ACADEMIA BRASILEIRA DE CIÊNCIAS

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PART. II

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Vol. 30 | RIO DE JANEIRO, 31 DE MARÇO DE 1958 | N.º 1

Quantum Mechanics and Geometry

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(Received June 25, 1957)

(PART II OF A SERIES OF FIVE)

L_n AND THE METRIC GEOMETRY

8. The introduction of the metric g_{jk} in the theory of L_n allows us to build the d_j and q^j: $\sqrt{2}\, d_j = \{I_j\} - g_{jk}\{I^k\}$, $\sqrt{2}\, q^j = \{I^j\} + g^{jk}\{I_k\}$. The q^j and $p_j = -\, i d_j$ are self-adjoint when $g_{jk} = \delta_{j,k}$. They can be identified to the coordinate and momentum operators of a particle in the three-dimensional case, by the introduction of a fundamental unit of length, as we showed in (SCHÖNBERG, M., 1956a). We shall denote by $Q_n^{(g)}$ the quasi-algebra generated by the q^j and by $P_n^{(g)}$ the quasi-algebra generated by the d_j. $P_n^{(g)}$ is obviously equivalent to the quasi-algebra $Q_n^{(-g)}$. $Q_n^{(g)}$ and $P_n^{(g)}$ correspond to the Clifford algebras $C_n^{(g)}$ and $C_n^{(-g)}$ of G_n, respectively. The commutative quasi-algebras $Q_3^{(\delta)}$ and $P_3^{(\delta)}$ of the euclidean three-dimensional space may be considered as quasi-algebras of the configuration and momentum spaces of a particle, respectively. *The theory of* $Q_n^{(g)}$ *and* $P_n^{(g)}$ *is equivalent to that of a certain kind of non-numerically valued functions of n non-numerical commutative variables.*

The elements $\exp\left(\sum_j c_j d_j\right)$, the c_j being numbers, constitute an abelian group depending on n parameters, which is essentially the translation group of the n-dimensional space in consideration. This group was already discussed at the end of section 6 in connection with the automorphisms $\underline{\Lambda} \longrightarrow \exp(-i\, p_V) \cdot \underline{\Lambda} \exp(i\, p_V)$. *We are now led to the quasi-algebra constituted by the Dirichlet series of the* d_j: $\sum_r a_r \exp\left(\sum_j c_j^{(r)} d_j\right)$, *the a and c denoting numerical coefficients.*

There is another kind of abelian groups associated to $P_n^{(g)}$: the groups constituted by the elements $\exp(c\, s^{jk}\, d_j\, d_k)$, s^{jk} being a symmetric contravariant tensor of the second order. *The one-parameter group associated to* $s^{jk} = g^{jk}$ *is particularly interesting: the sub-group corresponding* *to the real values of* c *describes the diffusion or heat conduction in a n-dimensional space, when the metric* g_{jk} *is euclidean; the sub-group corresponding to the imaginary values of* c *describes the quantal motion of a non relativistic n-dimensional free particle, in the case of the euclidean metric*. The differential equation $\frac{d}{dc}\exp(c\{s\}) = \{s\}\exp(c\{s\})$, with $\{s\} = s^{jk} d_j d_k$ plays a central rôle in the theory of the abelian group associated to the symmetric tensor s^{jk}. $\{g\}$ corresponds to the n-dimensional laplacian.

In the case of the euclidean metric, it is convenient to use orthogonal cartesian coordinates in order that $g^{jk} = \delta_{j,k}$. *Now* $\exp(c\{g\})$ *is simply the gaussian function* $\exp(c \sum_j d_j^2)$ *of the* d_j. We shall call gaussian group of the metric the group of the elements $\exp(c\{g\})$. This geometric group has a fundamental importance for physics, as shown by the above discussion. *It lies at the core of statistical physics, playing a central rôle in the theories of the brownian motion, heat conduction and quantum mechanics.*

The gaussian group is included in a very interesting quasi-algebra constituted by the series of the Hermite functions of the $p_j = -id_j$. The general form of the elements of this Hermite quasi-algebra is

$$F(p) = \sum_r^{0,\ldots,\infty} c_{r_1,\ldots,r_n} \prod_j h_{r_j}(p_j)\,,\quad h_r(u) = (\sqrt{\pi}\, r!\, 2^r)^{-1/2} \exp(-u^2/2)\, H_r(u) \qquad (1)$$

H_r denoting the Hermite polynomial: $H_r(u) = (-1)^r \exp(u^2)\, D^r \exp(-u^2)$. *These functions* F (p) *may not correspond to ordinary functions of numerical variables. In particular* $\sum_r \prod_j h_{r_j}(a_j)\, h_{r_j}(p_j)$ *corresponds to* $\prod_j \delta(p_j - a_j 1_{L_n})$, *the* a_j *being numbers. There are also elements* F(p) *corresponding to the derivatives of all orders of* $\prod_j \delta(p_j - a_j 1_{L_n})$, *which can be obtained by derivations with respect to the* a_j *of the terms of the series for* $\prod_j \delta(p_j - a_j 1_{L_n})$. In a similar way we can define a Hermite quasi-algebra of the functions F(q), which is included in $Q_n^{(\delta)}$.

The quasi-algebra of the F(q) gives a very convenient extension of the classical analysis of the functions of n real variables. The derivative with respect to q^j of F(q) can be defined as the element of the same Hermite quasi-algebra obtained by the substitution of the $h_r(q^j)$ by $[d_j, h_r(q^j)] = 2^{-1/2} \cdot (\sqrt{r}\; h_{r-1}(q^j) - \sqrt{r+1}\; h_{r+1}(q^j))$. *The derivatives of all orders of the elements of the Hermite quasi-algebra of the* F(q) *exist always and belong to it*. The Fourier transform of F(q) can be defined as $\sum_r c_{r_1,\ldots,r_n} (-i)^{r_1+\ldots+r_n} \cdot \prod_j h_{r_j}(q^j)$, by taking into account that the Fourier transform of $h_r(x)$ is $(-i)^r \cdot h_r(y)$. *All the elements of the Hermite quasi-algebra have Fourier transforms, which belong to the quasi-algebra.* The integral of F(q) over a region Ω can be defined as $\sum_r c_{r_1,\ldots,r_n} \int_\Omega \prod_j h_{r_j}(x^j)\, dx^1 \ldots dx^n$. *This integral is a number, which may exist even when* F(q) *does not correspond to an ordinary function F(x).*

Let F(q,a) be an element of the Hermite quasi-algebra depending on a parameter a. We define $\lim_{a \to a_0} F(q,a) = \Sigma_r \lim_{a \to a_0} c_{r_1,\ldots,r_n}(a) \Pi_j h_{r_j}(q^j)$. We have $\Pi_j \delta(q^j - a^j 1_{L_n}) = \lim_{\eta \to 0} (\pi \eta^2)^{-n/2} \exp\left(-\Sigma_j (q^j - a^j 1_{L_n})^2/\eta^2\right)$ as a consequence of the Mehler formula $(1-z^2)^{-1/2} \exp \frac{2xyz-(x^2+y^2)z^2}{1-z^2} = \Sigma_0^\infty (2^r r!)^{-1} z^r H_r(x) H_r(y)$. *In the Hermite quasi-algebra, the Dirac function is the true limit of the normalized gaussian, the normalization being meant in the sense of having integral equal to* 1.

We shall now prove that L_n *is generated by the* q^j *and* $\{P\}$. Let us introduce the element $\{P\} = \pi^{n/4} \exp(1/2 \Sigma_j (q^j)^2) \{P\}$, with the property $d_j\{P\} = 0$. It follows from the recurrence relation for the Hermite polynomials H(x) that $\{P^j\}^r \{P\} = 2^{-r/2} H_r(q^j) \{P\}$, hence $\{P^{j_1 \ldots j_p}\} = \Pi_k (2^{r_k} r_k!)^{-1/2} H_{r_k}(q^k)\{P\} = \Pi_j h_{r_k}(q^k)\{P\}$, h_r denoting the normalized Hermite function of the order r and r_k the number of indices of $\{P^{j_1,\ldots,j_p}\}$ equal to the integer k. In a similar way it is seen that $\{P_{j_1,\ldots,j_p}\} = \{P\} \Pi_k h_{r_k}(q^k)$. In all the above formulae the q^j were taken for the euclidean metric $g_{jk} = \delta_{j,k}$. We have proven that the generators $\{P^{j_1,\ldots,j_p}\}$, $\{P_{k_1,\ldots,k_q}\}$ can be built with the euclidean q^j and $\{P\}$. In a similar way it is seen that those generators can be built with the d_j and $\{P\}$. *It follows from the above results that all the* $\{\Psi\}$ *are of the form* $F(q)\{P\}$. *In particular* $\{P\}$ *is the* $\{\Psi\}$ *whose* $F(q) = \pi^{-n/4} \exp\left(-1/2 \Sigma_j (q^j)\right)^2$.

The above discussion shows the central rôle played by the Hermite quasi-algebras of $Q_n^{(g)}$ and $P_n^{(g)}$ in the theory of L_n. L_n *appears now as the mathematical analysis of the functions of n commutable non-numerical variables* q^j, *the* G(p) *corresponding to functions of the derivatives* $\frac{\partial}{\partial q^j}$: $G(p) F(q)\{P\} = \left(G\left(-i\frac{\partial}{\partial q}\right)F(q)\right)\{P\}$ · L_n is a generalization of the classical mathematical analysis, because the mathematical analysis of the functions of commutable non-numerical variables is more general than the corresponding theory for the ordinary functions of numerical variables.

