ACADEMIA BRASILEIRA DE CIÊNCIAS

QUANTUM MECHANICS AND GEOMETRY

(PART .III)

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SEPARATA DO VOL. 30 N.º 2 DOS "ANAIS DA ACADEMIA BRASILEIRA DE CIÊNCIAS"

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Quantum Mechanics and Geometry

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VECTOR QUASI-ALGEBRAS OF DIFFERENTIABLE MANIFOLDS

13. *The distribution quasi-algebra discussed in section* 11 *is not the same as the symmetric tensor quasi-algebra defined in the Introduction, although the algebraic formalism be the same in both cases for flat spaces.* It is convenient to use different notations for those two geometric quasi-algebras. We shall denote the symmetric tensor quasi-algebra by L_n^t and the quasi-algebra of the scalar distributions by L_n^d.

In curved spaces, the quasi-algebras L_n^t *and* L_n^d *are different formalisms.* We showed in section 11b that it is possible to define a single L_n^d for a whole riemannian or pseudo-riemannian space. This is obviously no more true for L_n^t, because in a curved space there is a different linear manifold of contravariant vectors at each point of the space. *There is a* L_n^t (M) *at each point M, the quasi-algebras corresponding to different points being isomorphic.* L_n^t (M) *is essentially the* L_n^t *of the flat tangent space at the point M.* The parallel displacement of the vectors and symmetric tensors allows to define isomorphisms between the symmetric tensor quasi-algebras associated to different points. These isomorphisms depend on the path used in the parallel displacement of the tensors, because of the non-integrability of the parallel displacement in a curved space.

It is convenient to use as generators of L_n^t(M) the basic-vector symbols $\{\mathbb{I}_j(M)\}$ and $\{\mathbb{I}^j(M)\}$ of the curvilinear coordinate system, the $\mathbb{I}_j(M)$ being defined by the condition $\mathbf{d}M = \mathbb{I}_j(x)\, dx^j$, and an element $\{P(M)\}$. The commutation rules are

$$\left[\{\mathbb{I}_j(M)\},\{\mathbb{I}^k(M)\}\right] = \delta_j^k 1_{L(M)}, \left[\{\mathbb{I}_j(M)\},\{\mathbb{I}_k(M)\}\right] = \left[\{\mathbb{I}^j(M)\},\{\mathbb{I}^k(M)\}\right] = 0 \quad (1)$$

the $\{\mathbb{I}(M)\}$ and $\{P(M)\}$ being related by the equations

$$\{\mathbb{I}_j(M)\}\{P(M)\} = 0, \ \{P(M)\}\{\mathbb{I}^j(M)\} = 0, \ \{P(M)\}^2 = \{P(M)\} \quad (2)$$

It is not necessary to introduce a riemannian or pseudo-riemannian metric in the theory of the $L_n^t(M)$. When the space is endowed with a metric $g_{jk}(x)$, we may introduce the $q^j(M)$ and $p_j(M)$: $\sqrt{2}\, q^j(M) = \{I^j(M)\} + g^{jk}(x)\{I_k(M)\}$, $\sqrt{2}\, p_j(M) = \{I_j(M)\} - g_{jk}(x)\{I^k(M)\}$. It is also possible to build the $L_n^t(M)$ by using as generators the $p_j(M)$ and $\{P(M)\}$, as we built L_n in section 10.

In a curved space there is also a different G_n *at each point.* The $G_n(M)$ of different points are all isomorphic. $G_n(M)$ contains the sub-algebra $C_n(M)$ generated by the $\gamma_j(M) = (I_j(M)) + g_{jk}(x)\,(I^j(M))$, the $(I(M))$ being the symbols of the basic vectors $I(M)$. $G_n(M)$ is essentially the G_n of the flat tangent space at the point M. The commutation rules of $G_n(M)$ are

$$\left[(I_j(M)),(I^k(M))\right]_+ = \delta_j^k\, 1_{G(M)},\ \left[(I_j(M)),(I_k(M))\right]_+ = \left[(I^j(M)),(I^k(M)\right]_+ = 0 \quad (3)$$

so that $\left[\gamma_j(M),\gamma_k(M)\right]_+ = 2\, g_{jk}(x)\, 1_{G(M)}$, with $\gamma_j(M) = (I_j(M)) + g_{jk}(x)\,(I^k(M))$.

The above definitions of $G_n(M)$ and $L_n^t(M)$ can be applied to any n-dimensional differentiable manifold. The introduction of an affine connection Γ^j_{hk} allows to define an isomorphism between $G_n(M+dM)$ and $G_n(M)$, in which

$$(I_j(M+dM)) = (I_j(M) - \Gamma^h_{jk}\, dx^k\,(I_h(M)),\ (I^j(M+dM)) \longrightarrow (I^j(M)) + \Gamma^j_{hk}\, dx^k\,(I^h(M)) \quad (4)$$

and between $L_n^t(M+dM)$ and $L_n^t(M)$ too. In this latter case, we have relations similar to (4) between the $\{I(M+dM)\}$ and $\{I(M)\}$ also and

$$\{P(M+dM)\} \longrightarrow \{P(M)\} \quad (5)$$

These isomorphisms may be conveniently called infinitesimal parallel-displacements of $G_n(M)$ and $L_n^t(M)$. On integrating the infinitesimal parallel displacements along a path C from M to M′, we get isomorphisms between $G_n(M)$ and $G_n(M')$, $L_n^t(M)$ and $L_n^t(M')$, which in general depend on the path C. In the case of euclidean and pseudo-euclidean spaces, those parallel-displacement isomorphisms do not depend on the path. *This explains why we may consider a single* G_n *and a single* L_n^t *for the whole flat space. It is nevertheless true that the set of the* $G_n(M)$ *associated to the different points of the flat space is a far richer structure than the single* G_n*, the same happening with the set of all the* $L_n^t(M)$ *and the single* L_n^t *of the whole space.*

In order that the parallel-displacement isomorphisms transform the $\gamma_j(M)$ as the $(I_j(M))$, and the $p_j(M)$ as the $\{I_j(M)\}$, it is necessary and sufficient that the metric tensor g_{jk} be invariant for parallel displacement. Thus we get the well known relations between the components of the affine connection Γ^j_{hk} and the first order derivatives of the components of the metric tensor, which are ordinarily obtained by the condition that the length of a vector be invariant for parallel displacement: $\frac{\partial}{\partial x^i} g_{jk} + g_{jh}\,\Gamma^h_{ki} + g_{hk}\,\Gamma^h_{ji} = 0$.

