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1957
N. 1 del *Supplemento* al Vol. 6, Serie X,
del *Nuovo Cimento* - pag. 356-380

Quantum Kinematics and Geometry.

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1. – Introduction.

The passage from the classical to the quantum mechanics is characterized by the introduction of the Planck constant, which allows to measure linear momenta in units cm^{-1}. In the relativistic quantum mechanics we have also the constant c so that all the physical quantities can be measured by powers of a length, by taking $\hbar = 1$ and $c = 1$. *This remarkable circumstance indicates the existence of a deep unity between physics and geometry, non equivalent to that of the general theory of relativity, since $\hbar$ is involved. The wave-particle duality leads essentially to the identification of the momentum to a geometric covariant vector, the dimension of the components of such a vector being precisely that of the inverse of a length.*

The above dimensional considerations led us to examine the relations between the quantum kinematics and the geometry of the space and the space-time. It is necessary to start from the affine geometry, in order to distinguish clearly the contravariant and covariant vectors. *We needed two kinds of affine vector calculus: one for spin-like discrete variables, another for the continuous position and momentum variables.* GRASSMANN [1] developed two kinds of vector calculus: one with an anticommutative product of contravariant vectors, another with a commutative product. The GRASSMANN calculi were not entirely adequate for our research, we had to extend them by the introduction of symbols for both the contravariant vectors $\boldsymbol{V}$ and covariant vectors $\boldsymbol{U}$. In our paper [2] « On the Clifford and Grassmann algebras I », we discussed a n-dimensional associative vector algebra G_n, in which the symbols $(\boldsymbol{V})$ and $(\boldsymbol{U})$ satisfy the anticommutation rules

$$(1)\qquad \begin{cases} [(\boldsymbol{V}), (\boldsymbol{V}')]_+ = 0\,, & [(\boldsymbol{U}), (\boldsymbol{U}')]_+ = 0\,, \\ [(\boldsymbol{V}), (\boldsymbol{U})]_+ = \langle \boldsymbol{V}, \boldsymbol{U}\rangle 1_{G_n}\,, & \langle \boldsymbol{V}, \boldsymbol{U}\rangle = V^j U_j\,, \end{cases}$$

1_{G_n} denoting the unity of G_n. *G_n is an extension of the anticommutative Grassmann algebra and also of the ordinary vector calculus.* In reference (2) we intro duced also an associative algebra L_n, in which the symbol $\{V\}$ and $\{U\}$ satisfy the following commutation rules

$$(2) \qquad [\{V\}, \{V'\}] = 0\,, \quad [\{U\}, \{U'\}] = 0\,, \quad [\{V\}, \{U\}] = \langle V, U\rangle 1_{L_n}\,;$$

1_{L_n} denoting the unity of L_n. L_n is an extension of the Grassman commutative algebra. *The direct product $G_n \times L_n$ is related to an extension of the ordinary vector analysis.*

We shall denote by I_j's the basic contravariant vectors and by I^j's the basic covariant vectors: $V = V^j I_j$, $U = U_j I^j$. We have the following commutation rules

$$(3) \qquad [(I_j), (I_k)]_+ = 0\,, \quad [(I^j), (I^k)]_+ = 0\,, \quad [(I_j), (I^k)]_+ = \delta_j^k 1_{G_n}\,,$$

$$(4) \qquad [\{I_j\}, \{I_k\}] = 0\,, \quad [\{I^j\}, \{I^k\}] = 0\,, \quad [\{I_j\}, \{I^k\}] = \delta_j^k 1_{L_n}\,.$$

Equations (3) *show that G_n is the Clifford algebra of a $2n$-dimensional pseudo-euclidean space S_{2n} with the metric quadratic form $\frac{1}{2}X^jY_j$, the X's and Y's denoting the $2n$ cartesian co-ordinates of a generic point of S_{2n}.* Since $4aX^jY_j = = \sum (X^j + aY_j)^2 - \sum (X^j - aY_j)^2$, the canonical form of the metric quadratic form of S_{2n} contains n positive and n negative squares. G_n over the complex numbers is the Clifford algebra of a $2n$-euclidean space. G_n is therefore of the order 2^{2n}. L_n is an infinite algebra associated to the symplectic geometry of S_{2n}, as we showed in reference [2].

The metric geometry is a part of the affine geometry. The introduction of the metric g_{jk} distinguishes sub-algebras of G_n: the Clifford algebras associated to the metrics g_{jk} and $-g_{jk}$. Indeed the $\gamma_j^{(+)}$'s and $\gamma_j^{(-)}$'s

$$(5) \qquad \gamma_j^{(\pm)} = (I_j) \pm g_{jk}(I^k)\,, \quad [\gamma_j^{(\pm)}, \gamma_k^{(\pm)}]_+ = \pm 2 g_{jk} 1_{G_n}, \quad [\gamma_j^{(+)}, \gamma_k^{(-)}]_+ = 0$$

generate two Clifford algebras. *The theory of the spinors of the n-dimensional spaces is included in the theory of G_n.* Let us introduce the elements q^j and d_j of L_n

$$(6) \qquad \sqrt{2}\, q^j = \{I^j\} + g^{jk}\{I_k\}\,, \qquad \sqrt{2}\, d_j = \{I_j\} - g_{jk}\{I^k\}\,.$$

Since

$$(7) \qquad [q^j, q^k] = 0\,, \quad [d_j, d_k] = 0\,, \quad [d_j, q^k] = \delta_j^k 1_{L_n}\,,$$

the commutation rules of the q's and d's are similar to those of the multipliers x^j and the derivatives D_j. *L_n is closely related to the differential calculus and* *to the Heisenberg quasi-algebra of the position and momentum operators: we can identify the λq^j's with the operators for the cartesian co-ordinates and the $-i\hbar\lambda^{-1}d_j$'s with the operators for the components of the linear momentum of a particle in the case of L_3, λ being a numerical constant with the dimension of a length. We shall take both λ and $\hbar$ equal to 1. The introduction of a fundamental length λ allows a geometrization of the quantum kinematics.*

Let us take $g_{jk} = \varepsilon_j \delta_{j,k}$ in equations (5), with $\varepsilon_j = \pm 1$. The element $\alpha \pm \gamma_1^{(+)} \ldots \gamma_n^{(+)} \gamma_1^{(-)} \ldots \gamma_n^{(-)}$ anticommutes with all the $\gamma^{(\pm)}$'s and $\alpha^2 = 1_{G_n}$. The elements $\alpha\gamma_j^{(-)}$ generate a Clifford algebra C_n' corresponding to the metric g_{jk}. All the elements of the algebra generated by the $\gamma^{(+)}$'s are commutable with those of C_n'. When n is even, the $\gamma_j^{(+)}$'s and $\alpha\gamma_j^{(-)}$'s are a set of generators of G_n, which is the direct product of two Clifford algebras corresponding to the same metric. *The consideration of the affine geometry of the space-time gives us two commutable algebras of Dirac γ-matrices, which may be associated with two different kinds of particles of spin $\frac{1}{2}$. G_4 contains as sub-algebras the Duffin-Kemmer-Petiau algebras for the particles with spin 0 and 1, as will be shown in Sect.* **2**. The algebra G_n is a straightforward extension of the vector calculus of the affine geometry of the n-dimensional space. Thus we see that the algebra for the spins 0, $\frac{1}{2}$ and 1 is the complete form of the ordinary affine vector calculus of the space-time.