The equivalence of L_n and the Heisenberg quasi-algebra of the position and momentum variables of a mechanical system with n degrees of freedom shows that the mathematical formalism of the quantal kinematics is more general than the classical mathematical analysis. This fact appeared since the development of the quantal transformation theory, although it was not understood conveniently. The present theory of the functions of commutable non-numerical variables seems to be the natural kind of mathematical analysis for the quantum mechanics of systems with a finite number of degrees of freedom that have a classical analogue. It is of course possible to use other systems of orthonormal functions besides the Hermite ones. Nevertheless the Hermite functions appear as a distinguished system in our geometric-algebraic theory.

The physical interest of the existence of this distinguished orthonormal system was already discussed in SCHÖNBERG, M., 1956a.

The elements $\{N^j_k\} = \{\mathbf{I}^j\}\{\mathbf{I}_k\}$ play an important part, because they come in in the expression of the element $\{\theta\}$ associated to the mixed tensor θ^k_j: $\{\theta\} = \theta^k_j\{N^j_k\}$. The most important of those tensors is δ^k_j, whose associated element $\{\delta\} = \sum_j \{\mathbf{I}^j\}\{\mathbf{I}_j\}$. The $\{N^j_k\}$ transform as the components of a tensor with the same indices for a change of the basic vectors $\mathbf{I}_j$ of the cartesian coordinate system. The $\{N_j\} = \{N^j_j\}$ are particularly interesting, because $\{N_k\}\{P^{j_1,\ldots,j_p}\} = r_k\{P^{j_1,\ldots,j_p}\}$, r_k denoting the number of the j equal to k. We have $2\{N_j\} = p_j^2 + q^{j\,2} - 1_{L_n}$, the p and q being those of the euclidean metric $g_{jk} = \delta_{j,k}$. *The* $\{N_j\}$ *correspond to hamiltonians of one-dimensional quantal harmonic oscillators without zero-point energy, as linear operators on the* $\{\Psi\}$. *With any choice of the metric we have*

$$2\{\delta\} = g^{jk}\, p_j\, p_k + g_{jk} q^j q^k - n\, 1_{L_n} \tag{2}$$

The hamiltonian of the isotropic harmonic oscillator is therefore a fundamental element of L_n.

The elements $\exp(c\{\delta\})$ constitute an important one-parameter group. Since $\exp(c\{\delta\})\{\mathbf{V}\}\exp(-c\{\delta\}) = e^{-c}\{\mathbf{V}\}$, the automorphisms of L_n induced by the similitudes are closely related to the group of the $\exp(c\{\delta\})$. *The subgroup corresponding to the imaginary values of* c *is equivalent to the group of the unitary transformations generated by the motion of the isotropic quantal harmonic oscillator of hamiltonian* $\{\delta\}$. It is easily seen that $\exp(c\{\delta\})\{\mathbf{U}\}\exp(-c\{\delta\}) = e^{c}\{\mathbf{U}\}$ and $\exp(c\{\delta\})\{\mathbf{P}\}\exp(-c\{\delta\}) = \{\mathbf{P}\}$ hence

$$\begin{aligned}\exp(c\{\delta\})\{P^{j_1,\ldots,j_p}\}\exp(-c\{\delta\}) &= \exp(c\{\delta\})\{P^{j_1,\ldots,j_p}\} = \\ &= \exp\Big(c\sum_k r_k\Big)\{P^{j_1,\ldots,j_p}\}\end{aligned} \tag{3}$$

$$\exp(i\pi/2\,\{\delta\})\prod_j h_{r_j}(q^j)\{P\} = i^{r_1+\ldots+r_n}\prod_j h_{r_j}(q^j)\{P\} \tag{4}$$

with $\sqrt{2}\, q^j = \{\mathbf{I}^j\} + \{\mathbf{I}_j\}$. It is well known that the Fourier transform of $h_r(x)$ is $(-i)^r h_r(x)$. Thereby we have $\exp(-i\pi/2\,\{\delta\})F(q)\{P\} = \hat{F}(q)\{P\}$, $\hat{F}(q)$ denoting the Fourier transform of $F(q)$: $\hat{F}(q) = \sum_r c_{r_1,\ldots,r_n}\prod_j(-i)^{r_j} h_{r_j}(q^j)$. Thus we are led to the generalization of the Fourier transformation introduced in (SCHÖNBERG, M., 1956a). *It is interesting to note that the above definition of* $\hat{F}(q)$ *does not require* F(q) *to correspond to an ordinary function* F(x) *of numerical variables.*

The above results show that the Fourier transformation may be considered as a geometric transformation associated to the euclidean-metric, although it be associated to the affine automorphism $\underline{\Lambda} \longrightarrow \exp(-i\,\pi/2\{\delta\})\,\underline{\Lambda}\,\exp(i\pi/2\,\{\delta\})$. *The reason of this remarkable circumstance is simply that the above automorphism transforms* q^j *into* $-p_j$ *and* p_j *into* q^j, *when the* p,q *are defined with* $g_{jk} = \delta_{j,k}$.

The introduction of the metric g_{jk} in the theory of L_n is associated with the replacement of the $\{\mathbf{l}_j\}, \{\mathbf{l}^j\}$ by the d_j, q^j, which satisfy commutation rules similar to those of the affine generators $\{\mathbf{l}_j\}, \{\mathbf{l}^j\}$. Let us consider the automorphism $\underline{A} \longrightarrow e^{cS} \underline{A}\, e^{-cS}$, with $2S = g_{jk}\{\mathbf{l}^j\}\{\mathbf{l}^k\} + g^{jk}\{\mathbf{l}_j\}\{\mathbf{l}_k\} = g_{jk} q^j q^k + g^{jk} d_j d_k$. It is easily seen that

$$e^{cS}\{\mathbf{l}_j\}e^{-cS} = \{\mathbf{l}_j\}\cos c - g_{jk}\{\mathbf{l}^k\}\sin c\,;\; d_j = \exp(\pi/4\, S)\{\mathbf{l}_j\}\exp(-\pi/4\, S) \quad (5)$$

$$e^{cS}\{\mathbf{l}^j\}e^{-cS} = \{\mathbf{l}^j\}\cos c + g^{jk}\{\mathbf{l}_k\}\sin c\,;\; q^j = \exp(\pi/4\, S)\{\mathbf{l}^j\}\exp(-\pi/4\, S) \quad (6)$$

Thus we get an important one-parameter group of automorphisms associated to the metric g_{jk}. The automorphism corresponding to $c = \pi/2$ transforms $\{\mathbf{l}_j\}$ into $-g_{jk}\{\mathbf{l}^k\}$ and $\{\mathbf{l}^j\}$ into $g^{jk}\{\mathbf{l}_k\}$; that corresponding to $c = \pi$ is the automorphism induced in L_n by the reflection $\mathbf{V} \longrightarrow -\mathbf{V}$.

In SCHÖNBERG, M., 1956b, we discussed an algebra K_n generated by elements λ_j satisfying the commutation rule $[\lambda_j, \lambda_k] = 2f_{jk}\, 1_{K_n}$, f_{jk} denoting an antisymmetric covariant tensor. In the case of even n, it is possible to take a tensor f_{jk} with a determinant $f \neq 0$. *The corresponding algebra* K_n *is the analogue of the Clifford algebra for the symplectic geometry.* We can take $\lambda_j = \{\mathbf{l}_j\} - f_{jk}\{\mathbf{l}^k\}$. We showed in SCHÖNBERG, M., 1956b, that L_n is the K_{2n} of the direct sum of the linear spaces of the contravariant and covariant vectors of the n-dimensional space, corresponding to a tensor F_{JK} of determinant different from 0. *There is such a 2n-dimensional tensor naturally associated to the n-dimensional metric* g_{jk}, *because we can take* $F_{JK} = 0$ *for* $J, K \leq n$ *and* $J, K \geq n$, $F_{JK} = g_{jk}$ *for* $J = j$, $K = k + n$, $F_{JK} = -g_{jk}$ *for* $J = j + n$, $K = k$. *By taking* $\Lambda_J = \sqrt{2}\, d_j$ *when* $J = j$ *and* $\Lambda_J = \sqrt{2}\, q_j = \{\mathbf{l}_j\} + g_{jk}\{\mathbf{l}^k\}$ *when* $J = j + n$, *we unify the three commutation rules of the p,q into the single rule* $[\Lambda_J, \Lambda_K] = 2F_{JK}\, 1_{L_n}$. L_n is not a K_{2n}, because of the existence of the generator $\{P\}$, but a new kind of 2n-dimensional symplectic quasi-algebra K_{2n}. It is interesting to note that we have also a natural metric tensor G_{JK} of the above 2n-dimensional space associated to g_{jk}: $G_{jk} = g_{jk}$, $G_{JK} = -g_{jk}$ when $J = j + n$, $K = k + n$, the remaining G_{JK} being all zero. The 2n elements $\Gamma_j = (\mathbf{l}_j) + g_{jk}(\mathbf{l}^k)$, $\Gamma_{j+n} = (\mathbf{l}_j) - g_{jk}(\mathbf{l}^k)$ with the commutation rule $[\Gamma_J, \Gamma_K]_+ = 2G_{JK}\, 1_{G_n}$ are the generators of a Clifford algebra C_{2n} which coincides with G_n.