13a. The association of a $G_n(M)$ and a $L_n^t(M)$ to each point M of a n-dimensional manifold does not yet lead to satisfactory geometric calculi: *it is clearly necessary to introduce sums and products of algebraic elements associated to different points. We are thus led to fuse all the $G_n(M)$ into a broader algebraic structure $\mathcal{G}_n$ and all the $L_n^t(M)$ into another structure $\mathcal{L}_n$.*

We shall assume that our differentiable manifold S is endowed with a measure of hypervolumes, in order to be able to define the integral $\int_S f(M)\, d\Omega$. When the manifold is a riemannian space we have simply $d\Omega = \sqrt{g}\, dx^1 \ldots . dx^n$. We shall introduce a symbolic function $\delta(M,M')$ such that $\int_S f(M')\, \delta(M,M')\, d\Omega = f(M)$ for the continuous functions. $\delta(M,M')$ may be considered as the formal series $\sum_r \psi_r(M)\, \psi_r^*(M')$ built with the functions ψ_r of a complete orthonormal system, $\int_S \psi_r(M)\, \psi_{r'}^*(M)\, d\Omega = \delta_{r,r'}$. We shall take $\mathcal{G}_n$ as the quasi-algebra generated by the elements $\langle \mathbf{I}_j(M)\rangle_+$, $\langle \mathbf{I}^j(M)\rangle_+$, $\langle P\rangle_+$ and $\mathcal{L}_n$ as the quasi-algebra generated by the elements $\langle \mathbf{I}_j(M)\rangle_-$, $\langle \mathbf{I}^j(M)\rangle_-$, $\langle P\rangle_-$, the multiplication rules being

$$[\langle \mathbf{I}_j(M)\rangle_\pm, \langle \mathbf{I}^k(M')\rangle_\pm]_\pm = \delta_j^k\, \delta(M,M')\, \langle 1\rangle_\pm,\ [\langle \mathbf{I}_j(M)\rangle_\pm, \langle \mathbf{I}_k(M')\rangle_\pm]_\pm = $$
$$= [\langle \mathbf{I}^j(M)\rangle_\pm, \langle \mathbf{I}^k(M')\rangle_\pm]_\pm = 0 \qquad (1)$$

$$\langle \mathbf{I}_j(M)\rangle_\pm \langle P\rangle_\pm = 0,\ \langle P\rangle_\pm \langle \mathbf{I}^j(M)\rangle_\pm = 0,\ \langle P\rangle_\pm^2 = \langle P\rangle_\pm \qquad (2)$$

the $\langle 1\rangle_\pm$ denoting unity elements. *$\mathcal{G}_n$ and $\mathcal{L}_n$ are obviously second quantization quasi-algebras for a vector field.*

In order to understand clearly the nature of $\mathcal{G}_n$ and $\mathcal{L}_n$, let us divide S into parts S_r of equal volume ω and associate a G_n and a L_n to each of the S_r, the generators $(\mathbf{I}(S_r))$ and $(I(S_{r'}))$ being anticommutable and the $\{\mathbf{I}(S_r)\}$, $\{\mathbf{I}(S_{r'})\}$ commutable for $r \neq r'$ and $[\{P(S_r)\}, \{P(S_{r'})\}] = 0$. The quasi-algebras $\mathcal{G}_n$ and $\mathcal{L}_n$ may be considered as the limiting cases of the above ones when the hypervolume $\omega \longrightarrow 0$, with $(\mathbf{I}(S_r))/\sqrt{\omega} \longrightarrow \langle \mathbf{I}(M)\rangle_+$, $\{\mathbf{I}(S_r)\}/\sqrt{\omega} \longrightarrow \langle \mathbf{I}(M)\rangle_-$, $\prod_r (P(S_r)) \longrightarrow \langle P\rangle_+$, $\prod_r \{P(S_r)\} \longrightarrow \langle P\rangle_-$ as the S_r tend to points M.

Let us introduce now two dual bases of contravariant and covariant vector fields $\mathbf{V}_{\alpha,r}(M)$, $\mathbf{U}_{\alpha,r}(M)$, $(\alpha = 1, \ldots, n)$

$$\sum_{\alpha,r} V_{\alpha,r}^j(M)\, U_{k;\alpha,r}(M') = \delta_k^j\, \delta(M,M')\ ;\ \int_S \sum_j V_{\alpha,r}^j(M)\, U_{j;\alpha',r'}(M)\, d\Omega = \delta_{\alpha,\alpha'}\, \delta_{r,r'}$$

We can take (3)

$$\langle \mathbf{I}_j(M)\rangle_\pm = \sum_{\alpha,r} \langle a_{\alpha,r}\rangle_\pm\, U_{j;\alpha,r}(M),\ \langle \mathbf{I}^k(M)\rangle_\pm = \sum_{\alpha,r} \langle a_{\alpha,r}\rangle_\pm^\dagger\, V_{\alpha,r}^k(M) \qquad (4)$$

with

$$[\langle a_{\alpha,r}\rangle_\pm, \langle a_{\alpha',r'}\rangle_\pm^\dagger]_\pm = \delta_{\alpha,\alpha'}\, \delta_{r,r'} \langle 1\rangle_\pm,\ [\langle a_{\alpha,r}\rangle_\pm, \langle a_{\alpha',r'}\rangle_\pm]_\pm =$$
$$= [\langle a_{\alpha,r}\rangle_\pm^\dagger, \langle a_{\alpha',r'}\rangle_\pm^\dagger]_\pm = 0 \qquad (5)$$

$$\langle a_{\alpha,r}\rangle_{\pm}\,\langle P\rangle_{\pm} = 0\,,\ \langle P\rangle_{\pm}\,\langle a_{\alpha,r}\rangle^{\dagger}_{\pm} = 0 \tag{6}$$

The vector fields $\mathbf{V}(M) = \sum_{\alpha,r} c^{*}_{\alpha,r}\,\mathbf{V}_{\alpha,r}(M)$ and $\mathbf{U}(M) = \sum_{\alpha,r} b_{\alpha,r}\,\mathbf{U}_{\alpha,r}(M)$ will be associated to the elements $\langle \mathbf{V}\rangle_{\pm}$ and $\langle \mathbf{U}\rangle_{\pm}$

$$\langle \mathbf{V}\rangle_{\pm} = \int_S \sum_j V^j(M)\,\langle \mathbf{I}_j(M)\rangle_{\pm}\, d\Omega = \sum_{\alpha,r} c^{*}_{\alpha,r}\,\langle a_{\alpha,r}\rangle_{\pm} \tag{7}$$

$$\langle \mathbf{U}\rangle_{\pm} = \int_S \sum_j U_j(M)\,\langle \mathbf{I}^j(M)\rangle_{\pm}\, d\Omega = \sum_{\alpha,r} b_{\alpha,r}\,\langle a_{\alpha,r}\rangle^{\dagger}_{\pm} \tag{8}$$

which have the commutation rules

$$\left[\langle \mathbf{V}\rangle_{\pm}\,,\ \langle \mathbf{U}\rangle_{\pm}\right]_{\pm} = \int_S \langle \mathbf{V}(M),\ \mathbf{U}(M)\rangle\, d\Omega\ \langle 1\rangle_{\pm} \tag{9}$$

$$\left[\langle \mathbf{V}_I\rangle_{\pm}\,,\ \langle \mathbf{V}_{II}\rangle_{\pm}\right]_{\pm} = \left[\langle \mathbf{U}_I\rangle_{\pm}\,,\ \langle \mathbf{U}_{II}\rangle_{\pm}\right]_{\pm} = 0 \tag{10}$$

The equations (1) are a particular case of (9)–(10). The $\langle \mathbf{I}_k(M)\rangle_{\pm}$ and $\langle \mathbf{I}^k(M)\rangle_{\pm}$ correspond to the fields $V^j(M') = \delta^j_k\ \delta(M,M')$ and $U_j(M') = \delta^k_j\ \delta(M,M')$.

