTZ *distributions.*

The passage to the affine geometry leads to an extension of the CLIFFORD algebra [1], in which there are symbols for the contravariant and covariant vectors. In the case of a n-dimensional space we have $2n$ generators (I_μ) and (I^μ) corresponding to the basic contravariant and covariant vectors of the cartesian coordinate system, with the commutation rules $[(I_\mu), (I_\nu)]_+ = 0$, $[(I^\mu), (I^\nu)]_+ = 0$, $[(I_\mu), (I^\nu)]_+ = \delta^\nu_\mu \cdot 1$. The CLIFFORD algebra is the sub-algebra of this (I) algebra generated by the elements $\gamma_\mu = (I_\mu) + g_{\mu\nu}(I^\nu)$. *In the case of the space-time, the* (I) *algebra can be represented by* 16×16 *matrices. This seems to indicate the existence of* 4 *types of fermions, which may perhaps be identified with the electron, the neutrino, the* μ *meson and the baryon, the eight kinds of baryons being regarded as different states of the same type of fermion.*

The consideration of the affine geometry of the flat space-time is a characteristic feature of our theory, which requires both the LORENTZ group and the full linear group for its development. *The latter group is likely to play an important part in the analysis of the inertial effects, since the components* $\Gamma^\lambda_{\mu\nu}$ *of the affine connection transform as those of a tensor for arbitrary non-singular linear transformations of the space-time coordinates, and* $\Gamma^\lambda_{\mu\nu} = 0$ *in any cartesian coordinate system of the flat space-time.* The LORENTZ group allows to distinguish the rest-masses, because $g^{\mu\nu} p_\mu p_\nu \psi = m^2 \psi$, m denoting the rest-mass and ψ the wave function of a particle. $g^{\mu\nu} p_\mu p_\nu$ *is the element of our* p, q *vector calculus associated to the tensor* $g^{\mu\nu}$.

The non-relativistic quantum mechanics allows the identification of the operators p_j of the linear momentum with the symbols of the basic contravariant vectors in the above geometric algebra, because it involves the constant $\hbar$ which may be taken as 1 in a convenient system of units. *In the relativistic quantum theory we have the natural system of units in which* $\hbar = 1$ *and* $c = 1$, *so that all the physical quantities have dimensions of powers of a length. The full geometrization of physics requires the two constants* h *and* c, *which are precisely given by the relativistic quantum theory.*

II

Geometrization of the Electric Charge and the Iso-Spin. In 1957 we introduced a special kind of orientation of the points of a manifold [3], by considering them as centres of bundles of incoming or outgoing curvilinear rays. This kind of orientation is closely related to that of a sphere by taking its radii oriented outwards or inwards, thereby it was called spherical orientation. We used in our point-calculus of a manifold three kinds of points: spherically non-oriented points and points with each of the two spherical orientations. The spherical orientation of points is obviously related to the old FARADAY picture of the point-charges as sources or sinks of lines of force. *We had shown* [3] *that there is a calculus of points on a manifold that gives the formalism of a quantized field of spinless neutral particles when applied to a space-like hypersurface of the space-time. The introduction of the spherically oriented points allows to obtain a quantized field of charged spinless particles as a calculus of* *spherically oriented points. By using also spherically non-oriented points we geometrized the theory of the quantized pion field, the sub-algebra of the spherically non-oriented points corresponding to the neutral pion field.*

The orientation of the radius of a sphere is a special case of that of a two-sided surface by the choice of an orientation of the normals. The definition of the normal depends of course on the metric but this kind of orientation is actually independent of the metric, for it amounts to a distinction of the two sides of the surface, i.e. to its outer orientation. We may define the spherical orientation of a volume by the choice of a side of its boundary. *The spherical orientation of a point can be defined through the spherical orientation of a volume containing the point, hence through the outer orientation of a surface.* The outer orientation of p-dimensional manifolds in a n-dimensional space is related to the antisymmetric covariant tensors of the order $n-p$. The spherical orientation of the three-dimensional volumes or points is therefore related to the antisymmetric covariant tensors of the second order of the space-time, because the boundary of the volume is a two-dimensional manifold of the space-time. *These considerations show that it is satisfactory to associate the electric charge to the spherical orientation of points in a three-dimensional space-like hypersurface of the space-time, since the electromagnetic field is described by an antisymmetric covariant tensor of the second order of the space-time.*

The identification of the electric charge with an index of spherical orientation gives a clue for the explanation of the fundamental fact that the charges of all the known charged elementary particles are either e or $-e$. *The charge conjugation is thus related to the reversal of the spherical orientation.*

In our geometric calculus of points for a manifold, the points not endowed with screw-orientation lead to the quantized scalar field. The calculus of the screw-oriented points leads to the quantized pseudo-scalar field. The introduction of points with the two spherical orientations leads to the replacement of the ordinary one-dimensional linear space of the pseudo-scalars by a two-dimensional linear space: the direct sum of the one-dimensional linear spaces corresponding to the two kinds of pseudo-scalars associated to points with opposite spherical orientations. We are thus led to the consideration of the full linear group in two variables. *The gauge group is obviously the rotation group of this two-dimensional linear space.*

The introduction of spherically non-oriented and spherically oriented points leads to a three-dimensional linear space: the direct sum of the one-dimensional linear spaces of the pseudo-scalars associated to the three kinds of points. *Thus we get the space of the iso-spin from our calculus of points.* The iso-spin space does not appear here associated naturally to a three-dimensional orthogonal group, but rather to the full linear group in three variables. The algebras of the iso-spin should therefore be algebras of the affine three-dimensional geometry, rather than algebras of the metric geometry. The introduction of a metric in the iso-spin space should lead to a kind of finer distinction. It is indeed so: *the iso-spin algebra of the pion field is the total matrix algebra of the iso-spin space: the iso-spin algebra of the kaon field is the total matrix algebra of the direct sum of the linear spaces of the scalars and vectors of the iso-spin space; the iso-spin algebra of the baryon field is the* (I) *algebra of the con-*

travariant and covariant vectors of the iso-spin space. This algebra is the three-dimensional analogue of the (I) algebra of the space-time discussed in section I. The introduction of the EUCLIDean metric in the iso-spin space leads to the distinction of the two kinds of kaon fields and the four kinds of baryon fields. The above (I) algebra is iso-morphic to a CLIFFORD algebra of a six-dimensional space, and its irreducible representation space has eight dimensions (spin-space of a six- or seven-dimensional space). A preliminary discussion of the geometrization of the iso-spin was given in reference [3], a more complete discussion will be given in a forthcoming paper.