The linear space of the contravariant vectors of S_{2n} is the direct sum of the linear spaces of the contravariant and covariant vectors of the associate n-dimensional affine space. S_6 is therefore analogous to the phase-space of a particle. By taking $\hbar = 1$, the dimension of a geometric covariant vector becomes the same as that of linear momentum; thereby we may identify S_6 with the phase-space of a particle. In a similar way we can identify S_8 with a kind of relativistic phase-space including different rest-masses. G_n is a Clifford algebra of S_{2n}; L_n is an algebra of the symplectic group of S_{2n}, as we showed in reference [2]. *The theory of G_4 and L_4 leads to the introduction of two new groups in the quantum mechanics: the group of the linear transformations of the 8 variables X^j, Y_j which leave invariant the symmetric bilinear form $X^jY_j + X^jY_j$ (28 parameters); the group of the linear transformations which leave invariant the skew bilinear form $X^jY_j - X^jY_j$, the symplectic group of S_8 (36 parameters).* These *linear transformations must involve a length, because of the different dimensions of the X's and Y's.* It is interesting to notice that the reduction of the indefinite metric quadratic form X^jY_j of S_8 to the ordinary canonical form $(4a)^{-1}\sum_j\{(X^j + aY_j)^2 - (X^j - aY_j)^2\}$ requires the introduction of a constant a with the dimension of the square of a length.

The introduction of the indefinite metric X^jY_j in S_{2n} does not depend on any choice of the n-dimensional metric. Equations (5) show that G_n is the Clifford algebra of *a* $2n$-dimensional space corresponding to the metric $g_{jk}x^jx^k - g^{jk}y_jy_k$, whose canonical form has also n positive and n negative squares. We can identify these x^j's with components of contravariant vectors of the n-dimensional space and the y_j's with components of covariant vectors, the x's and y's having the same dimension. In order to identify the y's with components of geometric covariant vectors, it is necessary to introduce again a constant multiplicative factor with the dimension of the square of a length.

2. – The spin algebra G_4.

The general form of the elements Γ of G_n is

$$\Gamma = \sum_{p,q}^{0,\dots,n} (p!\,q!)^{-1}\, C^{k_1\dots k_q}_{j_1\dots j_p}(\boldsymbol{I}^{j_1})\dots(\boldsymbol{I}^{j_p})(\boldsymbol{I}_{k_q})\dots(\boldsymbol{I}_{k_1}), \tag{1}$$

the C's being numerical coefficients antisymmetric with respect to the j's and k's separately. The C's transform as the components of tensors with the same indices for a change of the basic vectors $\boldsymbol{I}_j$. *G_n is a geometric calculus for the affine objects of the n-dimensional space described by sets of antisymmetric tensors $C^{k_1\dots k_q}_{j_1\dots j_p}$.*

The commutable idempotent elements $(N_j) = (\boldsymbol{I}^j)(\boldsymbol{I}_j)$ and $(\overline{N}_j) = (\boldsymbol{I}_j)(\boldsymbol{I}^j)$ play a central role in the theory of G_n. Let us introduce the elements $(P^{k_1\dots k_q}_{j_1\dots j_p})$

$$(P^{k_1\dots k_q}_{j_1\dots j_p}) = (\boldsymbol{I}^{k_1})\dots(\boldsymbol{I}^{k_q})(P)(\boldsymbol{I}_{j_p})\dots(\boldsymbol{I}_{j_1}), \qquad (P) = (\overline{N}_1)\dots(\overline{N}_n). \tag{2}$$

It is easily seen that $(\boldsymbol{I}_j)(P) = (P)(\boldsymbol{I}^j) = 0$ and

$$(P^{k_1\dots k_q}_{j_1\dots j_p})(P^{i_1\dots i_r}_{h_1\dots h_s}) = \delta_{p,r}\,\delta^{i_1\dots i_p}_{j_1\dots j_p}(P^{k_1\dots k_q}_{h_1\dots h_s}). \tag{3}$$

It follows from (3) that the 2^{2n} elements $(P^{k_1\dots k_q}_{j_1\dots j_p})$ with $p, q = 0, \dots, n$ and $j_1 < \dots < j_p$, $k_1 < \dots < k_q$ are linearly independent. Any Γ can be expressed as a linear combination of those 2^{2n} elements

$$\Gamma = \sum_{p,q} (p!\,q!)^{-1}\, A^{j_1\dots j_p}_{k_1\dots k_q}(P^{k_1\dots k_q}_{j_1\dots j_p}). \tag{4}$$

By taking the A's antisymmetrical with respect the j's and k's, separately, they transform as components of tensors with the same indices for any change of the $\boldsymbol{I}_j$'s. *Equation* (4) *shows that G_n is a total matrix algebra.* The left-multiplication of any element (Ψ) of $G_n(P)$ by a Γ is another (Ψ). G_n is therefore equivalent to the algebra of the linear transformation of the (Ψ)'s. *The (Ψ)'s may be considered as the contravariant spinors of the 2n-dimensional pseudo-euclidean space S_{2n} having G_n as its Clifford algebra; the covariant spinors of S_{2n} are the elements of the linear space $(P)G_n$.*

The invariant idempotent orthogonal elements $(\Pi_p) = (p!)^{-1} \sum_j (P^{j_1,\dots,j_p}_{j_1,\dots,j_p})$ are important, because $(\Pi_p)G_n(P)$ is equivalent to the linear space of the covariant antisymmetric tensors $A_{j_1,\dots,j_p}$ of the n-dimensional affine space. We have $\sum_p^{0,\dots,n} (\Pi_p) = 1_{G_n}$, therefore the spinor (Ψ) of S_{2n} is the sum of the elements $(\Pi_r)(\Psi)$ corresponding to antisymmetric covariant tensors of our affine space. *Thus we see how the spinors of a 2n-dimensional space are built with covariant antisymmetric tensors of a n-dimensional affine space.*

G_n over the complex numbers is the n-dimensional analogue of the Jordan-Wigner algebra of the emission and absorption operators of the second quantization for fermions. (P) corresponds to the projector of the vacuum, when $(\boldsymbol{I}_j)$ is assimilated to an absorption operator and $(\boldsymbol{I}^j)$ to an emission operator. (Ψ) corresponds to a state-vector of the fermion field. The (N_j)'s correspond to the occupation-number operators. The operator for the total number of particles corresponds to the element $(N) = \sum (N_j)$ associated to the tensor δ^k_j by the equation (1).