The second quantization quasi-algebras of the vector-fields are not strictly analogous to G_n and L_n. In order to get the exact analogues we must replace (9) by

$$\left[\langle \mathbf{V}\rangle_{\pm}\,,\ \langle \mathbf{U}\rangle_{\pm}\right]_{\pm} = \text{Average}\ \langle \mathbf{V}(M),\ \mathbf{U}(M)\rangle\ [1]_{\pm} \tag{11}$$

$[1]_{\pm}$ denoting the unity elements of the present quasi-algebras, the commutation rules (10) remaining unchanged. In a flat space, we have fields of constant vectors, which correspond to the free vectors. For such fields the commutation rule (11) becomes simply $\left[\langle \mathbf{V}\rangle_{\pm}, \langle \mathbf{U}\rangle_{\pm}\right]_{\pm} = \langle \mathbf{V}, \mathbf{U}\rangle\,[1]_{\pm}$ and we get the commutation rules of G_n and L_n.

Even in the case of the flat spaces, the consideration of the vector-fields, instead of the free vectors, leads to a more satisfactory theory, the free vectors corresponding to the uniform fields. *There is another important geometric reason for the introduction of the second quantization formalisms: the free contravariant vectors describe only translations; the general deformations are described by non-uniform vector-fields and lead to the quasi-algebras of the vector-fields.* It is interesting to notice that the analysis of the motions of classical continuous media requires the use of the vector-fields and leads therefore to the consideration of the quasi-algebras of those fields, i.e. to the second quantization formalisms for vector-fields. *This is implicitly done when the functional analysis is used.*

13b. The covariant vectors correspond to the linear functionals on the contravariant vectors, the application of the linear functional associated to $\mathbf{U}$ to a vector $\mathbf{V}$ giving the number $\langle \mathbf{V}, \mathbf{U}\rangle$. In the present theory we are dealing with vector fields $\mathbf{V}(M)$ and it is more convenient to consider linear functionals $\mathcal{U}$ on the elements of a linear space of vector-fields, than fields of covariant vectors. We shall denote by $\mathcal{U}[\mathbf{V}]$ the value of the functional $\mathcal{U}$ applied to the vector-field $\mathbf{V}(M)$.

The formalism of section 13a can be extended by associating elements $\langle \mathcal{U} \rangle_{\pm}$ to the linear functionals $\mathcal{U}[\mathbf{V}]$. The commutation rules (13a–9) and (13a–10) will be replaced by the following ones

$$[\langle \mathbf{V} \rangle_{\pm}, \langle \mathcal{U} \rangle_{\pm}]_{\pm} = \mathcal{U}[\mathbf{V}] \langle 1 \rangle_{\pm}, [\langle \mathbf{V}_{I} \rangle_{\pm}, \langle \mathbf{V}_{II} \rangle_{\pm}]_{\pm} = [\langle \mathcal{U}_{I} \rangle_{\pm}, \langle \mathcal{U}_{II} \rangle_{\pm}]_{\pm} = 0 \quad (1)$$

These commutation rules do not involve any measure of hyper-volumes on S. This indicates the possibility of having vector-field quasi-algebras on differentiable manifolds not endowed with such a measure. The effective construction of those algebras can be done by methods similar to those for scalar fields that will be discussed in section 14a.

In a riemannian space there is a functional $\mathcal{U}_{V}$ associated to the field $\mathbf{V}(M)$

$$\mathcal{U}_{V}[\mathbf{V}'] = \int_{S} g_{jk}(M)\, V'^{j}(M)\, (V^{k}(M))^{*} \sqrt{g}\, dx^{1} \ldots dx^{n} (\mathbf{V}', \mathbf{V})_{g} \quad (2)$$

When only real vector-fields are of interest, we may consider a quasi-algebra corresponding to the Clifford algebra, in which the vector fields $\mathbf{V}(M)$ are associated to elements $[\mathbf{V}]$ with the commutation rule

$$[[\mathbf{V}_{I}], [\mathbf{V}_{II}]]_{+} = 2(\mathbf{V}_{I}, \mathbf{V}_{II})_{g}\, [1] \quad (3)$$

$[1]$ denoting an unity element. We can build such a quasi-algebra by taking

$$[\mathbf{V}] = \langle \mathbf{V} \rangle_{+} + \langle \mathcal{U}_{V} \rangle_{+} \quad (4)$$

QUASI-ALGEBRAS OF SCALAR FIELDS AND WEIGHTED POINTS

14. The consideration of scalar and pseudo-scalar fields on a manifold is even more fundamental for geometry than that of the vector fields. The introduction of coordinates requires essentially the n scalar fields $x^{j}(M)$. The measure of the hyper-volumes is associated to a pseudo-scalar field $\theta(M)$: $d\Omega = \theta(M)\, dx^{1} \ldots dx^{n}$. We shall now consider quasi-algebras associated to scalar fields.

Let the $\psi_{r}(M)$ be the functions of a complete orthonormal set on a manifold S endowed with a measure of hyper-volumes: $\int_{S} \psi_{r}(M) \psi^{*}_{r'}(M)\, d\Omega = \delta_{r,r'}$, $\sum_{r} \psi_{r}(M) \psi_{r}(M') = \delta(M, M')$. We shall denote by λ_{r} the linear functional $\lambda_{r}[\Phi] = \int_{S} \Phi(M) \psi^{*}_{r}(M)\, d\Omega$. We shall associate to the ψ_{r} and λ_{r} elements $\langle \psi_{r} \rangle_{\pm}$ and $\langle \lambda_{r} \rangle_{\pm}$ with the commutation rules

$$[\langle \psi_{r} \rangle_{\pm}, \langle \psi_{r'} \rangle_{\pm}]_{\pm} = [\langle \lambda_{r} \rangle_{\pm}, \langle \lambda_{r'} \rangle_{\pm}]_{\pm} = 0, [\langle \lambda_{r} \rangle_{\pm}, \langle \psi_{r'} \rangle_{\pm}]_{\pm} = \delta_{r,r'} \langle 1 \rangle_{\pm} \quad (1)$$

and the idempotent elements $\langle P \rangle_{\pm}$

$$\langle \lambda_{r} \rangle_{\pm} \langle P \rangle_{\pm} = 0, \; \langle P \rangle_{\pm} \langle \psi_{r} \rangle_{\pm} = 0, \; \langle P \rangle^{2}_{\pm} = \langle P \rangle_{\pm} \quad (2)$$

The $\langle 1 \rangle_{\pm}$ are unity elements.