It is interesting to note that the spherical orientation and the iso-spin space appear related to the topological properties of space, rather than to its metric or affine properties.

The iso-spin seems to be a property of the elementary particles existing both in charged and neutral states: pions, kaons and baryons. Besides the iso-spin, it is necessary to consider another variable: the strangeness or the U of D'ESPAGNAT and PRENTKI [4]. The value of U for each of the known particles is the sum of the values of the index of spherical orientation in the different states of the particle, the iso-spin I defining the number $2I + 1$ of those states. *The present definition of U may also be applied to the muon, the leptons and the photon.*

III

Geometrization of the Quantized Pseudo-Scalar Fields. In reference [3] we discussed different types of geometric calculus corresponding to various kinds of quantized fields: PAULI-WEISSKOPF field, DIRAC field, MAXWELL and PROCA fields. *In all those cases the geometric calculi are those of a three-dimensional manifold, embedded in the four-dimensional space-time. This is related to the fact that each of those fields has quanta with a definite value of the restmass m.* In the relativistic wave mechanics, the D'ALEMBERTian operator is essentially a rest-mass operator, its eigenvalues giving the values of $-m^2$. The choice of a value of m leads to the FOCK-KLEIN-GORDON equation, which is the basic equation of motion. The wave functions of a particle are eigenfunctions belonging to a single eigenvalue of the D'ALEMBERTian and this leads to a three-dimensional geometric calculus. This is shown clearly by the fact that the energy-momentum manifold of a particle is three-dimensional, as a consequence of the fixation of the value of the rest-mass.

The WIGNER theory of the irreducible unitary linear representations of the inhomogeneous LORENTZ group [5] gives an abstract four-dimensional geometrical meaning to the fixation of the values of the spin and the rest-mass of the quanta of a field. *It is intuitive that the choice of a non-null rest-mass amounts to the introduction of a unit of length, in a covariant way. The choice of a value of the spin must correspond to a distinguished part attributed to some special geometric object, which must be of a purely spatial nature, because the spin is related to the spatial rotation group.*

The representations of the LORENTZ group corresponding to integral values of the spin are also representations of the full linear group in four variables. *The geometric objects associated to these representations must therefore be affine objects,* *whereas those associated to the half-integral spins must be objects of the metric geometry, not included in the affine geometry.*

The above discussion indicates that the fields with spin 0 ought to be associated to the simplest three-dimensional affine objects: *the points of space*. The spin 1 fields should be associated to the straight lines or the planes. The simplest three-dimensional EUCLIDean metric objects are the sphere and the rectangular trihedral. *The free rectangular trihedral is actually an object of the branch of geometry associated to the group of rotations and similarities. We may therefore expect the rectangular trihedrals to be associated to spin 1/2 fields with zero rest-mass of the quanta, the FOCK-KLEIN-GORDON equation with zero rest-mass being invariant for space-time similarities. These fields are described by two-component spinors.* We should expect the four-component spinors to be associated to spheres.

We discussed in detail in reference [3] the calculus of points of a manifold associated to a quantized scalar field. From the analytical point of view, this calculus deals with true and symbolic functions of the points of the manifold and with functionals on those functions. This calculus gives a powerful kind of functional analysis and presents also considerable interest from the geometrical point of view, since the introduction of coordinates on a manifold is based on the numerically-valued functions of the points of the manifold. The geometrical analysis of the pseudo-scalar quantized fields given in reference [3] is not satisfactory, for we were not able to identify clearly the nature of the geometric object associated to those fields: *the screw-oriented weighted points*. The relations between the vector fields and a three-dimensional calculus of planes were also analysed in [3]. The interpretation of the calculus of the two-components spinors as a calculus of free rectangular trihedrals was also given in [3] and applied to the analysis of the spin 1/2 fields.

IV

The geometric calculi corresponding to the quantized pseudo-scalar charged and neutral fields are associated to remarkably simple real objects of the three-dimensional EUCLIDean space. Let us consider firstly the pseudo-scalar neutral field. The contravariant antisymmetric tensors of the order p of a n-dimensional affine space describe p-dimensional free hypervolumes endowed with inner orientation, when they are simple p-vectors. The covariant antisymmetric tensors of order p corresponding to simple covariant p-vectors describe $(n-p)$-dimensional free hypervolumes endowed with an outer orientation. The simplest case is that of $p = n$, in which we get the two kinds of pseudo-scalars: *the contravariant ones describe free screw-oriented n-dimensional hypervolumes; the covariant ones describe free screw-oriented weighted points.* We may build a calculus for the free screw-oriented points and n-dimensional hypervolumes analogous to that for the free covariant and contravariant vectors corresponding to the HEISENBERG algebra of the position and momentum operators. We associate to the antisymmetric tensors of order n: $u_{j_1, \ldots, j_n}$ and $v^{j_1, \ldots, j_n}$ elements $\{u\}$ and $\{v\}$ of an algebra with the following commutation rules

$$[\{u\}, \{u'\}] = 0, \qquad [\{v\}, \{v'\}] = 0, \qquad [\{v\}, \{u\}] = u_{1, \ldots, n} v^{1, \ldots, n} \{1\}, \tag{1}$$

$\{1\}$ denoting a unity element. This algebraic structure is obviously of a basic nature, since it is related to the differential calculus of the functions of a single variable, being isomorphic to the x, d/dx algebra.