The two kinds of contravariant half-spinors of S_{2n} are the elements $(\Pi_\pm)(\Psi)$, (Π_+) denoting the sum of the (Π_p)'s corresponding to even values of p and (Π_-) the sum of those corresponding to odd p's. The element $(\omega) = (\Pi_+)-(\Pi_-)$ plays an important rôle. Since $(\Pi_p)(\boldsymbol{I}_j) = (\boldsymbol{I}_j)(\Pi_{p+1})$ and $(\boldsymbol{I}^j)(\Pi_p) = (\Pi_{p+1})(\boldsymbol{I}^j)$, $(\omega)(\boldsymbol{I})(\omega) = -(\boldsymbol{I})$: $\Gamma \to (\omega)\Gamma(\omega)$ is the automorphism of G_n induced by the reflection $\boldsymbol{V} \to -\boldsymbol{V}$ of the n-dimensional affine space. The two kinds of covariant half-spinors of S_{2n} are the elements of the linear space $(P)G_n(\Pi_+)$ and $(P)G_n(\Pi_-)$.

The elements $(\boldsymbol{I}^j)(\Pi_p)$, $(\Pi_p)(\boldsymbol{I}_j)$ and $(\Pi_p)+(\Pi_{p+1})$ generate a sub-algebra $D_{n,p}$. We shall write $(\Pi_{-1}) = (\Pi_{n+1}) = 0$, in order to extend the definition of D_n to $p = -1$ and $p = n$. *When n is even, all the $D_{n,p}$'s are irreducible Duffin-Kemmer-Petiau algebras; all the irreducible Duffin-Kemmer-Petiau algebras can be obtained in this way, when n is even.* The β-generators of $D_{n,p}$ are

$$(5) \qquad \beta^{(p)}_j = (\Pi_p)(\boldsymbol{I}_j) + g_{jk}(\boldsymbol{I}^k)(\Pi_p)\,, \qquad \beta^{(p)}_j\beta^{(p)}_h\beta^{(p)}_k + \beta^{(p)}_k\beta^{(p)}_h\beta^{(p)}_j = g_{hj}\beta^{(p)}_k + g_{hk}\beta^{(p)}_j\,,$$

when $p \neq (n-1)/2$. In the case of $n = 4$, $D_{4,0}$ and $D_{4,3}$ correspond to scalar and pseudo-scalar particles, respectively; $D_{4,2}$ and $D_{4,1}$ correspond to pseudo-vector and vector particles, respectively; $D_{4,-1}$ and $D_{4,4}$ are « trivial » algebras, with nil β-generators. The algebras $D_{n,p}$ and $D_{n,n-p-1}$ are anti-isomorphic. *When n is even, all the irreducible Duffin-Kemmer-Petiau algebras are affine, although their β-generators be dependent on the metric g_{jk}.*

G_n possesses and important involution $\Gamma \to \bar{\Gamma}$, the coefficients $\bar{A}$ in the expansion (4) of $\bar{\Gamma}$ being the dual tensors of the A's

$$(6) \qquad \bar{A}^{k_1,\dots,k}_{j_1,\dots,j_p} = (-1)^{(p-q)(p+q-1)/2}\{(n-p)!\,(n-q)!\}^{-1}\,\delta^{k_1\dots k_n}_{j_1,\dots,j_n} A^{j_{p+1},\dots,j_n}_{k_{q+1},\dots,k_n}\,.$$

This involution transforms (N_j) *into* $1_{G_n} - (N_j)$; *it corresponds to the charge-conjugation of the second quantization for fermions.*

All the above results are valid for G_n taken over the real numbers or the complex numbers. The possibility of using only real numbers is rather interesting, because the second quantization formalism of Jordan-Wigner seems to depend essentially on the use of complex numbers. It follows from (4) that the A's are the matrix elements of Γ considered as a linear operator on the (Ψ)'s, because the $(P^{j_1 \ldots j_p})$'s with $j_1 < \ldots < j_p$, $p = 0, \ldots, n$ are a basis for the (Ψ)'s and $\Gamma(P^{j_1 \ldots j_p}) = \sum_q (q!)^{-1} A^{j_1 \ldots j_p}_{k_1 \ldots k_q} (P^{k_1 \ldots k_q})$. The matrices of the $(\boldsymbol{I})$'s are not hermitian, that of $(\boldsymbol{I}_j)$ being the adjoint of the matrix of $(\boldsymbol{I}^j)$. This corresponds to the property of the absorption operators to be the adjoints of the corresponding emission operators. When $g_{jk} = \varepsilon_j \delta_{j,k}$, the $\gamma_j^{(+)}$'s corresponding to positive ε_j's have hermitian matrices, those corresponding to negative ε_j's have anti-hermitian matrices.

The central-affine transformation $\underline{x}^j = T^j_k x^k$ induces in G_n the automorphism $\Gamma \to \underline{\Gamma} = \exp[-(\tau)]\Gamma \exp[(\tau)]$, with $(\tau) = \tau^k_j(\boldsymbol{I}^j)(\boldsymbol{I}_k)$, the linear operator τ being such that $T = e^\tau$. The above rule of associating an element (τ) to a linear operator τ of the n-dimensional space corresponds to the well-known rule of association of a second quantization operator to an operator of the Hilbert space of the first quantization. If follows from the above rule of transformation of the Γ's that $(\underline{\Psi}) = \exp[-(\tau)](\Psi)$.

The results of this section can be extended to the case of $n = \infty$, by using a separable Hilbert-space, instead of the finite-dimensional affine space. The Hilbert space is not strictly sufficient, it is necessary to embed it in a convenient linear space. *Thus we see that the second quantization with anti-commutation rules is essentially a geometric procedure corresponding to a full development of the vector calculus of the separable Hilbert spaces based on the anticommutative product of vectors. The second quantization appears as a completion of the first quantization, which uses only an incomplete vector calculus of the Hilbert space.*