8a. We derived the fundamental equations

$$\{\mathbf{l}^{j_1}\}\ldots\{\mathbf{l}^{j_p}\}\{P\} = \prod_k 2^{-r_k/2} H_{r_k}(q^k)\{P\},\; \{P\}\{\mathbf{l}_{j_1}\}\ldots\{\mathbf{l}_{j_p}\} = \{P\}\prod_k 2^{-r_k/2} H_{r_k}(q^k) \quad (1)$$

In a similar way we get for the euclidean metric $g_{jk} = \delta_{j,k}$

$$\{\mathbf{l}^{j_1}\}\ldots\{\mathbf{l}^{j_p}\}\{P\} = \prod_k i^{-r_k} 2^{-r_k/2} H_{r_k}(p_k)\{P\},\; \{P\}\{\mathbf{l}_{j_1}\}\ldots\{\mathbf{l}_{j_p}\} = \{P\}\prod_k i^{r_k} 2^{-r_k/2} H_{r_k}(p_k)$$
$$(2)$$

We have $\{\Psi\} = F(q)\{P\} = \underset{\sim}{F}(p)\{\underset{\sim}{P}\}$, with $\underset{\sim}{F}(p) = \sum_r c_{r_1,\ldots,r_n}\prod_k i^{-r_k} h_{r_k}(p_k)$ and $\{\underset{\sim}{P}\} = \pi^{n/4}\exp(1/2\sum_j p_j^2)\{P\}$, the c denoting the coefficients of the expansion of F(q) in Hermite functions h_r. $\underset{\sim}{F}(y)$ *is the Fourier transform of* F(x), *the Fourier transform of a formal series of Hermite functions being defined* *as the series of the Fourier transforms of its terms.* Thus we get a new approach to the theory of the Fourier transformation in L_n. In the case of L_3, the F correspond to wave-functions in configuration space and the $\underset{\sim}{\mathrm{F}}$ to wave-functions in momentum space.

It is clear, by rotation invariance, that we must have

$$|\mathbf{U}|^{-r}\{\mathbf{U}\}^{r}\{P\} = 2^{-r/2} H_r(|\mathbf{U}|^{-1} U_j q^j)\{P\}, \{P\}|\mathbf{V}|^{-r}\{\mathbf{V}\}^{r} = \{P\} 2^{-r/2} H_r(|\mathbf{V}|^{-1} \sum_j V^j q^j) \tag{3}$$

as a consequence of the equations (1), with $|\mathbf{U}|^2 \Sigma = (U_j)^2$ and $|\mathbf{V}|^2 = \Sigma (V^j)^2$. Thus we get a geometric interpretation of the well known addition formula for the Hermite polynomials H_r

$$H_r(|\mathbf{U}|^{-1} U_j x^j) = |\mathbf{U}|^{-r} (r!) \sum_{s_1 + \ldots + s_n = r} \prod_j (s_j!)^{-1} (U_j)^{s_j} H_{s_j}(x^j) \tag{4}$$

since $\{\mathbf{U}\}^r = (r!) \sum_{s_1 + \ldots + s_n = r} \prod_j (s_j!)^{-1} (U_j)^{s_j} \{\mathbf{I}^j\}^{s_j}$.

We get from the second equation (2)

$$\{P\}\{\mathbf{V}\}^r = \{P\}|\mathbf{V}|^r i^r 2^{-r/2} H_r(|\mathbf{V}|^{-1} V^j p_j) \tag{5}$$

Hence

$$\{P\} \exp(\{\mathbf{V}\}) = \{P\} \exp(i\sqrt{2}\, V^j p_j + \tfrac{1}{2}|\mathbf{V}|^2) \tag{6}$$

by taking into account the well known expression of the generatrix function of the Hermite polynomials

$$\exp(2 V^j u_j - |\mathbf{V}|^2) = \sum_r \prod_k (r_k!)^{-1} (V^j)^{r_k} H_{r_k}(u_k) \tag{7}$$

We showed in section 7 that, in the case of a $\{\Psi\}$ corresponding to a function $\Psi(z)$, $\{P\} \exp(\{\mathbf{V}\})\{\Psi\}\{P\} = \Psi(V)\{P\}$. Hence

$$\Psi(z\sqrt{2})\{P\} = \{P\} \exp(2iz^j p_j + |\mathbf{z}|^2) F(q)\{P\}, \quad \{\Psi\} = F(q)\{P\} \tag{8}$$

Equation (8) relates the functions $\Psi(z)$ and $F(x)$. We shall see in section (9) that it corresponds to a Gauss-Weierstrass transformation.

8b. We shall now discuss some special circumstances of the theory of L_n as the quasi-algebra of a space with an indefinite metric. We shall take $g_{jk} = \varepsilon_j \delta_{j,k}$, $\varepsilon_j = 1$ or -1. *The* q^j, p_j *are self-adjoint when* $\varepsilon_j = 1$ *and anti-self-adjoint when* $\varepsilon_j = -1$. We shall introduce the self-adjoint elements $\hat{q}^j = \sqrt{\varepsilon_j}\, q^j$, $p_j = \sqrt{\varepsilon_j}\, \hat{p}_j$. When $\varepsilon_j = 1$, $\hat{p}_j = p_j$ and $\hat{q}^j = q^j$; when $\varepsilon_j = -1$, $\hat{p}_j = -ip_j$ and $\hat{q}^j = iq_j$. It is important to note that the commutation rule of the $\hat{p}$ and $\hat{q}$ is $[\hat{q}^j, \hat{p}_k] = i\,\delta_{j,k} 1_{L_n}$. *The* $\hat{p}$, $\hat{q}$ *come in naturally because*

$$\prod_j \{\mathbf{I}^j\}^{r_j}\{P\} = \prod_j (2\varepsilon_j)^{-r_j/2} H_{r_j}(\hat{q}^j)\{P\} \tag{1}$$

$$\prod_j \{\mathbf{I}^j\}^{r_j}\{P\} = \prod_j (-2\varepsilon_j)^{-r_j/2} H_{r_j}(\hat{p}^j)\{P\} \tag{2}$$

We shall now generalize the definition of $\{P\}$ and $\{\underset{\sim}{P}\}$ given for the euclidean metric $g_{jk} = d_{j,k}$

$$\{P\} = \pi^{n/4} \exp(g_{jk}\, q^j q^k/2)\{P\}\ ,\quad \{\underset{\sim}{P}\} = \pi^{n/4}\exp(g^{jk} p_j\, p_k/2)\{P\} \tag{3}$$

$$\hat{p}_j \{P\} = 0\ ,\quad \hat{q}_j \{\underset{\sim}{P}\} = 0 \tag{4}$$

The element $\{\Psi\} = \Psi(\{\mathbf{l}\})\{P\}$ can be expressed as follows

$$\{\Psi\} = F(\hat{q})\{P\} = \{\underset{\sim}{F}\}(\hat{p})\{\underset{\sim}{P}\},\ \underset{\sim}{F}(\hat{p}) = \underset{\sim}{F}(\hat{p}^1,\ldots,\hat{p}^n) \tag{5}$$

F and $\underset{\sim}{F}$ denoting formal series of products of Hermite functions, as a consequence of equations (1) and (2). $\underset{\sim}{F}$ *is the Fourier transform of* F. We have

$$i\ \hat{p}_j\ F(\hat{q})\{P\} = F'_j(\hat{q})\{P\} \tag{6}$$

$F'_j(x)$ denoting the formal series obtained by derivating term by term with respect to x^j the series $F(x)$.

The use of the $\hat{q}$, $\hat{p}$ is analogous to that of an imaginary fourth component of the four-vectors in the special theory of relativity. This procedure appears as especially adequate in the application of L_n to a pseudo-euclidean space.

8c. We have been using a definition of the adjuntion based on the conditions $\{\mathbf{l}_j\}^\dagger = \{\mathbf{l}^j\}, \{\mathbf{l}^j\}^\dagger = \{\mathbf{l}_j\}, \{P\}^\dagger = \{P\}$. Let us consider the symbolic vectors of components $\{\mathbf{l}_j\}$ and $\{\mathbf{l}^j\}$, respectively covariant and contravariant. The adjunction we have been using associates a contravariant symbolic vector to a covariant symbolic vector. *This association is invariant for euclidean orthogonal changes of the basic vectors* $\mathbf{l}_j$, *but not for general changes of the basic vectors. It is not adequate for the application of* L_n *to a pseudo-euclidean geometry. The satisfactory definition of the adjunction for the metric tensor* g_{jk} *is based on the covariant conditions*

$$\{\mathbf{l}_j\}^\dagger = g_{jk}\{\mathbf{l}^k\},\ \{\mathbf{l}^j\}^\dagger = g^{jk}\{\mathbf{l}_k\},\ \{P\}^\dagger = \{P\} \tag{1}$$

which give always

$$(q^j)^\dagger = q^j\ ,\quad (p_j)^\dagger = p_j \tag{2}$$

with $\sqrt{2}\ q^j = \{\mathbf{l}^j\} + g^{jk}\{\mathbf{l}_k\},\ \sqrt{2}\ i\ p_j = \{\mathbf{l}_j\} - g_{jk}\{\mathbf{l}^k\}$.