In the case of a general n-dimensional differentiable manifold, we must consider fields $u_{j_1,\ldots,j_n}(P)$, $v^{j_1,\ldots,j_n}(P)$, for it is no more possible to introduce free tensors. The above algebra of the $\{u\}$ and $\{v\}$ must now be replaced by a more powerful formalism generated by symbols $\langle u\rangle$ and $\langle v\rangle$, $\langle u\rangle$ being associated to the pseudo-scalar field $u(P)$ and v to a linear functional $v[u(P)]$ on the $u(P)$. The commutation rules of the symbols $\langle u\rangle$ and $\langle v\rangle$ are

$$[\langle u\rangle, \langle u'\rangle] = 0, \qquad [\langle v\rangle, \langle v'\rangle] = 0, \qquad [\langle v\rangle, \langle u\rangle] = v[u]\,\langle 1\rangle. \tag{2}$$

Let us assume that our manifold M is endowed with a measure of the hypervolumes. We may associate to $v[u]$ a contravariant pseudo-scalar field $v(P)$, such that $v[u] = \int\limits_M v^{1,\ldots,n}(P)\, u_{1,\ldots,n}(P)\, dP$, for sufficiently general linear functionals $v[u]$ and sufficiently general fields $u(P)$, dP denoting the element of hypervolume. $\langle v\rangle$ may now be regarded as the symbol of the contravariant pseudo-scalar field $v(P)$. The effective construction of the present geometric algebra can be done by means of a suitable restriction of the linear space of the $u(P)$. Let us assume that the allowed $u(P)$ constitute a separable BANACH space endowed with a complete bi-orthogonal set of functions $u_r(P)$ and linear functionals $v_r[u]$, $v_r[u_s] = \delta_{r,s}$. We take $v = \Sigma c_r^+ v_r$, $u(P) = \Sigma c_r u_r(P)$, so that $v[u] = \Sigma c_r^+ c_r$ (c_r^+ is not the complex conjugate c_r^* of c_r). The commutation rules (2) for the $\langle u\rangle$ and $\langle v\rangle$ are satisfied by taking $\langle u\rangle = \Sigma c_r \langle u_r\rangle$, $\langle v\rangle = \Sigma c_r^+\langle v_r\rangle$, the commutation rules for the $\langle u_r\rangle$, $\langle v_s\rangle$ being

$$[\langle u_r\rangle, \langle u_s\rangle] = 0, \qquad [\langle v_r\rangle, \langle v_s\rangle] = 0, \qquad [\langle v_r\rangle, \langle u_s\rangle] = \delta_{r,s}\langle 1\rangle. \tag{3}$$

When the manifold M is endowed with a measure of hypervolumes, we may introduce a bi-orthogonal complete set of functions $u_r(P)$, $v_r(P)$:

$$\int\limits_M v_r(P)\, u_s(P)\, dP = \delta_{r,s}, \qquad \Sigma v_r(P)\, u_r(P') = \delta(P, P'). \tag{4}$$

The symbolic function $\delta(P, P')$ is defined by the condition

$$\int\limits_M F(P)\, \delta(P, P')\, dP = F(P'),$$

$F(P)$ denoting an arbitrary continuous function. The functionals $v_r[u]$ are now defined as follows: $v_r[u] = \int\limits_M v_r(P)\, u(P)\, dP$. We have indeed $v_r[u_s] = \delta_{r,s}$. In the particular case of an EUCLIDean space, we may use only orthogonal cartesian coordinates. Thus it is no more necessary to distinguish the two kinds of pseudoscalar fields $u(P)$ and $v(P)$, which transform in the same way for orthogonal transformation of the cartesian coordinates. Instead of the general bi-orthogonal complete sets $u_r(P)$, $v_r(P)$, we shall use a complete orthonormed set $v_r(P) = u_r^*(P)$. We shall now denote $\langle u_r\rangle$ by $\langle v_r\rangle^+$. *The commutation rules* (3) *have now the form of those for* *the absorption and emission operators of a pseudo-scalar field corresponding to the states $v_r(P)$ of the quantum*

$$[\langle v_r\rangle, \langle v_s\rangle] = 0, \qquad [\langle v_r\rangle^+, \langle v_s\rangle^+] = 0, \qquad [\langle v_r\rangle, \langle v_s\rangle^+] = \delta_{r,s}\langle 1\rangle. \tag{5}$$

Let us introduce now the elements $\langle P\rangle$ and $\langle P\rangle^+$

$$\langle P\rangle = \Sigma\langle v_r\rangle\, u_r(P), \qquad \langle P\rangle^+ = \Sigma\langle u_r\rangle\, v_r(P) \tag{6}$$

which correspond to the absorption and emission operators at the point P

$$[\langle P\rangle, \langle P'\rangle] = 0, \quad [\langle P\rangle^+, \langle P'\rangle^+] = 0, \quad [\langle P\rangle, \langle P'\rangle^+] = \delta(P, P')\langle 1\rangle. \tag{7}$$

$\langle P\rangle$ and $\langle P\rangle^+$ have the transformation properties of a covariant and a contravariant pseudo-scalar field, respectively. *$\langle P'\rangle^+$ is the symbol of the „classical" field $u_{P'}(P) = \delta(P', P)$; $\langle P'\rangle$ is the symbol of the linear functional $v_{P'}[u] = u(P')$. We may interpret $\langle P\rangle$ as the symbol of the screw-oriented point P.* We have

$$\langle u\rangle = \int_M u(P)\langle P\rangle^+ dP, \qquad \langle v\rangle = \int_M v(P)\langle P\rangle dP. \tag{8}$$