It is convenient to formulate in an abstract way the theory of G_n, in order to see better its essential algebraic characteristics. Let Σ_n denote any n-dimensional linear space and Σ'_n its dual. We shall denote the elements of Σ_n by V's and those of Σ'_n by U's. The U's are linear functionals of Σ_n, the value of the functional U for the element V being denoted by $\langle V, U \rangle$. Let the I_j's denote n linearly independent V's, which can be taken as a basis of Σ_n. The dual basis of Σ'_n is constituted by the n functionals I^j defined by the conditions $\langle I_j, I^k \rangle = \delta^k_j$. The direct sum of the linear space Σ_n and Σ'_n is a $2n$-dimensional linear space Σ_{2n}, whose elements W may be taken as sums $V + U$. The linear spaces Σ_n and Σ'_n can be extended into anticommutative Grassmann algebras, by the introduction of anticommutative products $(V_a)(V_b)$ and $(U_a)(U_b)$, whose generators are the (I_j)'s and (I^j)'s, respectively. These two Grassmann algebras can be embedded into the Clifford algebra G_n of the sum-space Σ_{2n} endowed with the inner product $2(W_a, W_b) = \langle V_a, U_b \rangle + \langle V_b, U_a \rangle$, the commutation rule of the (W)'s being $[(W_a), (W_b)]_+ = 2(W_a, W_b)1_{G_n}$. This inner product corresponds to an indefinite metric, whose canonical quadratic form has n positive and n negative squares. *It is interesting to notice that, in the higher form of vector calculus given by G_n, there is a fusion of the two approaches of Grassmann and Hamilton.*

There is a natural symplectic geometry in Σ_{2n} defined by the symplectic product $2(W_a, W_b)_- = \langle V_a, U_b \rangle - \langle V_b, U_a \rangle$. There is a symplectic algebra L_n of Σ_{2n} with the commutation rule $[\{W_a\}, \{W_b\}] = 2(W_a, W_b)_- 1_{L_n}$. In the case of the L_1 of the phase-space S_{2n}, that commutation rule unifies the three commutation rules (1-2). *This approach to L_n shows immediately the relations between the quantum kinematics and the symplectic geometry of the classical phase-space discussed in reference* [2].

The theory of G_4 is clearly related to the well known method of fusion of De Broglie, which allows to derive algebras of particles with spins 0 and 1 by means of two Clifford algebras. Our geometric theory leads naturally to the direct product of two C_4's and shows clearly the geometric nature of the Duffin-Kemmer-Petiau algebras. We get from equation (5) the interesting formula $\gamma_j^{(+)} = \sum_p^{0,\dots,n} \beta_j^{(p)}$, which expresses the generators of a Clifford algebra in terms of those of Duffin-Kemmer-Petiau algebras. This shows in another way how the theory of the spinors can be derived from that of the antisymmetric tensors .

3. - The Heisenberg quasi-algebra $\underline{L}_3$.

The general form of the elements A of L_n is

$$(1) \qquad A = \sum_{p,q} (p!\,q!)^{-1} (r_1! \dots r_n!)(s_1! \dots s_n!)\, C^{k_1 \dots k_q}_{j_1 \dots j_p} \{I^{j_1}\} \dots \{I^{j_p}\}\{I_{k_1}\} \dots \{I_{k_q}\}\,,$$

the C's being symmetrical with respect to the j's and k's, separately; r_a is the number of j's equal to the integer a and s_a the number of k's equal to a. *In order to ensure always the existence of the product of Γ's, it is necessary to have only a finite number of terms in the right-hand side of equation* (1). It is convenient to drop this restriction; thus we get an algebraic structure $\overline{L}_n$, which is no more a linear associative algebra, but a quasi-algebra.

Equation (1) *shows that L_n is an algebra for the geometric objects described by finite sets of symmetric tensors C, only one for each kind of variance.* The sub-algebra generated by 1_{L_n} and the $\{I^j\}$'s is isomorphic to the ring of polynomials of n variables; the sub-algebra of $\overline{L}_n$ with the same generators is isomorphic to the ring of the formal power series of n variables. It is obvious that the theory of the affine algebra L_n leads to mathematical analysis. The equivalence of the Heisenberg and Schrödinger pictures of the quantum mechanics results precisely from the fact that the Heisenberg position-momentum quasi-algebra for a system with n degrees of freedom is a kind of algebraic infra-structure of the differential calculus of the functions of n variables. This fundamental fact follows directly from the geometric nature of the Heisenberg quasi-algebra, which is an extension of L_3.

The Heisenberg quasi-algebra is not equivalent to $\overline{L}_3$, but to another quasi-algebra $\underline{L}_3$, analogous to the total metric algebra of the (Ψ)-space discussed in Sect. **2**. $\underline{L}_n$ is the quasi-algebra of the $\underline{\Lambda}$'s

$$\underline{\Lambda} = \sum_{p,q}^{0,\dots,\infty} (p!\,q!)^{-1}(r_1!\dots r_n!\,s_1!\dots s_n!)A^{j_1,\dots,j_p}_{k_1,\dots,k_q}\{P^{k_0,\dots,k_q}_{j_1,\dots,j_p}\}\,. \tag{2}$$

The j's and k's run from 1 to n. The r's and s's have the same meaning as in (1). The A's are numerical coefficients symmetrical with respect to the j's and k's, separately. The elements $\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\}$ are symmetrical with respect to the j's and k's, separately, and have the following multiplication rule

$$\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\}\{P^{k'_1,\dots,k'_{q'}}_{j'_1,\dots,j'_{p'}}\} = \delta^{s_1'}_{r_1}\dots\delta^{s'_n}_{r_n}\{P^{k_1,\dots,k_q}_{j'_1,\dots,j'_{p'}}\}\,. \tag{3}$$

The unity of $\underline{L}_n$ is

$$1_{\underline{L}_n} = \sum_{p}^{0,\dots,\infty}\{\Pi_p\}\,, \qquad \{\Pi_p\} = \sum_{j}(p!)^{-1}(r_1!\dots r_n!)\{P^{j_1,\dots,j_p}_{j_1,\dots,j_p}\}\,. \tag{4}$$

The $\{\Pi_p\}$'s are orthogonal idempotent elements of $\underline{L}_n$. We shall define the $\{\boldsymbol{I}\}$'s as follows

$$\{\boldsymbol{I}_j\} = \sum_{p}^{0,\dots,\infty}\ \sum_{j_1,\dots,j_p}^{1,\dots,n}(p!)^{-1}(r_1!\dots r_n!)\{P_{j,j_1,\dots,j_p}^{\ \ j_1,\dots,j_p}\}(r_j+1)^{\frac{1}{2}}\,, \tag{5}$$

$$\{\boldsymbol{I}^j\} = \sum_{p}^{0,\dots,\infty}\ \sum_{j_1,\dots,j_p}^{1,\dots,n}(p!)^{-1}(r_1!\dots r_n!)\{P^{j,j_1,\dots,j_p}_{\ \ j_1,\dots,j_p}\}(r_j+1)^{\frac{1}{2}}\,, \tag{6}$$

r_a denotes the number indices $j_1, \dots, j_p$ equal to the integer a. It follows from the definitions (5)-(6) that

$$[\{\boldsymbol{I}_j\}, \{\boldsymbol{I}_k\}] = 0\,, \qquad [\{\boldsymbol{I}^j\}, \{\boldsymbol{I}^k\}] = 0\,, \qquad [\{\boldsymbol{I}_j\}, \{\boldsymbol{I}^k\}] = \delta^k_j 1_{\underline{L}_n}\,, \tag{7}$$

$$\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\} = (r_1!\dots r_n!\,s_1!\dots s_n!)^{-\frac{1}{2}}\{\boldsymbol{I}^{k_1}\}\dots\{\boldsymbol{I}^{k_q}\}\{P\}\{\boldsymbol{I}_{j_1}\}\dots\{\boldsymbol{I}_{j_p}\}\,. \tag{8}$$