Let $\lambda = \sum_r b_r \lambda_r$ denote a linear functional and $\psi(M) = \sum_r c_r^* \psi_r(M)$ a function or formal series. We shall introduce the elements $\langle\lambda\rangle_\pm$, $\langle\psi\rangle_\pm$

$$\langle\lambda\rangle_\pm = \sum_r b_r \langle\lambda_r\rangle_\pm , \quad \langle\psi\rangle_\pm = \sum_r c_r^* \langle\psi_r\rangle_\pm \tag{3}$$

Equations (1) and (2) can be generalized as follows

$$\left[\langle\psi\rangle_\pm , \langle\psi'\rangle\right]_\pm = \left[\langle\lambda\rangle_\pm , \langle\lambda'\rangle_\pm\right]_\pm = 0 , \left[\langle\lambda\rangle_\pm , \langle\psi\rangle_\pm\right]_\pm = \lambda\left[\psi\right]\langle 1\rangle_\pm \tag{4}$$

$$\langle\lambda\rangle_\pm \langle P\rangle_\pm = 0 , \langle P\rangle_\pm \langle\psi\rangle_\pm = 0 , \langle P\rangle_\pm^2 = \langle P\rangle_\pm \tag{5}$$

Let us denote by $\psi_\lambda = \sum_r b_r^* \psi_r$ a function or formal series associated to $\lambda = \sum_r b_r \lambda_r$. We may write $\lambda\left[\psi\right] = \int_S \psi(M)\psi_\lambda^*(M)\, d\Omega$ and

$$\langle\lambda\rangle_\pm = \int_S \langle\lambda_M\rangle_\pm \psi_\lambda^*(M)\, d\Omega , \quad \langle\psi\rangle_\pm = \int_S \psi(M) \langle\psi_M\rangle_\pm\, d\Omega \tag{6}$$

$$\lambda_M = \sum_r \psi_r(M)\, \lambda_r , \lambda_M\left[\psi\right] = \psi(M), \psi_M(M') = \psi_{\lambda_M}(M') = \delta(M, M') \tag{7}$$

The above quasi-algebras of scalar fields may be considered as calculi of weighted points on the manifold S, the symbols of the points M being the $\langle\lambda_M\rangle_\pm$. This description of the points by linear functionals on the $\psi(M)$ is closely related to that of section 12, which is thus extended to the differentiable manifolds endowed with a measure of hyper-volumes. *The n-dimensional manifold is now embedded into an infinite dimensional linear space, isomorphic to that of the* $\lambda = \sum_r b_r \lambda_r$. Each function $\psi(M)$ defines a linear functional $\psi\left[\lambda\right]$ on some of the λ defined by the equation $\psi\left[\lambda\right] = \lambda\left[\psi\right]$, the same holding also for the formal series $\psi(M)$. The replacement of $\psi(M)$ by $\psi\left[\lambda\right]$ extends the domain of definition of $\psi(M)$ into the above infinite-dimensional linear space, since $\psi(M) = \psi\left[\lambda_M\right]$. *The direct sum of the linear spaces of the* ψ *and* λ *gives a kind of extended phase-space of the manifold.*

We shall associate to the linear operator α on the ψ and λ

$$\alpha\psi = \sum_{r,s} \psi_r \alpha_{rs} c_s^* , \quad \lambda\alpha = \sum_{r,s} b_r \alpha_{rs} \lambda_s \tag{8}$$

the elements $\langle\alpha\rangle_\pm = \sum_{r,s} \langle\psi_r\rangle_\pm \alpha_{rs} \langle\lambda_s\rangle_\pm$. We have

$$\left[\langle\lambda\rangle_\pm , \langle\alpha\rangle_\pm\right] = \langle\lambda\alpha\rangle_\pm , \left[\langle\alpha\rangle_\pm , \langle\psi\rangle_\pm\right] = \langle\alpha\psi\rangle_\pm \tag{9}$$

In particular, the operator unity on the ψ corresponds to the elements $\langle N\rangle_\pm = \sum_r \langle N_r\rangle_\pm$

$$\langle N_r\rangle_\pm = \langle\psi_r\rangle_\pm \langle\lambda_r\rangle_\pm , \langle N\rangle_\pm = \int_S \langle\rho(M)\rangle_\pm\, d\Omega , \langle\rho(M)\rangle_\pm = \langle\psi_{\lambda_M}\rangle_\pm \langle\lambda_M\rangle_\pm \tag{10}$$

The $\langle N_1\rangle_\pm$ *are idempotent elements; the* $\langle N_r\rangle_\pm$ *are all commutable.*

Any function $\Phi(M)$ defines a linear operator $\alpha_\Phi : \alpha_\Phi\psi = \Phi\psi$. We have $\langle\alpha_\Phi\rangle_\pm = \int_S \Phi(M) \langle\rho(M)\rangle_\pm\, d\Omega$. We may associate elements $\langle\alpha_\Phi\rangle_\pm$ to formal series $\Phi(M)$ too, by means of the preceding formula. In particular $\langle \rho(M') \rangle_\pm$ are the elements $\langle \alpha_\Phi \rangle_\pm$ associated to $\delta(M,M')$. The elements $\langle N_r \rangle_\pm$ and $\langle \rho(M) \rangle_\pm$ are special cases of a more general kind of elements $\langle N_\psi \rangle_\pm$

$$\langle N_\psi \rangle_\pm = \langle \psi \rangle_\pm \langle \lambda_\psi \rangle_\pm , \quad \lambda_\psi [\Phi] = \int_S \Phi(M)\, \psi^*(M)\, d\Omega \tag{11}$$

14a. We have been considering a manifold endowed with a measure of hyper-volumes. In the case of pseudo-scalar fields, it is necessary to assume that the manifold is differentiable, because of the transformation law of the pseudo-scalars for changes of curvilinear coordinates. *In the case of scalar fields, it is not necessary to assume that the manifold S be differentiable.*

The commutation rules (14–4), (14–5) do not involve any measure of hyper-volumes. *This shows that such a measure is not necessary for the construction of our quasi-algebras of scalar fields, which may therefore be considered as related to the topology of manifolds.* The treatment of section 14 is convenient when there is a special reason to distinguish the Hilbert space of functions $\psi(M)$ with the inner product. $(\psi_I , \psi_{II}) = \int_S \psi_I(M)\, \psi^*_{II}(M)\, d\Omega$.

In order to see how the measure of hyper-volumes can be eliminated, let us consider a separable Banach space of functions $\psi(M)$ admitting a complete bi-orthogonal set of functions ψ_r and linear functionals λ_r: $\lambda_r[\psi_s] = \delta_{r,s}$. We associate to the ψ_r, λ_r elements $\langle \psi_r \rangle_\pm$, $\langle \lambda_r \rangle_\pm$ satisfying equations (14–1) and (14–2). The equations (14–4) and (14–5) follow immediately, with $\psi = \sum_r c^*_r \psi_r$ and $\lambda = \sum_r b_r \lambda_r$. We still have $\lambda_M = \sum_r \psi_r(M)\, \lambda_r$, $\langle \alpha \rangle_\pm = \sum_{r,s} \langle \psi_r \rangle_\pm \alpha_{rs} \langle \lambda_s \rangle_\pm$, $\langle N \rangle_\pm = \sum_r \langle \psi_r \rangle_\pm \langle \lambda_r \rangle_\pm$.

It follows from the above considerations that the embedding of a manifold S into an infinite-dimensional linear space does not require the introduction of a measure of hyper-volumes and can also be done in the case of a non differentiable S.

The present method of introducing quasi-algebras can be extended to sets S which are not manifolds. The M denote now the elements of the set and the $\psi(M)$ numerically-valued functions of the elements of the set. The quasi-algebras will depend both on the set S and the choice of the linear functional space constituted by the ψ. *In the case of a set with n elements, the linear space constituted by all the $\psi(M)$ is a n-dimensional linear space and the scalar-field quasi-algebras of the set, corresponding to the linear space of all the $\psi(M)$, are isomorphic to G_n and L_n.*

In the case of a manifold endowed with a measure of hyper-volumes, it is often convenient to use complete bi-orthogonal sets of functions ψ_r, Φ_r

$$\sum_r \psi_r(M)\, \Phi^*_r(M') = \delta(M, M'), \quad \int_S \psi_r(M)\, \Phi^*_s(M)\, d\Omega = \delta_{r,s} \tag{1}$$

We can now take

$$\lambda_r\left[\psi\right] = \int_S \psi(M)\, \Phi_r^*(M)\, d\Omega \qquad (2)$$

and apply the above treatment of the bi-orthogonal sets of functions and functionals.