The above discussion shows very clearly that the duality of the absorption and emission operators of the quantized pseudo-scalar field reflects the duality of the „classical" contravariant and covariant pseudo-scalar fields, when M is endowed with a measure of hypervolumes. We shall now assume only that M is parametrized by means of coordinates x^j. dP will now denote simply $dx^1, \ldots, dx^n$, instead of an invariant element of hypervolume. The $v_r(P)$ will now be taken as scalar fields. With the present definition of dP, the symbolic function $\delta(P, P')$ characterized by the condition $\int_M F(P)\,\delta(P, P')\, dP = F(P')$ has the transformation properties of $u(P)$. *Instead of the duality of the contravariant and covariant pseudo-scalar fields, we get naturally a duality of scalar and covariant pseudo-scalar fields, when no measure of hypervolumes is introduced.* It follows from (6), with the present transformation properties of the $u_r(P)$ and $v_r(P)$, that $\langle P\rangle$ has still the transformation properties of a covariant pseudo-scalar but $\langle P\rangle^+$ behaves as a scalar. *We may now use the „quantized" pseudo-scalar field $\langle P\rangle$ for the description of the screw-oriented points and the „quantized" scalar field $\langle P\rangle^+$ for the description of the non-oriented points.*

V

We shall now consider the case of a n-dimensional affine space A_n. The x^j denote cartesian coordinates and $d\vec{x} = dx^1, \ldots, dx^n$. We shall write $\langle\vec{x}\rangle = \langle P\rangle$ and $\langle\vec{x}\rangle^+ = \langle P\rangle^+$. Let us introduce the symbols $\langle\vec{k}\rangle$ and $\langle\vec{k}\rangle^+$

$$\langle\vec{k}\rangle = (2\pi)^{-n/2}\int_{-\infty}^{+\infty}\exp(i\,k_j\,x^j)\,\langle\vec{x}\rangle\, d\vec{x}, \quad \langle\vec{k}\rangle^+ = (2\pi)^{-n/2}\int_{-\infty}^{+\infty}\exp(-i\,k_j\,x^j)\,\langle\vec{x}\rangle^+ d\vec{x}; \tag{9}$$

$\langle\vec{k}\rangle$ behaves as a scalar and $\langle\vec{k}\rangle^+$ transforms as a contravariant pseudo-scalar for changes of the cartesian coordinates. *We can take the $\langle\vec{x}\rangle$ and $\langle\vec{k}\rangle^+$ as generators of the* *algebra of the „quantized" pseudo-scalar field.* The commutation rules are

$$[\langle\vec{x}\rangle, \langle\vec{x}'\rangle] = 0, \quad [\langle\vec{k}\rangle^+, \langle\vec{k}'\rangle^+] = 0, \quad [\langle\vec{x}\rangle, \langle\vec{k}\rangle^+] = (2\pi)^{-n/2} \exp(-i\, k_j\, x^j)\, \langle 1\rangle. \quad (10)$$

We may alternatively take the $\langle\vec{x}\rangle^+$ and $\langle\vec{k}\rangle$ as generators. The formalism of the quantized scalar or pseudo-scalar field appears now as a kind of calculus for the contravariant and covariant vectors of A_n. The first and second commutation rules (10) are similar to those of the $\{U\}$, $\{V\}$ algebra of section I, but the third rule is essentially different from the corresponding one of that algebra $[\{U\}, \{V\}] = i\, U_j\, V^j \cdot 1$.

In the case of the Euclidean spaces we may take conveniently $d\,P$ as a measure of non-oriented hypervolumes. $\langle\vec{k}\rangle$ and $\langle\vec{k}\rangle^+$ have now the transformation properties of the covariant and contravariant pseudo-scalars, respectively. *They transform, however, in the same way for the orthogonal substitutions of cartesian coordinates.*

We shall now replace the Euclidean k-space by the hypersurface of equation $k_0 = (\Sigma\, k_j^2 + m^2)^{1/2}$ of a k_0, k_j space endowed with the infinite metric $k_0^2 - \Sigma\, k_j^2$. The element of hypervolume on this hypersurface is $m\, k_0^{-1}\, d\,k_1, \ldots, d\,k_n$. It is now necessary to replace $\langle\vec{k}\rangle$ and $\langle\vec{k}\rangle^+$ by $\langle k\rangle = \sqrt{k_0/m}\,\langle\vec{k}\rangle$ and $\langle k\rangle^+ = \sqrt{k_0/m}\,\langle\vec{k}\rangle^+$, respectively. We have

$$[\langle k\rangle, \langle k'\rangle] = 0, \qquad [\langle k\rangle^+, \langle k'\rangle^+] = 0, \qquad [\langle k\rangle, \langle k'\rangle^+] = \frac{k_0}{m}\, \delta\,(\vec{k} - \vec{k}'). \quad (11)$$

These commutation rules are invariant for the homogeneous linear transformations of the k_μ, $\mu = 0, 1, \ldots, n$, which leave invariant $k_0^2 - \Sigma\, k_j^2$ and do not change the sign of k_0. In the case of $n = 3$, our hypersurface is the relativistic energy momentum manifold of a particle with rest-mass m. The n-dimensional Euclidean space of the x^j will be regarded as a hyperplane $x^0 = \text{const}$ of the pseudo-Euclidean space of dimensionality $n + 1$ with the metric $(x^0)^2 - \Sigma\,(x^j)^2$. Let us introduce the symbols $[x]$, $[x]^+$

$$\left.\begin{aligned} [x] &= (2\pi)^{-n/2}\, m \int_{-\infty}^{+\infty} \exp(-i\, k_\mu\, x^\mu)\, \langle k\rangle\, k_0^{-1} d\,k_1, \ldots, d\,k_n, \\ [x]^+ &= (2\pi)^{-n/2}\, m \int_{-\infty}^{+\infty} \exp(i\, k_\mu\, x^\mu)\, \langle k\rangle^+\, k_0^{-1} d\,k_1, \ldots, d\,k_n. \end{aligned}\right\} \quad (12)$$