Equations (7) show that $\underline{L}_n$ includes an algebra equivalent to L_n: In order to render $\underline{L}_n$ into an affine algebra, it suffices to assume that the elements $\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\}$ depend on the choice of the cartesian co-ordinate system and transform in the same way as components of tensors with the same indices; in particular $\{P\}$ must have the behaviour of a scalar. The equivalence with the matrix formalism for a system with n degrees of freedom is clearly seen by the introduction of the following notations

(9) $$A^{j_1,\dots,j_p}_{k_1,\dots,k_q} = a^{r_1,\dots,r_n}_{s_1,\dots,s_n}\,, \qquad \{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\} = \{\psi^{s_1,\dots,s_n}_{r_1,\dots,r_n}\}\,.$$

Equation (2) becomes

(10) $$\underline{A} = \sum_{r,s}^{0,\dots,\infty} a^{r_1,\dots,r_n}_{s_1,\dots,s_n}\{\psi^{s_1,\dots,s_n}_{r_1,\dots,r_n}\}\,.$$

The a's are matrix elements of the $\underline{A}$'s considered as linear operators of the vector space $\underline{L}_n\{P\}$, the application of $\underline{A}$ to any element $\{\Psi\}$ of this space being simply its left-multiplication by $\underline{A}$. *It is important to notice that the matrices of* $\{\boldsymbol{I}_j\}$ *and* $\{\boldsymbol{I}^j\}$ *are adjoint.*

$\underline{L}_n$ over the complex numbers is the n-dimensional analogue of the second quantization formalism for bosons, the $\{\boldsymbol{I}_j\}$'s corresponding to the absorption operators and the $\{\boldsymbol{I}^j\}$'s to the emission operators. $\{P\}$ corresponds to the projector of the state-vector of the boson field describing the vacuum. The elements $\{N_j\} = \{\boldsymbol{I}^j\}\{\boldsymbol{I}_j\}$ are analogous to the occupation-number operators of the boson field. *Thus we see that the boson second quantization formalism is simply a geometric algebra* $\underline{L}_\infty$ *of a separable Hilbert space.* It follows from equation (8) that the general form of the $\{\Psi\}$'s is

(11) $$\{\Psi\} = \sum_s a_{s_1,\dots,s_n}\{\psi^{s_1,\dots,s_n}\} = \sum_s \bar{a}_{s_1,\dots,s_n}\{\boldsymbol{I}^1\}^{s_1}\dots\{\boldsymbol{I}^n\}^{s_n}\{P\}\,.$$

The $\{\Psi\}$'s correspond to the wave functionals of the quantized boson field. Equation (11) is the analogue of the Fock expansion of the wave functionals of the boson field. *Each* $\{\Psi\}$ *is associated to a formal power series* $\sum_s \bar{a}_{s_1,\dots,s_n}\cdot(z^1)^{s_1}\dots(z^n)^{s_n}$ *of* n *variables* z^j. $\{\boldsymbol{I}_j\}\{\Psi\}$ *is associated to the series obtained from that corresponding to* $\{\Psi\}$ *by the derivation of its terms with respect to* z^j.

The $\{\Psi\}$'s for wich $\sum_s |a_{s_1,\dots,s_n}|^2 < \infty$ constitute a Hilbert space H_n. The power series associated to the elements of H_n converge for all the values of the z^j's and represent integral functions of the z^j's. We shall now see that H_n is equivalent to the Hilbert space of the states of a system with n degrees of freedom. Let us take $\sqrt{2}\,q^j = \{\boldsymbol{I}^j\} + \{\boldsymbol{I}_j\}$. It is easily seen that

(12) $$\{\psi^{s_1,\dots,s_n}\} = h_{s_1,\dots,s_n}(q)\{\underline{P}\}\,, \qquad \{P\} = \pi^{n/4}\exp\left[\tfrac{1}{2}\sum_j (q^j)^2\right]\{P\}\,.$$

The h's are normalized Hermite functions

(13) $$h_{s_1,\dots,s_n}(q) = \pi^{-n/4}\prod_j \{2^{s_j}(s_j!)\}^{-\frac{1}{2}}\exp\left[-\tfrac{1}{2}(q^j)^2\right]H_{r_j}(q^j)\,.$$

It follows from equation (12) and (11) that

$$\{\Psi\} = \sum_s a_{s_1,\dots,s_n} h_{s_1,\dots,s_n}(q)\{P\}\,. \tag{14}$$

$\{\Psi\}$ is associated to a formal series of Hermite functions of n real variables x^j: $\sum_s a_{s_1,\dots,s_n} h_{s_1,\dots,s_n}(x)$. When $\{\Psi\}$ belongs to H_n its associate Hermite series converges in the mean to a function $F(x)$ of Lebesgue integrable squared absolute value, over the whole x-space.

The $\{\Psi\}$'s not belonging to H_n correspond to functions of non square integrable absolute value and also to distributions. Thus the element $(2\pi)^{-n/2}\sum_s i^{s_1+\dots+s_n} h_{s_1,\dots,s_n}(u) h_{s_1,\dots,s_n}(q)\{\underline{P}\}$ corresponds to the function $\exp[i\sum_j u^j x^j]$ and $\delta_u = \sum_s h_{s_1,\dots,s_n}(u) h_{s_1,\dots,s_n}(q)\{\underline{P}\}$ to $\prod_j \delta(x^j - u^j)$. *The Fourier transformation can be easily extended to all the $\{\Psi\}$'s: the Fourier transform of $h_{s_1,\dots,s_n}$ being $i^{s_1+\dots+s_n} h_{s_1,\dots,s_n}$, we can take as Fourier transform of $\{\Psi\}$ the element $\sum_s i^{s_1+\dots+s_n}\cdot a_{s_1,\dots,s_n} h_{s_1,\dots,s_n}(q)\{\underline{P}\}$. The derivation can be defined for all the $\{\Psi\}$'s by means of the d_j's*

$$\sqrt{2}\, d_j = \{\boldsymbol{I}_j\} - \{\boldsymbol{I}^j\}\,, \qquad [d_j, q^k] = \delta_j^k 1_{\underline{L}_n}\,, \qquad d_j\{\underline{P}\} = 0, \tag{15}$$

$d_j\{\Psi\}$ is associated to the formal series obtained from that corresponding to $\{\Psi\}$ by derivation of its terms with respect to x^j. We can obtain a kind of integration by means of the inner product $(\{\Psi^{\mathrm{I}}\}, \{\Psi^{\mathrm{II}}\}) = \sum a^{\mathrm{I}}_{s_1,\dots,s_n}(a^{\mathrm{II}}_{s_1,\dots,s_n})^*$. This integration may have a meaning even when the $\{\Psi\}$'s do not correspond to functions. Thus we have $(\{\Psi\}, \delta_u) = F(u)$, when $\{\Psi\}$ is associated to the function $F(x)$.