THE COORDINATE QUASI-ALGEBRAS OF A MANIFOLD

15. The introduction of coordinates x^j in a manifold S leads to the consideration of sets of n point-functions $x^j(M)$, n denoting the dimensionality of the manifold. *This treatment of the coordinates in a n dimensional manifold* S *indicates the convenience of discussing more powerful quasi-algebras than those of the scalar-fields introduced in section 14: the quasi-algebras of a n-uple of scalar fields* $\psi^j(M)$. We shall associate to the $\psi^j(M)$ elements $\langle\psi^j\rangle_\pm$ and introduce other generators $\langle\lambda^j\rangle_\pm$ and $\langle P_n\rangle_\pm$, the $\langle\lambda^j\rangle_\pm$ corresponding to linear functionals $\lambda^j\left[\psi\right]$

$$\left[\langle\psi^j\rangle_\pm, \langle\psi'^k\rangle_\pm\right]_\pm = \left[\langle\lambda^j\rangle_\pm, \langle\lambda'^k\rangle_\pm\right]_\pm = 0, \left[\langle\lambda^j\rangle_\pm, \langle\psi^k\rangle_\pm\right] = \delta_{j,k}\, \lambda^j\left[\psi^k\right]\langle 1_n\rangle_\pm \qquad (1)$$

$$\langle\lambda^j\rangle_\pm \langle P_n\rangle_\pm = 0, \; \langle P_n\rangle_\pm \langle\psi^j\rangle_\pm = 0, \; \langle P_n\rangle_\pm^2 = \langle P_n\rangle_\pm \qquad (2)$$

The $\langle 1_n\rangle_\pm$ are unity elements of the two quasi-algebras. *The indices of the elements* $\langle\psi^j\rangle_\pm, \langle\lambda^j\rangle_\pm$ *run from* 1 *to n, but they are not tensor indices.*

We were led to the consideration of n-uples of scalar fields by the coordinatization of the manifold. A set of n independent real functions $\psi^j(M)$ may of course be associated to a system of coordinates $x^j(M) = \psi^j(M)$. The above quasi-algebras of the manifold deal with n-uples of functions, or even n-uples of formal series, not necessarily independent. We shall allow the functions $\psi^j(M)$ to be complex-valued. *We may say that the fundamental reason for the introduction of the present quasi-algebras is the necessity of using functionals of n independent functions* $\psi^j(M)$ *for a deeper geometrical analysis of the manifold, at least in the case of the quasi-algebra of the* $\langle\psi^j\rangle_-$ *and* $\langle\lambda^j\rangle_-$. It is well known from the quantum theory of fields that this quasi-algebra is related to the functionals of n functions.

In the case of a manifold endowed with a measure of hyper-volumes, we can take a complete orthonormal set of functions $\psi_r(M)$ and the associated linear functionals λ_r, defined in section 14. The n-uple of functions $\psi^j(M) = \sum_r c_r^{j*}\, \psi_r(M)$ corresponds to the n-uple of elements $\langle\psi^j\rangle_\pm = \sum_r c_r^{j*} \langle\psi_r^{(j)}\rangle_\pm$. *The 2n elements* $\langle\psi_r^{(j)}\rangle_\pm$, $\langle\lambda_r^{(j)}\rangle_\pm$, *corresponding to the same function* ψ_r *taken n times, have the commutation rules of the generators* $(\mathbf{I}^j)$, $(\mathbf{I}_j)$ *of* G_n, *in the case of the anticommutation rules, and those of the generators* $\{\mathbf{I}^j\}, \{\mathbf{I}_j\}$ of L_n, *in the case of the commutation rules with sign minus.*

At a point M of the manifold S, the n numbers $\psi^j(M)$ corresponding to a n-uple of functions may be regarded as the components of a contravariant vector $\vec{\psi}(M)$ of an euclidean space $\Sigma(M)$. *$\Sigma(M)$ will be called the local coordinate-space of the manifold at the point M. It is convenient to regard the $\Sigma(M)$ as the tangent spaces at the points M′ of a differentiable n-dimensional manifold S′, which will be called the global coordinate space of S.* It is possible to introduce an integrable parallel-displacement of vectors in S′ and to render it into a flat space, even when S is not such a space. This flattening of S′ allows to consider an orthogonal cartesian basic set of vectors $\mathcal{J}_j$ and to write $\vec{\psi}(M) = \sum_j \psi^j(M)\, \mathcal{J}_j$.

The quasi-algebras of n-uples of scalar fields of S are essentially the quasi-algebras of the vector fields of the n-dimensional differentiable manifold S′. *The flat euclidean S′ associated to the three-dimensional physical space seems to be related to the isotopic spin space.*

GEOMETRIC QUASI-ALGEBRAS AND FUNCTIONAL ANALYSIS

16. We shall now consider the linear spaces constituted by the formal series $\psi^{(m)}(M_1, \ldots, M_m) = \sum_r c^*_{r_1,\ldots,r_m} \psi_{r_1}(M_1) \ldots \psi_{r_m}(M_m)$, with coefficients c^* symmetrical or antisymmetrical with respect to the indices r, and linear functionals $\lambda^{(m)}\left[\psi^{(m)}\right] = \sum_r c_{\lambda, r_1,\ldots,r_m} c^*_{r_1,\ldots,r_m}$, the symmetry of the c_λ with respect to the indices r being the same as that of the c^*. We shall associate the elements $\langle\psi^{(m)}\rangle_\pm$ and $\langle\lambda^{(m)}\rangle_\pm$ to the $\psi^{(m)}$ and $\lambda^{(m)}$, respectively

$$\langle\psi^{(m)}\rangle_\pm = (m!)^{-1/2} \sum_r c^*_{r_1,\ldots,r_m} \langle\psi_{r_1}\rangle_\pm \ldots \langle\psi_{r_m}\rangle_\pm \quad (1)$$

$$\langle\lambda^{(m)}\rangle_\pm = (m!)^{-1/2} \sum_r c_{\lambda, r_1,\ldots,r_m} \langle\lambda_{r_1}\rangle_\pm \ldots \langle\lambda_{r_m}\rangle_\pm \quad (2)$$

The indices + and — correspond to the cases of antisymmetry and symmetry with respect to the indices r, respectively. $\lambda^{(m)}$ is associated to a formal series $\psi^{(m)}_\lambda$

$$\psi^{(m)}_\lambda(M_1, \ldots, M_m) = \sum_r c_{\lambda, r_1,\ldots,r_m} \psi_{r_1}(M_1) \ldots \psi_{r_m}(M_m) \quad (3)$$

We have

$$\langle\psi^{(m)}\rangle_\pm = (m!)^{-1/2} \int_S \psi^{(m)}(M_1, \ldots, M_m) \langle\psi_{M_1}\rangle_\pm \ldots \langle\psi_{M_m}\rangle_\pm \, d^{(m)}\Omega \quad (4)$$

$$\langle\lambda^{(m)}\rangle_\pm = (m!)^{-1/2} \int_S (\psi^{(m)}_\lambda(M_1, \ldots, M_m))^* \langle\lambda_{M_1}\rangle_\pm \ldots \langle\lambda_{M_m}\rangle_\pm \, d^{(m)}\Omega \quad (5)$$

The products $\langle\lambda_1\rangle_\pm \ldots \langle\lambda_m\rangle_\pm$ are particularly important $\langle\lambda^{(m)}\rangle_\pm$: the products $\langle\psi_1\rangle_\pm \ldots \langle\psi_m\rangle_\pm$ are important $\langle\psi^{(m)}\rangle_\pm$.