The $[x]$ and $[x]^+$ transform as $(n + 1)$-dimensional pseudo-scalars for the homogeneous linear transformations of the x^μ that do not change the sign of x^0 and leave invariant $(x^0)^2 - \Sigma\,(x^j)^2$. They satisfy the Fock-Klein-Gordon equations $\partial_\mu \partial^\mu [x] = -m^2 [x]$ and $\partial_\mu \partial^\mu [x]^+ = -m^2 [x]^+$. *When $n = 3$ the quantized neutral pseudo-scalar field with quanta of rest-mass m is $\langle\psi_x\rangle = (\sqrt{2}\, m)^{-1} ([x] + [x]^+)$, $(\sqrt{2}\, m)^{-1}\, [x]$ and $(\sqrt{2}\, m)^{-1}\, [x]^+$ being the positive and negative frequency parts of $\langle\psi_x\rangle$, respectively.* Indeed we have

$$[[x], [x']] = 0, \qquad [[x]^+, [x']^+] = 0, \qquad [[x], [x']^+] = 2\, i\, m\, \Delta^{(+)}(x - x')\, \langle 1\rangle, \quad (13)$$

$$[\langle\psi_x\rangle, \langle\psi_{x'}\rangle] = i\, \Delta\,(x - x')\, \langle 1\rangle. \quad (14)$$

Thus we have obtained a kind of geometric calculus for the space-time by means of the algebra of the pseudo-scalar fields and screw-oriented points of a three-dimensional manifold.

The $[x]$, $[x]^+$ algebra does not deal with the general pseudo-scalar fields of the MINKOWSKIan space-time, but only with those satisfying the FOCK-KLEIN-GORDON equation for the value m of the rest-mass of the quantum. The $[x]$ do not describe world-points in the same way as the $\langle\vec{x}\rangle$ describe the points of three-dimensional space. It follows from (12) that

$$[x]_{x^0=0} = \int_{-\infty}^{+\infty} G(m\,|\vec{x} - \vec{x}'|)\,\langle\vec{x}'\rangle d\vec{x}'. \tag{15}$$

The function G can be expressed in terms of HANKEL functions and tends to 0 exponentially when $m\,|\vec{x} - \vec{x}'| \gg 1$. Since $\langle\vec{x}'\rangle$ is the symbol of a point of the hyperplane $x^0 = 0$, $[x]_{x^0=0}$ describes a kind of blurred spherical region of radius $r \sim m^{-1}$ centred at the point of position vector $\vec{x}$. The same conclusion is obtained by taking the third commutation rule (13) for two world points whose interval s is space-like. In this case $\Delta^{(+)}$ can be expressed in terms of a HANKEL function of ms, which tends exponentially to 0 when $m\,|s| \gg 1$. *The $[x]$, $[x]^+$ geometric algebra of the MINKOWSKIan space-time gives us a covariant calculus of spatially extended „points", so that we have a kind of granular space-time.*

VI

Quantal Geometry. The aim of the geometrization of physics cannot be the description of the properties of matter by means of the concepts of the EUCLIDean geometry of the threedimensional space and the MINKOWSKIan geometry of the space-time or of the concepts of any kind of geometrical theory chosen a priori. The geometrization of physics involves necessarily a revision of the ideas of space and time aiming at the suppression of the philosophically insatisfactory separation of the spatial and temporal properties of matter from the other properties. We must use the knowledge of the properties of matter given by the quantum physics in order to improve the „materialization" of the space-time of the general theory of relativity.

The wave mechanics shows that the position and momentum variables are interlocked. The spatical location of matter can not be described by giving the coordinates of a set of point-like particles and the state of a system is no more described by the position of a point in its phase-space. The HAMILTONian formulation of the classical mechanics showed already that the momentum of a particle is actually a covariant vector, a kind of geometric object related to wave properties, whereas in the NEWTONian formulation the momentum appeared as a contravariant vector. The classical statistical mechanics showed the importance of the hypervolumes of the phase-space, which are scalars built with contravariant and covariant vectors. The association of the covariant and contravariant vectors in the description of the quantal motion of the particles lies at the core of the wave mechanics: *the momentum operators are $-i\,\partial_j$ in the SCHRÖDINGER wave formalism, when $\hbar$ is taken as* 1. Our identification of the HEISENBERG p, q algebra for a particle with an algebra of the covariant and contravariant vectors of the three-dimensional affine space [1], discussed in section I, constitutes therefore a development of geometrical theory corresponding to the new kind of spatial location introduced by the wave mechanics. We showed in references [1], [2], [3] that this algebra of the contravariant and covariant vectors is closely related to the symplectic geometry of the classical phase-space of a particle, associated to the POINCARÉ and LIOUVILLE integral invariants and the HAMILTONian form of the classical mechanics.