The above results show that $\underline{L}_n$ gives a powerful extension of the classical mathematical analysis and of the theory of the Hilbert space. It is noteworthy that most of those results do not depend essentially on the field of real or complex numbers over which $\underline{L}_n$ is taken. In particular, the continuity properties of those fields are irrelevant. We might even replace the fields by more general algebraic structures. It is possible that a further development of the quantal physics will require a deep revision of our intuitive picture of space and time. This does not necessarily mean that the tools of mathematical analysis will need essential modifications, because they seem to depend little on the properties of continuity.

It is clear that the concept of function is not the natural one for the quantal physics: the concept of element of a $\{\Psi\}$-space is certainly more adequate. From the analytical point of view, this concept is equivalent to that of a linear functional completely determined by its values for the functions $h_{s_1,\dots,s_n}$. Indeed, the coefficients $a_{s_1,\dots,s_n}$ may be considered as the values of such a functional for those functions. Conversely any such functional gives us a set of coefficients $a_{s_1,...,s_n}$, as its values for the functions $h_{s_1,...,s_n}$, which determine it completely and allow to define an associate $\{\Psi\}$.

The definition of the q's and d's used in this section corresponds to that given by (1-6), with the euclidean metric $g_{jk} = \delta_{j,k}$. *The q's are hermitian operators of the $\{\Psi\}$-sapce; the d's are anti-hermitian operators.* In the case of a pseudo-euclidean metric $g_{jk} = \varepsilon_j \delta_{j,k}$, the q^j's corresponding to negative ε_j's are anti-hermitian and the d_j's hermitian. The variable x^j must then be taken as imaginary.

The theory of $\underline{L}_n$ gives a remarkable fusion of the Schrödinger and Heisenberg formalisms, because the wave-functions and the operators for the observables are all associated to elements of the quasi-algebra. Moreover we have also the equivalent of symbolic functions such as Dirac's δ and its derivatives. It is interesting to notice that there is a simple closed expression for the element δ_u associated to the n-dimensional Dirac function

$$\delta_u = (4\pi)^{-n/4} \exp\left[-\tfrac{1}{2} \sum_j (d_j^2 + 2u^j d_j)\right]\{P\} . \tag{16}$$

A closed expression for the element associated to the derivative $D_1^{r_1} \ldots D_n^{r_n}$ of $\delta(\boldsymbol{x} - \boldsymbol{u})$ can be obtained from (16) by left-multiplication of both sides by $d_1^{r_1} \ldots d_n^{r_n}$.

Equation (2) shows that $\underline{L}_n$ is a calculus for the geometric objects described by finite or infinite sets of symmetric tensors $A^{j_1 \ldots j_p}_{k_1 \ldots k_q}$. Another such calculus is given by the quasi-algebra $\overline{L}_n$. The great advantage of $\underline{L}_n$ is obviously its equivalence with a quasi-algebra of infinite matrices. *It is remarkable that the geometric quasi-algebra of the symmetric tensors $\underline{L}_n$ be automatically a theory of functions, measures and distributions of n real variables.* The modern theory of the functions of real variables showed that the concept of continuous function of n real variables is a special case of the concept of measure of sets, i.e. of the distribution of mass or charge in a n-dimensional space. Not all the distributions of charge can be described by measures, the subtler ones correspond to the Schwarz distributions and perhaps to still more elaborate analytical concepts. *The geometric algebra $\underline{L}_n$ is a generalized theory of measures of the n-dimensional space, non equivalent to the Schwarz distribution theory.*

In the quantum mechanics, $\underline{L}_n$ appears as a tool for the description of probability distributions associated to dynamical systems with n degrees of freedom. These probability distributions describe ultimately non intuitive distributions of matter in space. *The use of probability in the quantum mechanics may be a means to relate the micro-geometric properties of matter at the atomic level wit the macroscopic geometric concepts. It is clear that the ordinary macro-geometric concepts of position etc. are not applicable to matter at the atomic level, which occupies space in very subtle ways described by the geometric algebras. A further clarification of the foundations of the quantum mechanics requires a* *better understanding of the picture of space involved in the geometric algebras.*

Our analysis of the symmetric tensors of the n-dimensional affine space led us to the introduction of an infinite dimensional linear space $\underline{L}_n\{P\}$. *The distinction of the $\{\Psi\}$'s of the Hilbert space H_n is not affine invariant: H_n is associated to the metric euclidean geometry of the n-dimensional space. The unitary metric of H_n is an extension of the euclidean metric of the n-dimensional space.* Indeed, the euclidean metric can be extended to the linear space of the symmetric covariant tensors of order q by taking as the norm of the tensor $A_{k_1,\dots,k_q}$ the square root of the sum $\sum_k (q!)^{-1}(s_1!\dots s_n!)|A_{k_1,\dots,k_q}|^2$. The square of the norm of a $\{\Psi\}$ is simply the sum of the squares of the norms of its tensors $A_{k_1,\dots,k_q}$, $q=0,\dots,\infty$.

The theory of $\underline{L}_n$ deals essentially with the linear space of the $\{\Psi\}$'s and its linear operators. The analysis of the properties of this linear space can be done with more depth by the introduction of its geometric algebras G and $\underline{L}$, i.e. of the second quantization formalisms for bosons and fermions. *The theory of the quantized fields is therefore associated with a higher stage of the geometric analysis of the space-time. In the case of the n-dimensional space, it is necessary to consider the direct product of G_n and $\underline{L}_n$, which is a generalization of the vector analysis. In a similar way, in the case of the infinite dimensional $\{\Psi\}$-space we must take the direct product of the two second quantization quasi-algebras, which is a quasi-algebra for interacting fermion and boson fields. The identification of $\underline{L}$ with the quantum kinematics requires a fundamental length λ, which is also involved in our geometric approach to the quantum field theory.*

4. – The harmonic oscillators of $\underline{L}_n$.