The $\langle\lambda^{(m)}\rangle_\pm$ can also be associated to symmetrical or antisymmetrical multi-linear functionals $\lambda^{(m)}\left[\Phi_1, \ldots, \Phi_m\right]$ of m functions $\Phi_k(M)$, namely to those obtained from the $\lambda^{(m)}\left[\psi^{(m)}\right]$ by taking $\psi^{(m)}$ as a symmetrized or anti-symmetrized product $\Sigma \pm \prod_k \Phi_k(M_h)$. *The* $\langle\lambda^{(m)}\rangle$ *can be associated to the homogeneous functionals* $\lambda^{(m)}[\Phi]$ *obtained by taking in* $\lambda^{(m)}[\psi^{(m)}]$, $\psi^{(m)}(M_1,..,M_m) = (m!)^{-1/2}\,\Phi(M_1)\ldots\Phi(M_m)$. *The elements of the form* $\langle\Lambda\rangle_- = \sum_m \langle\lambda^{(m)}\rangle_-$ *are associated to the non-linear functionals* $\Lambda[\Phi] = \sum_m \lambda^{(m)}[\Phi]$. The $\langle\Lambda\rangle_+ = \sum_m \langle\lambda^{(m)}\rangle_+$ are also important mathematical entities, as well as the $\langle\Psi\rangle_\pm = \sum_m \langle\psi^{(m)}\rangle_\pm$. *The* $\langle\Psi\rangle_-$ *can be associated to functionals* $\Psi[\Phi^*]$ *of the* $\Phi^*(M)$

$$\Psi[\Phi^*] = \sum_m (m!)^{-1/2} \int_S \psi^{(m)}(M_1,..,M_m)\,\Phi^*(M_1)\ldots\Phi^*(M_m)\,d^{(m)}\Omega \qquad (6)$$

It is easily seen that

$$[\langle\Lambda\rangle_-,\langle\psi\rangle_-] = \int_S \left\langle \frac{\delta\Lambda[\Phi]}{\delta\Phi(M)} \right\rangle_- \psi(M)\,d\Omega,\ [\langle\Lambda\rangle_-,\langle\psi_M\rangle_-] = \left\langle \frac{\delta\Lambda[\Phi]}{\delta\Phi(M)} \right\rangle_- \qquad (7)$$

$$[\langle\lambda\rangle_-,\langle\Psi\rangle_-] = \int_S \left\langle \frac{\delta\Psi[\Phi^*]}{\delta\Phi^*(M)} \right\rangle_- \psi^*_\lambda(M)\,d\Omega,\ [\langle\lambda_M\rangle_-,\langle\Psi\rangle_-] = \left\langle \frac{\delta\Psi[\Phi^*]}{\delta\Phi^*(M)} \right\rangle_- \qquad (8)$$

It is interesting to notice that the $\langle\psi\rangle_-$ *are associated to linear functionals* $\psi[\Phi^*] = \int_S \psi(M)\,\Phi^*(M)\,d\Omega$, *in particular* $\psi_M[\Phi^*] = \Phi^*(M)$. The second quantization formalism for scalar-boson fields leads naturally to the association of linear functionals $\psi[\Phi^*]$ to the functions $\psi(M)$ and to the consequent generalization of the concept of function. A similar association underlies the Schwarz *distribution theory. Since the above second quantization formalism deals also with non-linear functionals, it may be considered as a generalization of the distribution theory given by* L_n.

The above considerations show that the quasi-algebras of the scalar fields lead to the analysis of the functionals, because they deal not only with functions $\psi(M)$ of a single point, but also with functions $\psi^{(m)}(M_1,..,M_m)$ of sets of any number m of points. *In the quasi-algebra of the* $\langle\psi\rangle_-$, $\langle\lambda\rangle_-$ *we deal with non-orienred sets of points (symmetrical* $\psi^{(m)}$*): in the quasi-algebra of the* $\langle\psi\rangle_+$, $\langle\lambda\rangle_+$ *the sets of points are oriented (anti-symmetrical* $\psi^{(m)}$*).*

16a. In section 16 we did not consider the generators $\langle P\rangle_\pm$. By means of $\langle P\rangle_-$ we can associate to the functionals $\Psi[\Phi^*]$ and $\Lambda[\Phi]$ the elements $\{\Psi\}_- = \langle\Psi\rangle_-\langle P\rangle_-$ and $\{\Lambda\}_- = \langle P\rangle_-\langle\Lambda\rangle_-$. It follows from (16–8) that

$$\langle\lambda\rangle_-\{\Psi\}_- = \int_S \left\langle \frac{\delta\Psi[\Phi^*]}{\delta\Phi^*(M)} \right\rangle_- \psi^*_\lambda(M)\,d\Omega\,\langle P\rangle_-,\ \langle\lambda_M\rangle_-\{\Psi\}_- = \left\langle \frac{\delta\Psi[\Phi^*]}{\delta\Phi^*(M)} \right\rangle_- \langle P\rangle_- \qquad (1)$$

Thus we get a convenient algorithm for the derivation of functionals. *The main advantage of the introduction of* $\langle P \rangle_-$ *consists however in the possibility of integrating functionals.* We shall define $\int \Psi_1[\Phi^*] (\Psi_2[\Phi^*])^* d\Phi^*$ as follows

$$\Psi_1[\Phi^*] (\Psi_2[\Phi^*])^* d\Phi^* = \sum_m \int_S \psi_1^{(m)} (M_1, .., M_m) (\psi_2^{(m)} (M_1, .., M_m))^* d^{(m)}\Omega = \Lambda_{\Psi_2}[\Psi_1] \tag{2}$$

$\Lambda_{\Psi_2}[\Psi]$ is a linear functional on the Ψ associated to the non-linear functional $\Lambda_{\Psi_2}[\Phi] = \sum_m (m!)^{-1/2} \int_S (\psi_2^{(m)} (M_1, .., M_m))^* \Phi(M_1) \dots \Phi(M_m) d^{(m)} \Omega$ on the Φ.

We have

$$\{\Lambda_{\Psi_2}\}_- \{\Psi_1\}_- = \Lambda_{\Psi_2}[\Psi_1] \langle P \rangle_- \tag{3}$$

It is also convenient to introduce $\{\Psi\}_+ = \langle\Psi\rangle_+ \langle P \rangle_+$ and $\{\Lambda\}_+ = \langle P \rangle_+ \langle \Lambda \rangle_+$. The $\{\Lambda\}_+$ are associated to linear functionals $\Lambda[\langle\Psi\rangle_+]$ defined as follows

$$\{\Lambda\}_+ \{\Psi\}_+ = \Lambda[\langle\Psi\rangle_+] \langle P \rangle_+ \tag{4}$$

The linear spaces of the $\{\Psi\}_\pm$ can be used as representation-spaces for our quasi-algebras of scalar fields, the application of the linear operators corresponding to the elements $\langle\chi\rangle_\pm$ transforming $\{\Psi\}_\pm$ into $\langle\chi\rangle_\pm \{\Psi\}_\pm$.