We showed in reference [3] that the SCHRÖDINGER and HEISENBERG representations of the quantum kinematics of a particle lead to a geometric algebra of the EUCLIDean spaces of finite dimensionality, dealing with points and other kinds of geometric objects. *This geometric algebra describes the wave-mechanical type of spatial location by means of linear combinations of the symbols corresponding to the points.* This EUCLIDean metric algebra may also be regarded as an extension of the classical mathematical analysis, of the theory of the separable HILBERT space and of the theory of the temperate distributions, in which continuity and other topological properties play a minor part. The present metric algebra is isomorphic to the affine algebra L_3 discussed in reference [2], which is obtained from the $\{V\}$, $\{U\}$ algebra of section I by the introduction of another generator $\{P\}$ with the multiplication rules $\{P\}^2 = \{P\}$, $\{V\}\{P\} = 0$, $\{P\}\{U\} = 0$. *$\{P\}$ describes a point, the introduction of $\{P\}$ corresponds to the passage from an algebra of the three-dimensional vector space to an algebra of the three-dimensional affine space.* The symbols of the point $P + \vec{V}$ are $\exp(-\{V\})\{P\}\exp(\{V\}) = \{P\}\exp\{(V)\}$. The TAYLOR series of 3 variables play a central rôle in the affine algebra. *They are replaced by the series of HERMITE functions in the metric algebra.*

The metric algebra associated to the non-relativistic wave mechanics gives a new kind of description of the EUCLIDean three-dimensional space, in which the linear combinations of the symbols of the points constitute a linear space of infinite dimensionality. *The properties of the symbols of the points in this algebra are therefore different from those of the ordinary calculus of points.* We showed in reference [3] that the symbol $\{Q\}$ of the point Q describes essentially the linear functional λ_Q such that $\lambda_Q[F] = F(Q)$. *The description of the points by their λ_Q is a fundamental feature of the quantal geometry.* We met it already in section IV, in the calculus of points associated to the quantized pseudo-scalar field. In the present algebra, the elements $S\{Q\}S^{-1}$ obtained from the $\{Q\}$ by an inner automorphism, corresponding to a suitable element S, have the same properties as the $\{Q\}$, even when they do not describe points (the „suitable" S correspond to the unitary operators). *Those allowed automorphisms constitute a much larger group than the group of the EUCLIDean displacements that characterizes the equivalence of the points in the ordinary picture of the EUCLIDean space: the abstract group of the three-dimensional EUCLIDean displacements is replaced by the abstract group of the unitary transformations of the HILBERT space of the $F(Q)$ with $\int_{-\infty}^{+\infty} |F(Q)|^2\, dQ < \infty$.* This means that in the description of space given by the present algebra there are other kinds of geometric objects with rôles similar to that of the points. Such geometric objects are not considered in the usual elementary form of the EUCLIDean geometry, but they are nevertheless EUCLIDean objects. In this connection it is necessary to recall that the concept of geometrical object is not equivalent to that of set of points, even in extremely simple cases. Two essentially different geometric objects may be constituted by the same set of points. The set of the planes and the set of the straight lines contain both all the points of space, but even their dimensionalities are different.

The FOURIER transformation plays a central part in the above algebra of the EUCLIDean geometry. It is associated to the isomorphism of the linear spaces of the contravariant and covariant vectors defined by the EUCLIDean metric, which induces a unitary automorphism $A \to S_F\, A\, S_F^{-1}$ in the algebra. *The elements* $S_F\,\{Q\}\,S_F^{-1}$ *describe the geometric objects corresponding to the points of the „momentum space", whose „position" vectors are covariant vectors of the three-dimensional EUCLIDean space.* This geometrization of the FOURIER transformation is one of the most interesting mathematical aspects of the quantal geometry.

The description of the points by the linear functionals λ_Q shows clearly that there are also geometric objects corresponding to other kinds of linear functionals. λ_Q is a linear functional associated to the DIRAC symbolic function:

$$\lambda_Q[F] = \int_{-\infty}^{+\infty} F(Q')\,\delta(Q', Q)\,dQ'.$$

The geometric objects corresponding to the linear functionals

$$\lambda[F] = \int_{-\infty}^{+\infty} F(Q)\,\psi^*(Q)\,dQ \quad \text{with} \quad \int_{-\infty}^{+\infty} |\psi^*(Q)|^2\,dQ = 1$$

are particularly interesting, because of their relation with the quantal states of a particle described by the wave functions $\psi(Q)$. The symbol of the geometric object corresponding to the linear functional λ is $\{\lambda\} = \int_{-\infty}^{+\infty} \psi^*(Q)\,\{Q\}\,dQ$. *The* $\{\lambda\}$ *are therefore linear combinations of the symbols of the points.*

We shall not discuss here the quantal geometry corresponding to the non-relativistic kinematics of the particles with spin 1/2. In this form of the quantal geometry, the points appear bound to rectangular trihedrals. This follows from the well known fact that the two-component spinors are associated to complex null vectors $\vec{V}_1 + i\,\vec{V}_2$. The real vectors $\vec{V}_1$ and $\vec{V}_2$ are orthogonal and have the same length, because $(\vec{V}_1 + i\,\vec{V}_2)^2 = V_1^2 - V_2^2 + 2\,i\,\vec{V}_1 \cdot \vec{V}_2 = 0$. $\vec{V}_1, \vec{V}_2$ and $\vec{V}_1 \wedge \vec{V}_2$ define a rectangular trihedral associated to the spinor.

VII

The relativistic quantum mechanics requires the introduction of quantized fields. The quantized field formalisms are related to the geometry of the differentiable manifolds, as is clearly shown by the discussion of section IV. We shall now discuss the geometrical interpretation of the algebra corresponding to the quantized electromagnetic field.