The elements $\{N_j\}=\{I^j\}\{I_j\}$ of $\underline{L}_n$ correspond to the occupation-number operators of the boson second quantization. They play a central part in the theory of $\underline{L}_n$, because $(\{N_j\}-s_j1_{\underline{L}_n})\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\}=0$, $\{P^{k_1,\dots,k_q}_{j_1,\dots,j_p}\}(\{N_j\}-r_j1_{\underline{L}_n})=0$. Since $\{N_j\}=\frac{1}{2}((q^j)^2-(d_j)^2-1_{\underline{L}_n})$, $\{N_j\}$ is analogous to the hamiltonian of a one-dimensional harmonic oscillator. *$\underline{L}_n$ is essentially the quantum mechanics of a system of n one-dimensional uncoupled harmonic oscillators.*

The $\{N_j\}$'s are a special case of the elements $\{N_V\}=\{V\}^\dagger\{V\}$, V being a real unit vector $\sum_j V^{j^2}=1$; $\{V\}^\dagger=\sum V^j\{I^j\}$ is the adjoint of $\{V\}$ as an operator of the $\{\Psi\}$-space. The eigenvalues of $\{N_V\}$ are the non negative integers, because $[\{V\},\{V\}^\dagger]=1_{\underline{L}_n}$. $\{N_V\}$ does not depend on the sense of V, it is associated to its direction. The $\{N_j\}$'s correspond to the co-ordinate axes.

The existence of elements with non negative integer eigenvalues associated to the directions of the n-dimensional space is a very remarkable propertiy of $\underline{L}_n$, still enhanced by the fundamental part played by these elements. *It is* *important to notice the metric nature of the* $\{N_V\}$*'s. The definition of a vector of length unity, presupposes a choice of the unit of length. We have the natural unit of length* λ, *which was identified to* 1. *Strictly speaking the eigenvalues of the* $\{N_V\}$*'s are the non negative integral multiplus of* λ^2. *There is a kind of quantization of the square of the lenght of a vector.* In the case of a Hilbert space, this quantization of the square of the norm gives the integral numbers of bosons in the one-boson states. The $\{N_V\}$'s corresponding to orthogonal $\{V\}$'s are commutable, otherwise not.

There are natural relations between $\underline{L}_n$ *and the theory of probability. The eigenvalues of* $\{N_V\}^{\frac{1}{2}}$ *are the mean square displacements of a particle in a one dimensional random walk of step* 1, *after integral numbers of steps.* It is interesting to notice that the fundamental element $\{P\}$ is associated to the gaussian function $\pi^{-n/4} \exp[-\frac{1}{2}\sum_j x^{j^2}]$. The true nature of those relations between $\underline{L}_n$ and the theory of probability is not yet clear. It is well known that the theory of probability is closely related to that of the averages of functions of infinite numbers of variables (see PAUL LÉVY: *Problèmes concrets d'analyse fonctionnelle* (Paris, 1951); especially the third part). *The central part played by the gaussian and the Hermite functions in the theory of* $\underline{L}_n$ *indicates clearly that the stage of the analysis of the n-dimensional space given by the present form of the theory of* $\underline{L}_n$ *corresponds to some kind of averaging over a linear space of infinite dimensionality. As a matter of fact, such a space appears already in our treatment: the* $\{\Psi\}$*-space . The point of co-ordinates* x^j *of the n-dimensional space is associated to the element* δ_x *of the* $\{\Psi\}$*-space.* $\underline{L}_n$ *is a link between the micro-geometry and the macro-geometry of the ordinary objects. The probabilistic results seem to be due to the averaging over the microscopic properties which lead to the familiar geometric properties. This suggests that the micro-geometric properties begin to appear more clearly in physics at the level of the quantum theory of fields.*

The linear transformations $\boldsymbol{I}_j \to e^{a_j}\boldsymbol{I}_j$ correspond to changes of the unit vectors of the co-ordinate axes. They induce in $\underline{L}_n$ a n-parameter abelian group of automorphisms $\underline{A} \to \exp[-\sum_j a_j\{N_j\}]\underline{A}\exp[\sum_j a_j\{N_j\}]$. In particular we have $\{\Psi\} \to \exp[-\sum_j a_j\{N_j\}]\{\Psi\}$. *The* $\{N_j\}$*'s are essentially the infinitesimal transformations of that abelian group of automorphisms.* By taking imaginary values for the a_j's, we get the n-dimensional analogue of the change of the phases of the basic vectors in a Hilbert space, well known from the theory of the gauge transformation of the quantum mechanics. *The consideration of real values of the* a_j*'s shows clearly the geometric nature of the group underlying the gauge transformations.*

By taking all the a_j's equal to a, we get the important one parameter group of automorphisms $\underline{A} \to \exp[-a\{N\}]\underline{A}\exp[a\{N\}]$, with $\{N\} = \sum\{N_j\}$, induced in $\underline{L}_n$ by the similitudes $V \to e^a V$ of the n-dimensional space. The element $\{N\}$ does not depend on the choice of the $\boldsymbol{I}_j$'s, because it is the contracted product $\langle V_I, U_I \rangle$ of the symbolic vectors V_I and U_I of components $\{I^j\}$ and $\{I_j\}$, respectively. $\{N\}$ is the analogue of the element (N) of G_n, being associated to the fundamental tensor δ_j^k and to the operator unity on the V's. $\{N\}$ corresponds to the operator for the total number of particles of the boson second quantization.

It is easily seen that $\exp[(i\pi/2)\{N\}]\{\Psi\}$ is the Fourier transform of $\{\Psi\}$ defined in Sect. 3. *The imaginary similitude $V \to -iV$ induces the Fourier transformation in the $\{\Psi\}$-space; $\{\Psi\} \to \exp[a\{N\}]\{\Psi\}$ and $\{\Psi\} \to \exp[\sum a_j\{N_j\}]\{\Psi\}$ are generalizations of the Fourier transformation.* This association of the Fourier transformation to a complex similitude is very remarkable and gives a simple geometric foundation for the theory of the Fourier transformation. The automorphism $\Gamma \to \exp[(i\pi/2)(N)]\Gamma \exp[(-i\pi/2)(N)]$ of G_n is analogous to the Fourier automorphism of $\underline{L}_n$; it transforms the $\gamma_j^{(+)}$'s into the $-i\gamma_j^{(-)}$'s.

$\{N\}$ is the hamiltonian, without zero-point term, of a n-dimensional isotropic harmonic oscillator. The automorphisms of $\underline{L}_n$ corresponding to the unitary transformations $\underline{\Lambda} \to \exp[-it\{N\}]\underline{\Lambda} \exp[it\{N\}]$ generated by the motion of the isotropic oscillator are powers of the Fourier transformation of the $\underline{\Lambda}$'s. The inverse Fourier transformation corresponds to $t = \pi/2$ and the spatial reflection to $t = \pi$. *Our geometric theory gives the deep reason of the relations between the spatial reflection and the iterated Fourier transformation.* Those automorphisms correspond to the changes of phase $V \to e^{ti}V$ of the vectors of the n-dimensional space.