There is no difficulty in developing similar considerations for the quasi-algebras of section 15. *Thus the quasi-algebra of the* $\langle\psi^j\rangle_-$, $\langle\lambda^j\rangle_-$, $\langle P_n \rangle_-$ *appears as a kind of generalization of the analysis of the functionals of n-uples of functions* $\psi^j(M)$, *applicable also to functionals on p-uples of functions, with* $1 \le p \le n$. It is important to notice that those functionals can also be applied to formal series $\psi^j(M) = \sum_r c_r^{j*} \psi_r(M)$, as it happens in the theory of section 16.

16b. Let us replace in (16-6) $\Phi(M)$ by its expansion $\Phi(M) = \sum_r a_r^* \psi_r(M)$. Thus we get for $\Psi[\Phi^*]$ a power series of the a_r

$$\psi^{(m)}[\Phi^*] = (m!)^{-1/2} \sum_r c^*_{r_1, \dots, r_m} a_{r_1} \dots a_{r_m}, \quad \Psi[\Phi^*] = \sum_m \psi^{(m)}[\Phi^*] \tag{1}$$

The $\{\Psi\}_-$ are therefore associated to power series of an infinite number of complex variables

$$\{\Psi\}_- = \sum_m (m!)^{-1/2} \sum_r c^*_{r_1, \dots, r_m} \langle\psi_{r_1}\rangle_- \dots \langle\psi_{r_m}\rangle_- \langle P \rangle_- = \Psi[\langle\psi_M\rangle_-] \langle P \rangle_- \tag{2}$$

In equation (2), $\Psi[\langle\psi_M\rangle_-]$ denotes the functional Ψ applied to the symbolic function $\Phi^*(M) = \langle\psi_M\rangle_-$. The quasi-algebra of the $\langle\psi\rangle_-$, $\langle\lambda\rangle_-$, $\langle P \rangle_-$ is the L_∞ of an infinite-dimensional affine space, whose basic contravariant vectors are the $\langle\lambda_r\rangle_-$ and the basic covariant vectors the $\langle\psi_r\rangle_-$. We can develop a calculus of points for this space, analogous to that of section 7 for the n-dimensional affine space. In this calculus, the $\langle P_\lambda \rangle_- = \langle P \rangle_- \exp \langle \lambda \rangle_-$ correspond to the points, $\langle P \rangle_-$ corresponding to the origin of the coordinates. We have the remarkable relation

$$\langle P_\lambda \rangle_- \{\Psi\}_- = \Psi[\psi_\lambda^*] \langle P \rangle_- \tag{3}$$

We have discussed in section 14 the extension of the manifold S into an infinite-dimensional linear space isomorphic to that of the $\langle \lambda \rangle_-$. The above considerations lead to the transformation of the linear space of the $\langle \lambda \rangle_-$ into the infinite-dimensional affine space of the $\langle P_\lambda \rangle_-$. Our calculus of weighted points extends this infinite-dimensional affine space into the linear space of the $\{\Lambda\}_-$, which are essentially linear combinations of the $\langle P_\lambda \rangle_-$, as we shall now prove. Let us introduce an integration with respect to the ψ

$$\int \Lambda_I[\Phi] \left(\Lambda_{II}[\Phi]\right)^* d\Phi = \Lambda_I[\Psi_{II}], \quad \Lambda_{II}[\Psi] = \Lambda_{\Psi_{II}}[\Psi] \tag{4}$$

Since $\exp \langle \lambda \rangle_- = \sum_m (m!)^{-1} \langle \lambda \rangle_-^m = \sum_m (m!)^{-1} \int_S \psi_\lambda^*(M_1) \ldots \psi_\lambda^*(M_m) \langle \lambda_{M_1} \rangle_- \ldots$ $\ldots \langle \lambda_{M_m} \rangle_- d^{(m)}\Omega$ we have

$$\langle \Lambda \rangle_- = \int \Lambda[\psi_\lambda] \exp \langle \lambda \rangle_- \, d\psi_\lambda, \quad \{\Lambda\}_- = \int \Lambda[\psi_\lambda] \langle P_\lambda \rangle_- \, d\psi_\lambda \tag{5}$$

The above considerations show that a first extension leads to the embedding of S into a linear space isomorphic to that of the linear functionals $\lambda[\psi]$, which may be associated to particle wave-functions ψ_λ, and a further extension to a linear space isomorphic to that of the functionals $\Lambda[\psi]$, which may be interpreted as wave-functionals of a field of uncharged spinless particles.

THE EXTENDED PHASE-SPACE OF A MANIFOLD

17. We have associated to the n-dimensional manifold S the two isomorphic linear spaces of the λ and ψ. *The direct sum of the linear spaces of the λ and ψ will be called the extended phase-space Θ of the manifold.* The elements of Θ will be denoted by R's. The R associated to the *sum* $\lambda \dot{+} \psi$ corresponds to the elements $\langle R \rangle_\pm = \langle \lambda \rangle_\pm + \langle \psi \rangle_\pm$ of our quasi-algebras of scalar-fields. There is a natural definition of an inner product $(R_I, R_{II})_\pm$ and a symplectic product $(R_I, R_{II})_-$ in Θ

$$2\,(R_I, R_{II})_\pm = \lambda_I[\psi_{II}] \pm \lambda_{II}[\psi_I] \tag{1}$$

The two quasi-algebras of the scalar-fields are closely related to the metric and symplectic geometries of Θ defined by the above products. The commutation rules of the $\langle \psi \rangle_\pm$, $\langle \lambda \rangle_\pm$ are special cases of the general rules

$$[\langle R_I \rangle_\pm, \langle R_{II} \rangle_\pm]_\pm = 2\,(R_I, R_{II})_\pm \langle 1 \rangle_\pm \tag{2}$$

which show the importance of the inner and symplectic products of elements R. The situation is similar to that in the theories of G_n and L_n, with respect to the metric and symplectic geometries of the 2-dimensional phase-space S_{2n} introduced in section 1.

Let α denote a linear operator on the ψ. We shall define αR as the *sum* $\lambda \alpha^{\dagger} \dotplus \alpha \psi$ when $R = \lambda \dotplus \psi$. It is easily seen that the linear operators α on the R, corresponding to unitary operators α on the ψ, leave invariant the inner and symplectic products of elements R of Θ. *The orthogonal linear transformations with respect to the inner product* $(R_I, R_{II})_+$ *may be regarded as generalizations of the unitary linear transformations of the* ψ *with respect to the inner products* $(\psi_I, \psi_{II}) = \int_S \psi_I(M) \psi_{II}^*(M) \, d\Omega$ *of the ψ-space.* We shall call symplectic linear transformation of Θ the linear transformations leaving invariant the symplectic products $(R_I, R_{II})_-$. *These symplectic tranformations may also be regarded as generalizations of the unitary linear transformations of the* ψ.

In order that the inner and symplectic products $(R_I, R_{II})_\pm$ be defined for any pair R_I, R_{II}, it is necessary to restrict Θ to the sub-space constituted by the R whose λ and ψ are such that $\int_S |\psi_\lambda|^2 \, d\Omega < \infty$ and $\int_S |\psi|^2 \, d\Omega < \infty$. This sub-space is obviously the direct sum of the Hilbert spaces constituted by those λ and ψ.