We want to build a vector calculus for a three-dimensional differentiable manifold M, not endowed with an affine connection nor a metric. The vectors are now localized objects, so that we need to consider basic contravariant and covariant vectors at each point P of M, $\vec{I}_j(P)$ and $\vec{I}^j(P)$. The comparison with the case of the pseudo-scalar fields shows that a satisfactory analogue of the $\{V\}$, $\{U\}$ algebra of section I is obtained by taking the following commutation rules for the symbols $\langle I_j(P)\rangle$ and $\langle I^j(P)\rangle$ of the basic vectors at the point P

$$[\langle I_j(P)\rangle, \langle I_k(P')\rangle] = 0, \qquad [\langle I^j(P)\rangle, \langle I^k(P')\rangle] = 0, \tag{16}$$

$$[\langle I_j(P)\rangle, \langle I^k(P')\rangle] = \delta_j^k\, \delta(P, P')\, \langle 1\rangle. \tag{17}$$

These commutation rules are obviously related to those of the potentials of a radiation field, with the special choice of the gauge which renders $A_0 = 0$, M being taken as the three-dimensional Euclidean *space. The $\langle I_j(P)\rangle$ correspond to absorption operators and the $\langle I^j(P)\rangle$ to emission operators. The duality of the contravariant and covariant vectors is reflected in that of the emission and absorption operators.*

We shall now assume that our three-dimensional manifold M is embedded into a four-dimensional space-time and we shall consider four-vector fields on our manifold, instead of the three-vector fields. There are now four basic contravariant vectors $I_\mu(P)$ and four basic covariant vectors $I^\mu(P)$, whose symbols satisfy commutation rules similar to (16) and (17), the latin indices being replaced by greek indices taking the values 0, 1, 2, 3. Let us now take M as a three-dimensional Euclidean space $x^0 = \text{const}$ of the pseudo-Euclidean space-time. The potentials of the quantized electromagnetic field can be taken as follows:

$$A_\mu(x) = (2\pi)^{-3/2} \int_{-\infty}^{+\infty} \{\exp(-i\, k_\alpha x^\alpha)\, \langle I_\mu(\vec{k})\rangle + \exp(i\, k_\alpha x^\alpha)\, g_{\mu\nu}\, \langle I^\nu(\vec{k})\rangle\}\, (2 k_0)^{-1/2}\, d\vec{k} \tag{18}$$

with $k_0 = |\vec{k}|$ and

$$\left.\begin{aligned} \langle I_\mu(\vec{k})\rangle &= (2\pi)^{-3/2} \int_{-\infty}^{+\infty} \exp(-i\,\vec{k}\cdot\vec{x})\, \langle I_\mu(P)\rangle\, d\vec{x}, \\ \langle I^\mu(\vec{k})\rangle &= (2\pi)^{-3/2} \int_{-\infty}^{+\infty} \exp(i\,\vec{k}\cdot\vec{x})\, \langle I^\mu(P)\rangle\, d\vec{x}. \end{aligned}\right\} \tag{19}$$

It is interesting to note that by taking $k_0 = \sqrt{\vec{k}^2 + m^2}$ we do not get the correct relativistic commutation rules for a Proca *field with neutral quanta of rest-mass m, since $[A_\mu(x), A_\nu(x')] = i\, g_{\mu\nu}\, \Delta(x{-}x')\, \langle 1\rangle$. The electromagnetic field is therefore the only spin 1 quantized field related in a natural way to the vector calculus on a space-like manifold.*

The present calculus of vectors on a three-dimensional space-like manifold can be extended by the introduction of a new generator Ω, with the following multiplication rules

$$\langle I_\mu(\vec{k})\rangle\, \Omega = 0, \qquad \Omega\, \langle I^\mu(\vec{k})\rangle = 0, \qquad \Omega^2 = \Omega. \tag{20}$$

Ω describes the vacuum state of the electromagnetic field. The introduction of Ω corresponds to the introduction of the new generator $\{P\}$ in the $\{V\}$, $\{U\}$ algebra. The algebra of the $A_\mu(x)$ and Ω deals also with the states of the quantized electromagnetic field. We showed in reference [3] that it leads naturally to the indefinite metric of Gupta. The geometrical meaning of Ω is not yet known.

We can also build directly a geometric algebra for the axial and polar vector fields of a three-dimensional space, related to the operators for the components of the electric and magnetic fields.

VIII

The Spinors and the Sphere. The vectors and tensors are affine geometric objects, whereas the spinors are objects of the metric geometry, since they describe the flat manifolds contained in the null hypercone $g_{\mu\nu} x^\mu x^\nu = 0$ [6]. In the physical interesting cases of the EUCLIDean three-dimensional space and the four-dimensional MINKOWSKIan space-time the situation is particularly simple. Thus, in the case of the ordinary space, we are led to the discussion of the complex null vectors, for the flat manifolds of maximal dimensionality on the null cone are now complex lines. The components of a null vector $\vec{V} = \vec{V}_{\mathrm{re}} + i\,\vec{V}_{\mathrm{im}}$ can be expressed in terms of the components u_1, u_2 of a spinor u

$$V^1 = \frac{1}{2}(u_1^2 - u_2^2), \qquad V^2 = \frac{i}{2}(u_1^2 + u_2^2), \qquad V^3 = -u_1 u_2. \tag{21}$$

The spinors u and $-u$ correspond to the same $\vec{V}$. It is possible to associate to the spinor u the rectangular trihedral whose edges are parallel to the three mutually orthogonal real vectors $\vec{V}_{\mathrm{re}}$, $\vec{V}_{\mathrm{im}}$, $\vec{V}_{\mathrm{re}} \wedge \vec{V}_{\mathrm{im}}$ [7]. *This association depends on the choice of a screw-sense, which is required by the definition of the vector product* $\vec{V}_{\mathrm{re}} \wedge \vec{V}_{\mathrm{im}}$. Since $|\vec{V}_{\mathrm{re}}| = |\vec{V}_{\mathrm{im}}|$, as a consequence of the condition $\vec{V}^2 = 0$, the complex vector $\vec{V}$ gives a free rectangular trihedral with a weight $|\vec{V}_{\mathrm{re}}|$. *The correspondence between the weighted rectangular trihedrals, with a given screw-orientation, and the two component spinors can be rendered one to one by giving a sign to the weight of the trihedral.* The calculus of the two-component spinors is therefore a calculus of weighted screw-oriented rectangular trihedrals with the same screw-sense [3]. *The existence of two opposite screw-senses of the trihedrals accounts for that of the two kinds of two-component spinors u and v.*