The p_j correspond to the infinitesimal translations along the coordinate axes and the $m^{jk} = q^j g^{kh} p_h - q^k g^{jh} p_h$ to the infinitesimal rotations in the j, k coordinate plane. It follows from (2) that the m^{jk} are self-adjoint with the definition of the adjunction based on the conditions (1). *The covariant definition of the adjunction leads to "unitary" representations of the translation-rotation group of the n-dimensional space endowed with the metric* g_{jk}, *the "unitary" elements of* L_n *being those which satisfy the condition* $\mathcal{U}^\dagger = \mathcal{U}^{-1}$, *with the definition of the adjunction based on the covariant conditions* (1).

In the case of the space-time, the above translation-rotation group is the inhomogeneous Lorentz group. *The application of L_4 to the space-time is naturally associated with "unitary" representations of the inhomogeneous Lorentz group in its $\{\Psi\}$-space. This result is of great interest because it leads to a geometrization of the Klein-Gordon equation.* Indeed, the element $g^{jk}\, p_j\, p_k$ is commutable with the generators p_j, m^{jk} of the inhomogeneous Lorentz group. In order to have an irreducible representation of the Lorentz group we must take a sub-space constituted by $\{\Psi\}$ satisfying the equation

$$g^{jk}\, p_j\, p_k\, \{\Psi\} = c\{\Psi\} \qquad (3)$$

which is the Klein-Gordon equation when the number c is positive. The solutions of equation (3) are eigenvectors of the elements $\exp(\alpha\, g^{jk}\, p_j\, p_k)$ of the gaussian group. *Thus we see the interplay of the two groups associated to the metric: the non-abelian translation-rotation group and the abelian gaussian group. It is interesting to note that our algebraic description of space leads to the quantum dynamics of free particles without the use of curved space-time.*

L_n and the Gauss-Weierstrass Transformation

9. The gaussian exponential plays a central rôle in the metric euclidean theory of L_n, as we showed in section 8. We should expect the Gauss-Weierstrass transformation

$$\mathcal{G}_z^{\alpha}\left[F\right] = (2\pi\alpha)^{-n/2} \int_{-\infty}^{+\infty} F(x)\, \exp\left(-\sum_j (x^j - z^j)^2/(2\alpha)\right) dx^1 \ldots dx^n \qquad (1)$$

to play an important part in that theory. By taking into account that

$$\exp(-(\mathbf{X} - \mathbf{Z})^2) = \exp(-\mathbf{X}^2) \sum_r \prod_j (r_j!)^{-1} (z^j)^{r_j}\, H_{r_j}(x^j) \qquad (2)$$

we get the fundamental formula

$$\mathcal{G}_u^{1/2}\left[H_r\right] = (2u)^r \qquad (3)$$

which shows that the operator $\mathcal{G}_z^{1/2}$ transforms a series of Hermite polynomials into a power series.

Let $\{\Psi\} = F(q)\{P\} = \Psi(\{\mathbf{I}\})\{P\}$. We may write $\{\Psi\} = F(q)\{P\}$, F denoting a formal series of Hermite polynomials. It follows from (8a–1) and (3) that

$$\Psi(z\sqrt{2}) = \mathcal{G}_z^{1/2}[F] = \pi^{-n/2} \int_{-\infty}^{+\infty} F(x)\, \exp(-(\mathbf{X} - \mathbf{Z})^2)\, dx^1 \ldots dx^n \qquad (4)$$

the integral of a formal series being taken as the sum of the integrals of its terms. *$\Psi(z\sqrt{2})$ is the Gauss-Weierstrass transform of index $1/2$ of* $\pi^{n/4} \exp(1/2\, \mathbf{X}^2) \cdot F(x)$. When the series $\Psi(z\sqrt{2})$ converges, we may use equation (8a–8) to obtain the Gauss-Weierstrass transform.

It follows from equation (4) that

$$\mathcal{G}^1_z[\mathrm{F}] = \psi(z), \; \psi(z\sqrt{2}) = 2^{-n/2}\,\pi^{-n/4}\,\exp(-\tfrac{1}{2}\,\mathbf{z}^2)\,\Psi(z) \tag{5}$$

Since $\exp(u^2/2) = \sqrt{2}\,\sum_r (2^{2r}(r!))^{-1}\,H_{2r}(u)$, as a special case of the Mehler formula

$$\{P\} = 2^{n/2}\,\pi^{n/4}\,\exp(\tfrac{1}{2}\sum_j \{\mathbf{I}^j\}^2)\{\mathrm{P}\} \tag{6}$$

Hence $\{\Psi\} = \mathrm{F}(q)\{P\} = \psi(\{\mathbf{I}\})\{P\}$, with $\{\mathbf{I}^j\} = \sqrt{2}\,\{\mathbf{I}^j\}$. When the series $\psi(z)$ converges, we get from (8a–8)

$$2^{n/2}\,\pi^{n/4}\,\psi(z)\{\mathrm{P}\} = \{\mathrm{P}\}\exp(iz^j\,p_j)\,\mathrm{F}(q)\{P\} \tag{7}$$

The function ψ is obtained from F by a kind of smoothing out which replaces $\delta(\mathbf{X}-\mathbf{a})$ by the normalized gaussian $(2\pi)^{-n/2}\exp(-\tfrac{1}{2}(\mathbf{Z}-\mathbf{a})^2)$. This means that a punctual distribution in the x-space corresponds to an extended distribution in the z-space. The $\psi(z)$ associated to the $\{\Psi\}$ do not allow to isolate points in the z-space, because their linear space does not include $\delta(\mathbf{Z}-\mathbf{a})$, there being no power series $\psi(z)$ such that $z^j\,\psi(z) = a^j\,\psi(z)$.

The transformation $\mathrm{F}(x) \longrightarrow \psi(z)$ is not unitary in the ordinary sense. It is unitary with the following definition of the inner product of the ψ:

$(\psi_1, \psi_2) = \int_{-\infty}^{+\infty} \mathrm{F}_1(x)\,(\mathrm{F}_2(x))^*\,dx^1 \ldots dx^n$. It is easily seen that

$$(\psi_1, \psi_2) = \pi^{-n/2}\int_{-\infty}^{+\infty} \psi_1(z)\,(\psi_2(z))^*\,\exp\{(\mathbf{z}^2 + \mathbf{z}^{*2} - 2\,|z|^2)/4\}\,d\mathbf{U}\;d\mathbf{V} \tag{8}$$

$\psi(z)$ being considered as a function of the real variables u^j and v^j, with $z^j = u^j + i\,v^j$. It is important to note that $\psi(z)$ is an integral analytic function of the z^j when $\int_{-\infty}^{+\infty} |\mathrm{F}(x)|^2\,dx^1 \ldots dx^n < \infty$. Ir follows from (8) that $(\psi, \mathrm{D}_a) = \psi(a^*)$ with $\mathrm{D}_a(z) = (2\sqrt{\pi})^{-n}\exp(-(\mathbf{Z}-\mathbf{a})^2/4)$. The function $\mathrm{D}_a(z)$ plays now the part of the Dirac symbolic function.

The x^j and z^j are dimensionless numbers. In order to identify L_3 with the Heisenberg quasi-algebra of the coordinate and momenta operators of a particle, it is necessary to introduce a constant λ with the dimension of a length, such that the $\lambda\,q^j$ be the coordinate operators and the $\lambda^{-1}\,p_j$ the momentum ones. It is obvious that the smoothing out involved in the transformation $\mathrm{F} \longrightarrow \psi$ is an averaging over a region of the x-space of the order of λ^3. Let φ and G denote the Fourier transforms of ψ and F, respectively. It follows from (5) that $\varphi(w) = (2\pi)^{-n/2}\exp-(\mathbf{W}^2/2)\,G(w)$. *We have a gaussian cut-off of the momenta at the value λ^{-1} in the transformation $\mathrm{F} \longrightarrow \psi$.*

We have $[d_j, \{\mathbf{I}^k\}] = \delta^k_j\,1_{L_n}$ with $\sqrt{2}\,d_j = \{\mathbf{I}_j\} - \{\mathbf{I}^j\}$ for the euclidean metric $g_{jk} = \delta_{j,k}$. Thereby $d_j\{\psi\}$ is associated to the corresponding derivatives $\frac{\partial \mathrm{F}}{\partial x^j}$ and $\frac{\partial \psi}{\partial z^j}$. This is of course a consequence of the fact of the Gauss-Weierstrass transformation being of the convolution kind. *When* F(x) *satisfies a linear differential equation with constant coefficients,* $\psi(x)$ *is also a solution of the same equation. This happens when* F(x) *is a free particle wave-function.* The situation is essentially different when F(x) satisfies a linear differential equation with variable coefficients, as shown by the equation of the isotropic harmonic oscillator: $i \frac{\partial}{\partial t} F(t, x) = \frac{1}{2} (\mathbf{x}^2 - \Delta - n) F(t, x)$ corresponds to $i \frac{\partial}{\partial t} \psi(t, z) = (\mathbf{z}, \partial_z) \psi(t, z)$, ∂_z denoting the gradient with respect to $\mathbf{z}$.