THE QUANTIZED SCHRÖDINGER FIELDS

18. The simplest kinds of quantized fields are obtained by the second quantization of the non-relativistic Schrödinger equation for a particle of zero spin. The manifold S will be taken as the euclidean three-dimensional space. The quantized wave-function of the particle will be identified to $\langle \lambda_M \rangle_-$ in the case of the Bose statistics and to $\langle \lambda_M \rangle_+$ in the case of the Fermi statistics. The $\langle \lambda_M \rangle_\pm$ will be identified to the adjoint of the quantized wave-functions.

The $\langle \lambda \rangle_\pm$ associated to normalized $\psi_\lambda(M)$ can be identified to absortion operators of the particle-state $\psi_\lambda(M)$ and the $\langle \psi \rangle_\pm$ of normalized $\psi(M)$ to emission operators of the particle-state $\psi(M)$, the two signs $\pm$ corresponding to the two statistics.

The $\{\Psi\}_\pm$ play the part of the states of the quantized Schrödinger fields. In particular, the $\langle P \rangle_\pm$ correspond to the vacuum-states of the two statistics. The $\langle \psi^{(m)} \rangle_\pm \langle P \rangle_\pm$ correspond to the m-particle states described by the wave-functions $\psi^{(m)}(M_1, \ldots, M_m)$, symmetrical in the case of the Bose statistics and antisymmetrical in the case of the Fermi statistics.

It is interesting to notice that the quasi-algebra corresponding to functions of non-oriented sets of points is associated to the Bose statistics, that corresponding to oriented sets of points being associated to the Fermi statistics. *There seems to be a close relation between the orientation properties of space and the Fermi statistics.*

The embedding of the euclidean space into the λ-space is naturally related to the quantal description of the particles by wave funtions. The states of a spinless non-relativistic particle are described by functionals λ, since each λ is characterized by a ψ_λ. *The linearity of the λ-space corresponds to the superposition principle of the quantum mechanics.* Instead of the λ-space, it is preferable to use the essentially equivalent ψ-space, the ψ corresponding to the wave functions. *The quantum kinematics of a spinless non-relativistic particles is geometrized by the calculus of weighted points associated to the extension of the three-dimensional euclidean space by the infinite-dimensional λ-space. The extension of this calculus given by the quasi-algebras of the scalar-fields, introduces the many-particle systems constituted by identical indistinguishable particles, with the corresponding kinds of statistics, Bose or Fermi statistics, through the second quantization of the Schrödinger wave-functions.*

18a. We showed in section 8 that the Schrödinger equation of a free particle is closely related to a geometric group: the gaussian group of the euclidean metric. We shall now consider the fundamental group of the euclidean geometry, constituted by the translations, rotations and similitudes. The ψ-space is the representation-space of an irreducible representation of the euclidean group. The basic infinitesimal transformations of this group correspond to the partial derivatives ∂_j with respect to the orthogonal cartesian coordinates x^j, the rotation-operators $x^j\,\partial_k - x^k\,\partial_j$ and the dilatation-operator $x^j\,\partial_j$. *The representation of the rotation-translation group is unitary but reducible, because of the commutability of the laplacian with the operators for the infinitesimal rotations and translations. Irreducible unitary representations of that sub-group of the euclidean group are induced in the sub-spaces of the ψ-space corresponding to the solutions of the stationary wave equation for a free particle for the various values of the energy E.*

$$\Delta\,\psi(E,x) = 2\,\mu\,E\,\psi(E,x),\ \mu = \text{mass of the particle} \tag{1}$$

The similitude $x^j \longrightarrow a\,x^j$ transforms the sub-space of the $\psi(E,x)$ into the sub-space of the $\psi(a^2\,E,x)$.

The most fundamental geometrical feature of the non-relativistic quantum kinematics is of course the identification of the momentum-operators with the hermitian operators for the infinitesimal translations and that of the angular momentum operators with the hermitian operators for the infinitesimal rotations, in the ψ-space representation of the translation-rotation group. The possibility of such a geometrization of the basic dynamical quantities results from the introduction of a fundamental constant with the dimensions of an action, as we discussed already in reference (SCHÖNBERG, M). In the relativistic quantum mechanics there is the possibility of a deeper geometrization of physics because of the existence of a fundamental velocity, which, together with the Planck constant, allows to measure all the physical quantities with the unit of length, as well known.

In the geometric quasi-algebras of the scalar-fields of an euclidean space, the elements $\langle\psi_p\rangle_\pm$ associated to the exponential functions $\psi_p(x) = (2\pi)^{-n/2} \exp(i\, p_j x^j)$ and the corresponding $\langle\lambda_p\rangle_\pm$ associated to the Fourier linear functionals λ_p

$$\lambda_p[\psi] = (2\pi)^{-n/2} \int_S \exp(-i\, p_j x^j)\, \psi(x)\, d\Omega$$

play a part as important as the $\langle\psi_M\rangle_\pm$, $\langle\lambda_M\rangle_\pm$. $\lambda_p[\psi]$ is obviously the Fourier transform of $\psi(M)$. It is interesting to note that $\psi[\lambda_p] = \lambda_p[\lambda]$, so that the Fourier transform of ψ is simply the value of the function $\psi[\lambda]$ for the element λ_p of the λ-space, as $\psi(M) = \psi[\lambda_M]$. We have

$$\langle\psi\rangle_\pm = \int_{-\infty}^{+\infty} \psi[\lambda_p] \langle\psi_p\rangle_\pm\, d\mathfrak{p}, \quad \langle\lambda\rangle_\pm = \int_{-\infty}^{+\infty} \lambda[\psi_p] \langle\lambda_p\rangle_\pm\, d\mathfrak{p} \qquad (3)$$

18b. A scalar field $\psi(M)$ can be extended into the λ-space by taking $\psi[\lambda]=\lambda[\psi]$, as we have seen. Since $\psi[\lambda]=\int_S \psi(M)\, \psi_\lambda^*(M)\, d\Omega$, the consideration of the $\psi[\lambda]$ allows to "smooth" out the $\psi(M)$. The quantized fields $\langle\psi_M\rangle_\pm$ can also be extended into the λ-space, their values at the *point* λ being simply $\langle\psi_\lambda\rangle_\pm$ Since $\langle\psi_\lambda\rangle_\pm = \int_S \psi_\lambda(M)\ \langle\psi_M\rangle_\pm\ d\Omega$, the $\langle\psi_\lambda\rangle_\pm$ allow to *smooth* out the quantized field $\langle\psi_M\rangle_\pm$. The quantized field $\langle\lambda_M\rangle_\pm$ can also be extended into the λ-space, their values at the *point* λ being taken as $\langle\lambda\rangle_\pm$.

The above extension of the quantized Schrödinger fields into the λ-space suggests that similar extensions can also be done in the case of relativistic fields, where the *smoothing* out is often required to avoid divergence difficulties. It seems that the extension of fields into the λ-space is not simply a mathematical trick, but rather a significant physical procedure in the quantum theory of fields, related to the necessity of using extended testing bodies in the field measurements. The extension of the domain of definition of the relativistic scalar-fields will be discussed in sections 19a and 20.

(To be continued)