The simplest metric object is the vector of length unity, rather than the rectangle trihedral. *A unit vector $\vec{W}$ can be conveniently described by a complex number $w = (W^1 + i\,W^2)/(1 - W^3)$, the W^j denoting orthogonal components.* w is the complex number associated to the point of the RIEMANN sphere of position vector $\vec{W}$. It is well known that any rotation around the centre of the sphere induces a linear fractionary transformation $w' = (\alpha\, w + \beta)/(\gamma\, w + \delta)$ with a unitary unimodular matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ and conversely. *w is the ratio of the two components of a spinor v, for $v_1' = \alpha\, v_1 + \beta\, v_2$ and $v_2' = \gamma\, v_1 + \delta\, v_2$. The reflection with respect to the $x^1 x^3$ plane transforms w into w^*. w^* is the ratio of the two components of a spinor u.*

The sphere is a quadric with two conjugate complex systems of rulings. The equations of the two rulings through the point defined by w are

$$x^1 + i\,x^2 = w(1 - x^3), \qquad w(x^1 - i\,x^2) = 1 + x^3, \tag{I}$$

$$x^1 - i\,x^2 = w^*(1 - x^3), \qquad w^*(x^1 + i\,x^2) = 1 + x^3. \tag{II}$$

It is readily seen that the line defined by the equations (II) is parallel to the null vector $\vec{V}$ defined by (21), with $w^* = u_1/u_2$. The line (I) is of course parallel to the complex conjugate vector $\vec{V}^*$. *The two kinds of spinors v and u are associated to the two systems of rulings of the* RIEMANN *sphere.*

The consideration of the two systems of rulings of the sphere requires obviously the introduction of the real and complex points of the sphere. The two rulings through a complex point of the sphere have the following equations

$$v_1(x^1 - i\,x^2) = v_2(1 + x_3), \qquad v_2(x^1 + i\,x^2) = v_1(1 - x^3), \qquad (\mathrm{I}')$$

$$u_1(x^1 + i\,x^2) = u_2(1 + x_3), \qquad u_2(x^1 - i\,x^2) = u_1(1 - x^3). \qquad (\mathrm{II}')$$

The four-component spinor is actually a pair v, u of two-component spinors of different kinds. It follows from equations (I′) and (II′) that a four-component spinor defines a pair of null vectors parallel to the two rulings through a complex point of the RIEMANN sphere. It is $v = u^*$ in the case of a real point.

Let us take now an arbitrary vector $\vec{W}$, instead of a unit vector. The metric allows to define $|\vec{W}|$ and thus to obtain the complex number

$$w = \frac{(W^1 - i\,W^2)}{(|\vec{W}| - W^3)}.$$

w depends only on the direction and sense of $\vec{W}$, but not on its length. *It is clear that we have a metric calculus of directions of the three-dimensional space given by the complex numbers, which underlies the spinor calculus.*

The complex sphere is invariant for complex rotations around the origin. These rotations constitute a continuous group depending on six real parameters, isomorphic to the proper LORENTZ group. They are obtained by means of the transformations $w' = (\alpha w + \beta)/(\gamma w + \delta)$, with a general unimodular matrix. The complex rotations exchange the rulings of each system separately. This corresponds to the well known fact that the components of a two-component spinor undergo a linear transformation, not only for real spatial rotations, but also for all the proper LORENTZ transformations. On the other hand, a reflection with respect to the $x^1\,x^3$ plane exchanges the two systems of rulings, as shown by equations (I′) and (II′). This corresponds to the exchange of the two kinds of spinors.

We had to consider both real and complex vectors of the three-dimensional space in the above discussion. *It is important to note that the complex vectors on the rulings of the sphere are neither polar nor axial: the real part is polar and the imaginary part axial in the case of the vector* $\vec{V}$ *defined by equations* (21), *which is of the same kind as the complex vector* $\vec{E} + i\,\vec{H}$ *built with the fields of a plane electromagnetic wave.* The vectors $\vec{V}_{\mathrm{re}}$ and $\vec{V}_{\mathrm{re}} \wedge \vec{V}_{\mathrm{im}}$ are therefore polar, whereas $\vec{V}_{\mathrm{im}}$ is axial. The imaginary unity is used here in the same way as the dual unity ε, with $\varepsilon^2 = 0$, in the theory of the motors, in order to combine a polar vector and an axial one into a single complex vector.

The isomorphism between the complex rotation group and the proper LORENTZ group is certainly a most remarkable property of the physical universe. *It could not occur for any dimensionality different from* 3. *The above relation between the complex numbers and the directions of space is another remarkable property of the three-dimensional space. The possibility of describing spatial directions by complex numbers w* tial being perhaps the isomorphism of its complex rotation group with the proper LORENTZ group and the fact that both the rotation and the translation groups depend on the same number of parameters. The former property renders the system of the complex numbers an algebra of the directions, as was explained in section VIII, the latter gives a special rôle to the dual numbers. *The tridimensionality of space gives a distinguished status to the straight line, which is projectively self-dual.* On the other hand, it leads to the remarkable isomorphism between the projective line-geometry and the metric sphere-geometry discovered by SOPHUS LIE. *A truly satisfactory integration of physics and geometry must give a central part to the characteristic properties of the tridimensional EUCLIDean space.*

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