We have considered until now the case of the euclidean metric $g_{jk} = d_{j,k}$. In the case of the space-time $g_{44} = -1$, so that $\sqrt{2}\, q^4 = \{\mathbf{I}^4\} - \{\mathbf{I}_4\}$ We showed in section 8b that $\{\Psi\} = \Psi\{(\mathbf{I})\}\{P\} = F(\hat{q})\{P\}$, with $\hat{q}^j = q^j$ for $j = 1,2,3$ and $\hat{q}^4 = i\, q^4$, $F(\hat{q}) = \pi^{n/4} \exp(g_{jk}\, q^j q^k/2)\, F(\hat{q})$. It is easily seen that

$$\Psi(z^1, z^2, z^3, -iz^4) = \mathcal{G}^{1/2}_{z/\sqrt{2}}[F] \tag{10}$$

We shall now see that the Gauss-Weierstrass transformation comes in in another remarkable way in the theory of L_n. We have

$$\prod_j (d_j)^{r_j} \{P\} = \pi^{-n/4} \prod_j (d_j)^{r_j} \exp(-\tfrac{1}{2}\, \mathbf{q}^2)\{P\} = \prod_j 2^{-r_j/2} (-1)^{r_j} H_{r_j}(q^j/\sqrt{2})\{P\} \tag{11}$$

as a consequence of the definition of the Hermite polynomials. Let us write

$$i\, q^j = \sqrt{2}\, d_j, \; i d_j = q^j/\sqrt{2}, \; [d_j, q^k] = \delta^k_j\, 1_{L_n} \tag{12}$$

It follows from (11) that

$$\prod_j (q^j/2)^{r_j}\{P\} = \prod_j (2i)^{-r_j} H_{r_j}(id_j)\{P\} \tag{13}$$

Here comes in again the Gauss-Weierstrass transformation, because $\prod_j (2i)^{-r_j} H_{r_j}(iy^j) = \mathcal{G}^{1/2}_y \left[\prod_j (x^j)^{r_j}\right]$.

9a. *The Gauss-Weierstrass transformation is closely related to the gaussian group of* L_n *discussed in section 8.*

$$\exp(-\tfrac{1}{2}\, \alpha\, \mathbf{p}^2)\, F(q)\{P\} = \mathcal{G}^{\alpha}_q[F]\{P\} \tag{1}$$

This follows from the well known fact that $\frac{\partial}{\partial \alpha} \mathcal{G}^{\alpha}_x[F] = \frac{1}{2} \Delta\, \mathcal{G}^{\alpha}_x[F], \mathcal{G}^{0}_x[F] = F(x)$ and the equation

$$[F(q), \mathbf{p}^2] = \Delta F(q) + 2 \sum_j [d_j, F]\, d_j \tag{2}$$

We may define the Gauss-Weierstrass transform of index α *of any* $\{\Psi\}$ *as* $\exp(-\frac{1}{2}\, \alpha\, \mathbf{p}^2)\{\Psi\}$ *for all the complex values of* α *such that the real part of* α *be larger than* $-|\alpha|^2$, *because the G. W. transforms of all the Hermite functions exist for those values of the index* α. *The imaginary values of* α *are particularly interesting because they correspond to the* *unitary transformations defined by the Schrödinger equation for a free particle, in the case of $n=3$.*

The present approach to the Gauss-Weierstrass transformation can be extended to a very general class of convolution transforms. Let $K(x)$ be a function whose Fourier transform $\underset{\sim}{K}$ is integral analytic. Since $\int_{-\infty}^{+\infty} K(x-u)\, F(x)\, dx^1 \ldots dx^n = \int_{-\infty}^{+\infty} K(x) \exp((\mathbf{X}, \partial_\mu)\, d\mathbf{X}\, F(u) = (2\pi)^{n/2} \cdot$ $\cdot \underset{\sim}{K}(i\, \partial_\mu)\, F(u)$, the K-transform of $\{\Psi\} = F(q)\{P\}$ can be defined as $(2\pi)^{n/2} \cdot$ $\cdot \underset{\sim}{K}(-p)\{\Psi\}$. The above treatment is also applicable when $\underset{\sim}{K}$ can be expanded in Hermite functions, or associated to a formal series of Hermite functions. Since $\{\Psi\} = \underset{\sim}{F}(p)\{\underset{\sim}{P}\}$, the K-transform of $\{\Psi\}$ exists when $\underset{\sim}{K}(-p)\, \underset{\sim}{F}(p)$ can be expressed as a formal series of Hermite functions of the p_j.

We can generalize the convolution transforms by considering formal series $\underset{\sim}{K}(-x)$ which are not Fourier transforms of true functions, $(2\pi)^{n/2} \cdot$ $\cdot \underset{\sim}{K}(-p)\{\Psi\}$ being taken as the transform of $\{\Psi\}$. This generalization is particularly interesting for the treatment of linear hyperbolic partial differential equations with constant coefficients. The Gauss-Weierstrass transformation is the convolution transformation associated to the parabolic heat equation in the case of an infinite medium. When it is taken for imaginary values of the index, the associated equation is the Schrödinger equation for a free n-dimensional particle.

Metric Construction of L_n

10. We introduced L_n as an affine vector calculus dealing with both the contravariant and covariant vectors. In the metric geometry it is possible to obtain the covariant vectors from the contravariant ones by means of the metric tensor g_{jk}: $U_j = g_{jk} V^j$. Thereby it should be possible to use only generators corresponding to contravariant vectors in the construction of the vector calculi associated to the metric geometry. This happens indeed in the case of the Clifford algebra C_n, whose generators can be taken as the unity 1_{C_n} and the γ_j, which correspond to the basic contravariant vectors $\mathbf{I}_j$. The elements $c\, 1_{C_n}$ correspond to the scalars; 1_{C_n} is therefore a generator associated to the scalars. *It follows from the equations* (8a–2) *that* L_n *is generated by the* p_j and $\{P\}$. The p_j may be considered as the elements associated to the basic vectors $\mathbf{I}_j$, the element associated to $\mathbf{V}$ being taken as $p_V = V^j p_j$; the $c\{P\}$ are elements associated to the scalars. We can take now as basic multiplication rules of L_n

$$[p_j, p_k] = 0\ ,\ \{P_{j_1, \ldots, j_p}\}\{P^{j'_1, \ldots, j'_q}\} = \delta_{p,q}\, \delta_{r_1, r'_1} \ldots \delta_{r_n, r'_n} \{P\} \tag{1}$$

with the definitions

$$\{P^{j_1, \ldots, j_p}\} = \prod_k (2^{r_k}(r_k!))^{-1/2}\, i^{-r_k}\, H_{r_k}(p_k)\{P\} \tag{2}$$

$$\{P_{j_1, \ldots, j_p}\} = \{P\} \prod_k (2^{r_k}(r_k!))^{-1/2}\, i^{r_k}\, H_{r_k}(p_k) \tag{3}$$

Equations (2) and (3) are obviously equivalent to the equations (8a–2). The matrix units of L_n are the $\{P^{j_1,\dots,j_p}_{k_1,\dots,k_q}\} = \{P^{j_1,\dots,j_p}\}\{P_{k_1,\dots,k_q}\}$. Thus we have completed the construction of L_n as the quasi-algebra of the matrices of the linear space of the $\{\Psi\}$. There is however still an important point to be examined. Let us denote by $\{\Pi_q\}$ the sum of all the distinct $\{P^{j_1,\dots,j_q}_{j_1,\dots,j_q}\}$. The $\{\Pi_q\}$ are orthogonal and idempotent; the element $\sum_q^{0,\dots,\infty} \{\Pi_q\}$ is idempotent and commutable with the generators p_j and $\{P\}$. *We may therefore impose the further conditions*

$$p_j \sum_q \{\Pi_q\} = \sum_q \{\Pi_q\}\, p_j = p_j \tag{4}$$

and take $1_{L_n} = \sum_q \{\Pi_q\}$. *It is important to note that in the construction of* L_n *by means of the* $\{\mathbf{l}_j\}, \{\mathbf{l}^j\}, \{P\}$ *and* 1_{L_n} *it is also necessary to impose the condition* $1_{L_n} = \sum_q \{\Pi_q\}$.

We can now define the $\{\mathbf{l}_j\}$ and $\{\mathbf{l}^j\}$ and take $\sqrt{2}\, q^j = \{\mathbf{l}_j\} + \{\mathbf{l}^j\}$

$$\{\mathbf{l}_k\} = \sum_{p,j} (p!)\,(r_1!\dots r_n!)^{-1}\sqrt{r_k+1}\,\{P^{j_1,\dots,j_p}_{k,j_1,\dots,j_p}\} \tag{5a}$$

$$\{\mathbf{l}^k\} = \sum_{p,j} (p!)\,(r_1!\dots r_n!)^{-1}\sqrt{r_k+1}\,\{P^{k,j_1,\dots,j_p}_{j_1,\dots,j_p}\} \tag{5b}$$

It is easily seen that the expression (1–8) of the matrix units $\{P^{j_1,\dots,j_p}_{k_1,\dots,k_q}\}$ in terms of the $\{\mathbf{l}\}$ is a consequence of equations (1) and (5). The approach of section 1 has the advantage of showing directly the affine covariance of the quasi-algebra L_n. The equations (2) and (3) are covariant for orthogonal linear transformations of the p_j, as a consequence of the addition theorem (8a–4) of the Hermite polynomials, $\{P\}$ being assumed as invariant for such transformations. This orthogonal covariance is an immediate consequence of the structure of the generating function of the Hermite polynomials

$$\exp(\mathbf{y}^2 - 2i(\mathbf{x},\mathbf{y})) = \sum_r \prod_k H_{r_k}(x^k)\,(-iy^k)^{r_k}/(r_k!) \tag{6}$$

which shows that the expression $\sum_j S_{j_1,\dots,j_p} \prod_k (r_k!)^{-1} H_{r_k}(x^k)$ is invariant for linear orthogonal transformations, when S is a symmetric tensor.

It is convenient to avoid the use of complex numbers in the above metric construction of L_n. This can be easily done by taking as generators the d_j and $\{P\}$, instead of the p_j and $\{P\}$. The association of the element $V^j d_j$ to the vector $\mathbf{V}$ is actually more natural than that of p_V, as shown by the consideration of the infinitesimal translations. Equation (2) may be written as follows

$$\{P^{j_1,\dots,j_p}\} = \prod_k (2^{r_k}(r_k!))^{-\frac{1}{2}}\,\hat{H}_{r_k}(d_k)\{P\}, \quad \hat{H}_r(u) = i^{-r} H_r(-iu) \tag{7}$$

The $\hat{H}_r$ are polynomials with real coefficients

$$\hat{H}_r(u) = (-1)^r \exp(-u^2)\,\frac{d^r}{du^r}\exp(u^2) \tag{8}$$

Equation (3) may be written as follows

$$\{P_{j_1,\ldots,j_p}\} = \{P\} \prod_k (2^{r_k}(r_k!))^{-\frac{1}{2}} \hat{H}_{r_k}(-d_k) \tag{9}$$

The metric construction of L_n can be done in a more elegant way by using, instead of $\{P\}$, the generator $\{\underset{\sim}{R}\} = \{\underset{\sim}{P}\}\{\underset{\sim}{P}\}^\dagger = \pi^{n/2} \exp(-\mathbf{d}^2/2)\{P\}\cdot$ $\cdot \exp(-\mathbf{d}^2/2)$, $\mathbf{d}$ denoting the symbolic vector of components d_j. We can take as multiplication rules

$$[p_j, p_k] = 0 , \quad \{\underset{\sim}{R}\} F_1(p)F_2(p)\{\underset{\sim}{R}\} = \int_{-\infty}^{+\infty} F_1(x)\, F_2(x)\, dx^1 \ldots dx^n \{\underset{\sim}{R}\} \tag{10}$$

The second equation (1) follows immediately from the second, equation (10) by taking

$$F_1(x) = \prod_k (2^{r_k}(r_k!))^{-\frac{1}{2}} i^{r_k} H_{r_k}(x^k) , F_2(x) = \prod_k (2^{r'_k}(r'_k!))^{-\frac{1}{2}} i^{-r'_k} H_{r'_k}(x^k)$$

with the definitions (2)–(3) and $\{P\} = \pi^{-n/2} \exp(-\mathbf{p}^2/2)\{\underset{\sim}{R}\}\exp(-\mathbf{p}^2/2)$. *The present approach has the advantage of showing explicitly the basic rôle of the generalized kind of integration involved in the theory of L_n, as well as that of the functions of non-numerical commutative variables.* When one or both the F in the second equation (10) are non-convergent formal series of Hermite functions, the integral in the right-hand side is defined as the numerical series obtained by integrating term by term the formal series $F_1(x)\, F_2(x)$. We must of course impose also the conditions (4) in the present approach to the construction of L_n.

The integration in the second equation (10) is a generalization of the inner product of vectors $\mathbf{V}$. Let us write $[\mathbf{V}] = \sqrt{2}\, \pi^{-n/4} \exp(-\mathbf{p}^2/2)\, p_V$; it follows from (10) that

$$\{\underset{\sim}{R}\}[\mathbf{V}_1][\mathbf{V}_2]\{\underset{\sim}{R}\} = \sum_j \mathbf{V}_1^j \mathbf{V}_2^j \{R\} \tag{11}$$

We may associate to the symmetric tensor $T^{j_1,\ldots,j_p}$ the element $[T]$

$$[T] = \sum_j (p!)^{-1}\, T^{j_1,\ldots,j_p} \prod_k (r_k!)\, h_{r_k}(p_k) \tag{12}$$

the h_r denoting the normalized Hermite functions. We have $\{\underset{\sim}{R}\}[T_1][T_2]\{\underset{\sim}{R}\} =$ $= \sum_j (p!)^{-1} (r_1! \ldots r_n!)\, T_1^{j_1,\ldots,j_p}\, T_2^{j_1,\ldots,j_p} \{R\}$, T_1 and T_2 being two symmetric tensors of the same order p. F(p) is a sum of elements [T] for $p = 0, 1, 2, \ldots$. It is interesting to note that a scalar T is associated to $T \prod_k h_0(p_k)$, i.e. to the gaussian function $\pi^{-n/4}\, T \exp(-\mathbf{p}^2/2)$.

L_n as a Theory of Distributions

11. In the metric construction of L_n we may use as generators the q^j and $\{R\}$

$$\{R\} = \{P\}\{P\}^\dagger = \pi^{n/2} \exp(\mathbf{q}^2/2)\{P\}\exp(\mathbf{q}^2/2), \quad \{R\}^\dagger = \{R\} \tag{1}$$

instead of the p_j and $\{\underset{\sim}{R}\}$, the multiplication rules being now

$$[q^j, q^k] = 0, \quad \{R\} F_1(q)\, F_2(q) \{R\} = \int_{-\infty}^{+\infty} F_1(x)\, F_2(x)\, d\mathbf{X} \{R\} \tag{2}$$

The two approaches to the metrical construction of L_n are related by the automorphism $\underline{\Lambda} \longrightarrow \exp(i\,\pi/2\{\delta\}\; \underline{\Lambda}\; \exp(-i\,\pi/2\{\delta\})$, discussed in section 8 in connection with the Fourier transformation, because

$$p_j = \exp(i\pi/2\{\delta\})\, q^j \exp(-i\pi/2\{\delta\}); \; \{\underset{\sim}{R}\} = \exp(i\pi/2\{\delta\}) \{R\} \exp(-i\,\pi/2)\{\delta\}) \tag{3}$$

Since $p_j\{R\} = \{R\}\, p_j = 0$, we have $F(p)\{R\} = \{R\}\, F(p) = F(0)\{R\}$. In particular

$$\exp(i\, p_V)\{R\} \exp(-i\, p_V) = \{R\}, \; \exp(c\, \mathbf{p}^2)\; \{R\} \exp(-c\, \mathbf{p}^2) = \{R\} \tag{4}$$

$\{R\}$ *is invariant for the translation and gaussian automorphisms.*

We shall now introduce an element $\{Q\}$

$$\{Q\} = (2\pi)^{-n/2}\, \{P\}\; \{\underset{\sim}{P}\}^\dagger = 2^{-n/2} \exp(\mathbf{q}^2/2)\{P\} \exp(\mathbf{p}^2/2) \tag{5}$$

$\{Q\}$ has the following properties $p_j\{Q\} = 0, \{Q\}q^j = 0$ and $\{Q\}^2 = \{Q\}$. $\{Q\}$ *is related to the* d_j, p^j, *in the same way as* $\{P\}$ *to the* $\{\mathbf{l}_j\}, \{\mathbf{l}^j\}$. *This is still true in the case of pseudo-euclidean metrics if* $\{Q\}$ *be defined as follows*

$$\{Q\} = 2^{-n/2} \exp(g_{jk} q^j\, q^k/2)\; \{P\} \exp(g^{jk}\, p_j\, p_k/2) \tag{6}$$

with $\sqrt{2}\, q^j = \{\mathbf{l}^j\} + g^{jk}\{\mathbf{l}_k\}$, $\sqrt{2}\; d_j = \{\mathbf{l}_j\} - g_{jk}\{\mathbf{l}^k\}$, $p_j = -i\, d_j$ We have

$$\exp(i\, p_V)\{Q\} \exp(-i\, p_V) = \{Q\} \exp(-i\, p_V), \; F(p)\{Q\} = F(0)\{Q\} \tag{7}$$

$$\{Q\}\, F(q) = F(0)\{Q\}, \; \{Q\} \exp(i\, p_V)\; F(q) = F(V)\{Q\} \exp(i\, p_V) \tag{8}$$

In section 7 we associated the element $c\{P\} \exp(\{\mathbf{V}\})$ to the point of coordinates V^j, endowed with a weight c. Thus we obtained an affine calculus of points associated to L_n. *In order to obtain a metric calculus of weighted points, we shall associate the element* $c\{Q\} \exp(i\, p_V)$ *to the point of coordinates* V^j, *endowed with the weight* c. More generally, the element $\sum_r c_r\{Q_{V_r}\}$, with $\{Q_V\} = \{Q\} \exp(i\, p_V)$, corresponds to the system of weights c_r localized at the points of coordinates V_r^j. In order to describe continuous distributions we shall define $\int_{-\infty}^{+\infty} c(V) \exp(i\, p_j\, V^j)\, d\mathbf{V}$ as follows

$$\int_{-\infty}^{+\infty} c(V) \exp(ip_j\, V^j)\, d\mathbf{V} = (2\pi)^{n/2} \sum_r \prod_k i^{r_k} h_{r_k}(p_k) \int_{-\infty}^{+\infty} c(V)\, h_{r_k}(V^k)\, d\mathbf{V} \tag{9}$$

because $\exp(i\, p_V) = (2\pi)^{n/2} \sum_r \prod_k i^{r_k} h_{r_k}(V^k)\, h_{r_k}(p_k)$. *The weight distribution of density* $c(V)$ *will be associated to* $\{Q\} \int_{-\infty}^{+\infty} c(V) \exp(i\, p_V)\, d\mathbf{V}$.

We showed in section 8a that $\underset{\sim}{F}(p)\{\underset{\sim}{P}\} = F(q)\{P\}$, $\underset{\sim}{F}$ being the Fourier transform of F. Hence $\{\underset{\sim}{P}\}^{\dagger}(\underset{\sim}{F}(p))^{\dagger} = \{\underset{\sim}{P}\}^{\dagger} F^{*}(q)$, and by taking into account that the Fourier transform of the function $F^{*}(x)$ is $(\underset{\sim}{F}(-y))^{*}$, we get

$$\{Q\}\, \underset{\sim}{F}(p) = (2\pi)^{-r/2} \{R\}\, F(-q) \tag{10}$$

In particular

$$\{Q_V\} = \{R\}\, \delta(\mathbf{q} - \mathbf{V}) \tag{11}$$

It follows from (10) that

$$\{Q\} \int_{-\infty}^{+\infty} c(V) \exp(i\, p_V)\, d\mathbf{V} = \{R\}\, c(q) \tag{12}$$

The element $\{R\}$ c(q) *describes a weight-distribution of density* c(x). It is important to notice that c(q) exists even when c(x) is no true function. In particular c(q) may describe a weighted point, as in equation (11), or a point-multipole, the corresponding c(q) being of the form $\alpha \prod_j d_j^{r_j} \delta(\mathbf{q} - \mathbf{V})$. It is well known that the Schwarz theory of distributions was developed as a method to deal with subtle distributions not describable by ordinary functions. The geometric quasi-algebra L_n gives an alternative method for the theory of such subtle distributions, not entirely equivalent to the Scwarz theory.

Equation (12) shows that the association of $c\{Q\}\exp(i\, p_V)$ to the point of coordinates V^j, endowed with the weight c, leads to satisfactory results. It is also possible to associate to the weight distribution $c(V)$ the element $\int_{-\infty}^{+\infty} c(V) \exp(-i p_V)\, d\mathbf{V} \{Q\}^{\dagger} = c(q)\, \{R\}$. We have been using the linear space of the $\{\Psi\} = F(q)\{\underset{\sim}{P}\}$ to represent the functions or formal series of Hermite functions F(x). We may as well use the linear space of the $\{\Psi\}\,\{P\}^{\dagger} = F(q)\{R\}$, whose elements have a simple interpretation as weight distributions.

The definition (1) of $\{R\}$ can be generalized as follows

$$\{R\} = \pi^{n/2} \exp(g_{jk}\, q^j q^k/2)\, \{P\}\, \exp(g_{jk}\, q^j q^k/2) \tag{13}$$

for any flat space. It follows from equations (8b–3) that

$$\{R\} = \{P\}\,\{P\}^{\dagger},\ \{Q\} = (2\pi)^{-n/2}\, \{P\}\,\{\underset{\sim}{P}\}^{\dagger} \tag{14}$$

with the general definitions of $\{P\}$ and $\{Q\}$, the adjunction being the covariant one of section 8c, which renders the p_j and q^j self-adjoint for any metric of the flat space; $\{R\}^{\dagger} = \{R\}$, with the covariant adjunction. It follows from (13) that $p_j\{R\} = \{R\}\, p_j = 0$, with $i\sqrt{2}\, p_j = \{\mathbf{l}_j\} - g_{jk}\{\mathbf{l}^k\}$. $\{R\}$ is therefore always invariant for the translation and gaussian automorphisms

$$\exp(i\, p_V)\{R\}\exp(-i\, p_V) = \{R\},\ \exp(c\, g^{jk} p_j p_k)\{R\}\exp(-c\, g^{jk} p_j p_k) = \{R\} \tag{15}$$

Although we can define $\{R\}$ *and* $\{Q\}$ *for a pseudo-euclidean space, the above calculus of weighted points can not be extended in a* *straightforward way to such spaces.* The basic equation (10) must be modified, when the metric is indefinite. It is easily seen that, with $g_{jk} = \varepsilon_j \delta_{j,k}$, $\varepsilon_j = \pm 1$, we have

$$\{Q\} \underset{\sim}{F}(\hat{p}) = (2\pi)^{-n/2} \{R\} F(-\hat{q}) \tag{16}$$

with $\hat{q}^j = \sqrt{\varepsilon_j}\, q^j$, $\hat{p}_j = p_j/\sqrt{\varepsilon_j}$. Instead of (11), we have now

$$\{Q\} \exp(i\ \hat{p}_j V^j) = \{R\}\ \delta(\hat{q} - V) \tag{17}$$

The second equation (2) is no more valid with the pseudo-euclidean metrics, being replaced by the following one

$$\{R\} F^{I}(\hat{q}) F^{II}(\hat{q}) \{R\} = \sum_r c^{I}_{r_1,\dots,r_n} c^{II}_{r_1,\dots,r_n} \{R\} = \int_{-\infty}^{+\infty} F^{I}(x) F^{II}(x)\, dX \{R\} \tag{18}$$

The indefinite metric of the x*-space is associated to a definite metric in the linear space of the functions* F(x).

The element $\{Q\}$ is idempotent for euclidean and pseudo-euclidean metrics and $p_j \{Q\} = 0, \{Q\} q^j = 0$. We can take the q^j, p_j and $\{Q\}$ as generators of L_n. *These generators have the same multiplication rules as the* $\{\mathsf{l}^j\}, \{\mathsf{l}_j\}$ *and* (P). The new choice of generators is advantageous in some cases because of the covariant self-adjointness of the q and p. The equation $\{Q\}^2 = \{Q\}$ follows immediately from (18) and the following equation

$$\exp(u^2/2) = \sqrt{2} \sum_r 2^{-2r} (r!)^{-1} H_{2r}(u) \tag{19}$$

which allows to obtain the expansion of 1 in series of Hermite functions.

It follows from (19) and equations (8b–1), (8b–2) that

$$\{P\} = 2^{n/2} \pi^{n/4} \exp(g_{jk} \{\mathsf{l}^j\}\{\mathsf{l}^k\}/2) \{P\}, \{\underset{\sim}{P}\}^{\dagger} = 2^{n/2} \pi^{n/4} \{P\} \exp(-g^{jk}\{\mathsf{l}_j\}\{\mathsf{l}_k\}/2) \tag{20}$$

and

$$\{Q\} = 2^{n/2} \exp(g_{jk}\{\mathsf{l}^j\}\{\mathsf{l}^k\}/2) \{P\} \exp(-g^{jk}\{\mathsf{l}_j\}\{\mathsf{l}_k\}/2) \tag{21}$$

$$\{R\} = 2^{n} \pi^{n/2} \exp(g_{jk}\{\mathsf{l}^j\}\{\mathsf{l}^k\}/2) \{P\} \exp(g^{jk}\{\mathsf{l}_j\}\{\mathsf{l}_k\}/2) \tag{22}$$

11a. In section 7 we discussed an affine calculus of weighted points, in which $\{P\} \exp(\{V\})$ plays a part corresponding to that of $\{Q\} \exp(i\, p_V)$ in the present metric calculus of distributions. It follows from the second equation (8–5) that

$$i\ p_v = \exp(\pi S/4) \{V\} \exp(-\pi S/4) \tag{1}$$

We shall now prove that

$$\{Q\} = \exp(\pi S/4) \{P\} \exp(-\pi S/4) \tag{2}$$

Let us denote by Λ the element in the right-hand side of (2). Λ is of the form A{P}B, A denoting a power series of the $\{\mathsf{l}^j\}$ and B a power series of the $\{\mathsf{l}_j\}$.

It follows from (8–5) and (8–6) that $p_j \Lambda = 0$, $\Lambda q^j = 0$. These relations lead to the following equations for A and B

$$[\{I_j\}, A] = g_{jk}\{I^k\} A, \ [B, \{I^j\}] = -g^{jk} B\{I_k\} \tag{3}$$

which show that $A = c' \exp(g_{jk}\{I^j\}\{I^k\}/2)$, $B = c'' \exp(-g_{jk}\{I_j\}\{I_k\}/2)$ c' and c'' denoting numbers. The product $c' c''$ is determined by the condition $\Lambda^2 = \Lambda$. By taking into account that $\{Q\}^2 = \{Q\}$ and the expression (11–21) of $\{Q\}$, we get $\Lambda = \{Q\}$.

Equations (1) *and* (2) *show that the affine and euclidean metric calculi of weighted points are related by the automorphism* $\underline{\Lambda} \rightarrow \exp(\pi S/4) \cdot \underline{\Lambda} \exp(-\pi S/4)$, *which depends on the metric* g_{jk}. In the affine calculus, we cannot introduce continuous distributions of weight, because in the affine geometry it is not possible to define a density.

Let $\{\Psi\} = \Psi(\{I\})\{P\} = F(q)\{P\}$. The above automorphism transforms $\{\Psi\}$ into $\Psi(q)\{Q\} = \{\Psi\}$. The linear space of the $\{\Psi\}$ can be used as a representation space for L_n, in the same way as the $\{\Psi\}$ – space. The part of the $\{P^{j_1,\dots,j_p}_{k_1,\dots,k_q}\}$ is now played by the $\{Q^{j_1,\dots,j_p}_{k_1,\dots,k_q}\}$

$$\{Q^{j_1,\dots,j_p}_{k_1,\dots,k_q}\} = (r_1!\dots r_n!\, s_1!\dots s_n!)^{-1/2} q^{j_1}\dots q^{j_p}\{Q\} d_{k_1}\dots d_{k_q} \tag{4}$$

We have $\Psi(\{I\})\{P\} = F(q)\{P\}$ with $\Psi(z\sqrt{2}) = \mathcal{G}_z^{1/2}[F]$ (equation (9–4)). The transformation $\{\Psi\} \rightarrow \{\psi\}$ is therefore associated to a Gauss-Weierstrass transformation $F(x) \rightarrow \Psi(x\sqrt{2})$. *Both the Fourier and Gauss-Weierstrass transformations are associated to automorphisms of* L_n *as a quasi-algebra of the metric euclidean geometry.*

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11. We have been interpreting L_n as a quasi-algebra associated to a flat n-dimensional space, the q^j corresponding to cartesian coordinates. We may obviously consider also L_n as an abstract algebraic formalism that can be applied as a kind of generalized mathematical analysis to problems involving functions or symbolic functions of n variables. We shall now discuss the application of the L_n formalism to distributions on a n-dimensional differentiable manifold S, whose points have curvilinear coordinates x^j. We shall use the generators q^j, p_j, $\{Q\}$ with the multiplication rules $[q^j, q^k] = 0$, $[p_j\ p_k] = 0$, $[q^j, p_k] = i\, 1_{L_n}$, $p_j\{Q\} = 0$, $\{Q\}q^j = 0$, $\{Q\}^2 = \{Q\}$. The element Q will be associated, in the present calculus, to the point whose coordinates are all 0. The point of coordinates a^j will be associated to the element $\{Q_a\} = \{Q\}\exp(ia^j p_j)$, with the weight 1; $c\{Q_a\}$ denotes the same point with the weight c. *The indices of the* q^j *and* p_j *do not indicate any tensor law of transformation. In a general differentiable manifold the tensors are localized geometric objects; our q and p are not associated to any point of the manifold S, but to the manifold as a whole.* For the sake of simplicity we shall assume that the x^j vary from $-\infty$ to $+\infty$.

A weight distribution on S is described by a scalar-density $c(x)$ and can be associated to the element $\{Q\} \int_{-\infty}^{+\infty} \exp(i x^j p_j) c(x)\, dx^1 \ldots dx^n = \{R\} c(q)$. Thus the formalism developed for the n-dimensional euclidean space can be applied to the weight distributions on S. *L_n is not being used as a geometrical quasi-algebra, but as an analytical tool, in the same way as mathematical analysis is applied to geometrical problems.* We may say that S is mapped on a n-dimensional euclidean space, and the quasi-algebra L_n of the latter space applied to the description of weight distributions on S.

Instead of using the L_n of an euclidean space, we may proceed in a different way, building a $L_n(S)$ for the manifold itself. The q^j will now be regarded as associated to a coordinate system on S. The change of coordinates $\bar{x}^j = f^j(x)$ will be associated to the change $q^j \longrightarrow \bar{q}^j = f^j(q)$, $p_j \longrightarrow \bar{p}_j = \frac{1}{2}(a_j^k(q)\, p_k + p_k\, a_j^k(q))$ with $a_k^j(x) = \dfrac{\partial x^j}{\partial \bar{x}^k}$. It is now convenient to take as generators the elements q^j, p_j and $\{R\}$ and to represent the transformation of the generators as a contact transformation: $\bar{q}^j = U^{-1} q^j U$, $\bar{p}_j = U^{-1} p_j U$, $\overline{\{R\}} = U^{-1} \{R\} U$. The equation $\{R\}\, F(q)\, \{R\} = \int_{-\infty}^{+\infty} F(x)\, dx^1 \ldots dx^n \{R\}$ corresponds essentially to the second equation (11–2) for an euclidean space. It is transformed by the above automorphism into $\overline{\{R\}} F(\bar{q}) \overline{\{R\}} = \int_{-\infty}^{+\infty} F(x)\, dx^1 \ldots dx^n \overline{\{R\}}$. The quasi-algebra $L_n(S)$ *is interesting, but does not lead to a satisfactory description of the distributions on the general differentiable manifold S.* Such a distribution is described by a scalar density $c(x)$ and can be associated to the element $\{R\}\, c(q)$. The automorphism $\Lambda \longrightarrow U^{-1} \Lambda U$ transforms $\{R\}\, c(q)$ into $\{R\}\, c(\bar{q})$. *This is not satisfactory, because the weight-distribution is described in the new coordinate-system by* $c(x)D$, *D denoting the jacobian of the old coordinates with respect to the new ones.*

The same difficulties discussed above in the extension of L_n to a riemannian manifold appear also in the euclidean spaces, when general transformations of coordinates are allowed. L_n was built as a free-vector quasi-algebra, not as a quasi-algebra of scalar or pseudo-scalar fields. It turned out that, in the case of a flat-space and cartesian reference frames, its algebraic formalism can also be used to describe weight-distributions. The scalar and pseudo-scalar fields are kinds of geometric objects of a more complicated nature than those considered in the preceding sections. *A convenient approach to the geometric quasi-algebras of such objects cannot be found within the frame of the affine geometry. We shall see in section 14 that the quasi-algebras of the scalar-fields are related to the topology of the continuous manifolds.* In the case of pseudo-scalar fields it is of course necessary to consider differentiable manifolds, because of the transformation laws of the pseudo-scalars, which involve the jacobian D.

The geometric quasi-algebras of the scalar-fields are closely related to those of the quantum theory of the scalar-fields, as will be seen in later sections of this paper. The development of the quantum theory led from the Heisenberg quasi-algebra to the introduction of the quantized fields, which give a more satisfactory mathematical description of the particle-wave duality of the quantum mechanics. *We are being led to the quasi-algebras of the quantized fields by the extension of the geometric calculi of affine spaces to more general manifolds.*

The theory of the affine spaces is a convenient starting-point for the discussion of the geometric-algebraic formalisms, by reasons of simplicity. On the other hand, the affine geometry corresponds to a very complex level of the geometric theory, the simplest in principle being topology and non-topological set theory. It is clear that the analysis of the algebraic formalisms of the lower levels of geometry must lead to a more complete understanding of the quantal kind of geometry associated to the geometric algebras. It is thus seen that the simplicity of the affine algebraic formalisms corresponds to a drastic impoverishment of the description of the algebraic-geometric properties of space.

L_n AS A THEORY OF LINEAR FUNCTIONALS

12. The discussion of section 11 shows that L_n may be considered as a metric calculus of points, which associates to a n-dimensional affine space the linear space of infinite dimensionality constituted by the *points* $\{Q_V\}$ and their linear combinations $\int_{-\infty}^{+\infty} \{Q_V\}\ c(V)\ dV$, $c(V)$ denoting in general a formal series of Hermite functions. This linear space is isomorphic to the Wiener differential space associated to the expansions in series of Hermite functions, which is constituted by all the formal series of Hermite functions. The Grassmann calculus of weighted points extends the n-dimensional affine space by embedding it in a (n+1)-dimensional linear space constituted by the weighted points and free contravariant vectors. *The theory of L_n given in section 11 leads to an extension of the euclidean n-dimensional space into a linear space of infinite dimensionality isomorphic to the Wiener space of the $c(V)$.* The Grassmann linearization of the affine space is closely related to the projective geometry; the linearization of section 11 leads to a new kind of geometric theory, closely related to the quantum kinematics. *In the case of the ordinary three-dimensional space, the elements $\{R\}\ \phi^*(q)$ corresponding to normalized functions $\phi(x)$ can be associated to quantal states of a non-relativistic spinless particle, when $\phi(x)$ is everywhere continuous and differentiable up to the second order.*

The elements $F(q)\,\{R\}$ can be associated to the formal series $F(x)$ and the $\{R\}\,\phi^*(q)$ to linear functionals λ_ϕ on the $F(x)$

$$\lambda_\phi\left[F\right] = \int_{-\infty}^{+\infty} F(x)\ \phi^*(x)\ dX \qquad (1)$$

the generalized integral being defined as $\sum_r c_{r_1,\dots,r_n}\ b^*_{r_1,\dots,r_n}$, the b denoting the coefficients of the formal series of Hermite functions ϕ and the c those of F. The second equation (11–2) may be written as follows

$$\{R\}\ \phi^*(q)\ F(q)\{R\} = \lambda_\phi\left[F\right]\{R\} \qquad (2)$$

The element $\{Q_V\}$ corresponding to the point of coordinates V^j is associated to the linear functional λ_V : $\lambda_V\left[F\right] = F(V)$.

To the F(x) correspond linear functionals $F(\lambda_\phi)$ on the λ_ϕ defined as follows

$$F\left[\lambda_\phi\right] = \lambda_\phi\left[F\right] \qquad (3)$$

On replacing the function F(V) by the linear functional $F\left[\lambda_\phi\right]$, *we extend the domain of definition of F(V) from our n-dimensional space into its infinite dimensional extension as a* λ*-space, given by the theory of* L_n. We have $F(V) = F\left[\lambda_V\right]$. It is interesting to note that the series F(x) correspond to functionals $F\left[\lambda_\phi\right]$, whose domains of definition do not include all the λ_V, when the F do not converge for all the x.

(*To be continued*)