A complete basis for the infinitesimal transformations of the $\{\Psi\}$-space induced by central-affinities of the n-dimensional space is constituted by the $\{M_{jk}\} = \{I^j\}\{I_k\} - \{I^k\}\{I_j\} = q^j d_k - q^k d_j$ and the $\{S_{jk}\} = \{I^j\}\{I_k\} + \{I^k\}\{I_j\}$. The $\{M_{jk}\}$'s correspond obviously to the infinitesimal n-dimensional rotations and the $\{S_{jk}\}$'s to the infinitesimal deformations. We have $\{S_{jj}\} = 2\{N_j\}$. Our theory introduces the deformations together with the rotations, whereas ordinarily only the rotations are considered. *We see that the infinitesimal dilatations play a central rôle in the theory of $\underline{L}_n$.* The analogue of $\{M_{jk}\}$ in G_n is $(M_{jk}) = (I^j)(I_k) - I^k)(I_j) = \frac{1}{2}(\gamma_j^{(+)}\gamma_k^{(+)} - \gamma_j^{(-)}\gamma_k^{(-)})$.

The fundamental importance of the isotropic harmonic oscillator associated to the basic length indicates that it must be of significance in physics. It seems satisfactory to assume that the basic length is the Compton wave length of the π-meson. In the shell model of the nucleus, the isotropic harmonic oscillator is of importance. *It seems therefore reasonable to associate the shell model of the nucleus to the isotropic oscillator of $\underline{L}_3$.* The spin-orbit coupling cannot be included in the theory of $\underline{L}_3$, which does not contain spin-like variables.

The non relativistic theory of the spin $\frac{1}{2}$ particles requires the introduction of the direct product of the Pauli spin algebra by $\underline{L}_3$. The Pauli algebra is a sub-algebra of the reducible C_3. C_3 is of the order 8 and may be generated by the element $\theta = i\gamma_1\gamma_2\gamma_3$ and the three components of the spin vector $\boldsymbol{\sigma}$, $\sigma_k = i\gamma_h\gamma_j$, (h, j, k) denoting a circular permutation of $(1, 2, 3)$. The element of C_3 associated to the euclidean tensor A_{jk} is $\sum A_{jk}\gamma_j\gamma_k$. The spin-orbit element $\sum_{j,k} \{M_{jk}\}\gamma_j\gamma_k$ is therefore associated to the symbolic antisymmetric tensor of components $\{M_{jk}\}$. The element $\mathcal{N}$ associated to the symbolic tensor $\{\boldsymbol{I}^j\}\{\boldsymbol{I}_k\}$ is $\mathcal{N} = \sum \{\boldsymbol{I}^j\}\{\boldsymbol{I}_k\}\gamma_j\gamma_k = \{N\} + \frac{1}{2}\sum \{M_{jk}\}\gamma_j\gamma_k$. Thus we get quite naturally an isotropic oscillator with an exceedingly strong spin-orbit coupling, by considering the complete symbolic tensor of the infinitesimal central-affine transformations. The involution of G_3 discussed in Sect. **2** induces an involution in $C_3 \times \underline{L}_3$, which transforms $\mathcal{N}$ into $\overline{\mathcal{N}} = \{N\} - \frac{1}{2}\sum \{M_{jk}\}\gamma_j\gamma_k$ and changes the sign of the spin-orbit interaction. The above considerations do not give a satisfactory spin-orbit interaction for the shell-model oscillator. It is likely that the nuclear spin-orbit interaction depends essentially on relativistic effects. The above results are nevertheless interesting because they show how naturally the geometric algebras lead to the kinds of forces involved in the shell-model.

5. – Isotopic spin and projective geometry.

An algebra of vectors cannot give a satisfactory affine calculus, because the vectors are non localized objects. The anticommutative Grassmann algebra of vectors is included in the algebra of points, the contravariant vectors being considered as differences of points. This algebra of points can also be applied to the projective geometry, as well known. We shall now develop an algebra, which is an extension of the Grassmann algebra of points. Let the X^μ's, μ running from 0 to n, denote polyhedric co-ordinates of the points of a n-dimensional space. The projective transformations are given by homogeneous linear transformations of the X's. The polyhedric co-ordinates of hyperplanes will be denoted by Y_μ's. We shall introduce two $(n+1)$-dimensional dual linear spaces whose vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ have the components X^μ and Y_μ, respectively. The G-algebra associated to this pair of dual linear spaces will be denoted by G'_n. G'_n is isomorphic to a G_{n+1}; it is generated by the elements $(\mathcal{J}_\mu)$. $(\mathcal{J}^\mu)$ corresponding to the basic vectors of two dual basis of the above linear spaces. The commutation rules of the $(\mathcal{J})$'s are

$$(1) \qquad [(\mathcal{J}_\mu), (\mathcal{J}_\nu)]_+ = 0\,, \qquad [(\mathcal{J}^\mu), (\mathcal{J}^\nu)]_+ = 0\,, \qquad [(\mathcal{J}_\mu), (\mathcal{J}^\nu)]_+ = \delta^\nu_\mu 1_{G'_n}\,.$$

The projectivity $X^\mu \to T^\mu_\nu X^\nu$ induces in G'_n the inner automorphism $\Gamma' \to \exp[-(S)]\Gamma' \exp[S]$, Γ' denoting a generic element of G'_n and (S) the element $S^\nu_\mu(\mathcal{J}^\mu)(\mathcal{J}_\nu)$, the matrix S^ν_μ being any determination f the logarithm of the non-singular matrix T^{ν}_{μ} of the projectivity. The linear transformations of X's into Y's correspond to correlations, they do also induce inner automorphisms in G'_n. The affine transformations are a particular case of the projective ones, characterized by leaving invariant the hyperplane at infinity.

For the discussion of the affine transformations, it is convenient to take the X^{μ}'s and Y_{μ}'s as homogeneous cartesian point and hyperplane co-ordinates. Thus X^0 and Y_0 behave as scalars and the X^j's and Y_j's as components of contravariant and covariant vectors, respectively, for central-affine transformations. Since $x^j = X^j/X^0$, the translation $x^j \to x^j + V^j$ corresponds to $X^0 \to X^0$, $Y_0 \to Y_0 - Y_j V^j$, $X^j \to X^j + X^0 V^j$, $Y_j \to Y_j$. The $\mathcal{J}_j$'s can be identified with the $\boldsymbol{I}_j$'s, but the $\mathcal{J}^j$'s do not transform in the same way as the $\boldsymbol{I}^j$'s for translations. $\mathcal{J}_0$ corresponds to the origin of the co-ordinates and $\mathcal{J}^0$ to the hyperplane at infinity. *The sub-algebra of G'_n generated by those $(\mathcal{J}_j)$'s and $(\mathcal{J}^j)$'s is isomorphic to G_n, but cannot be identified with G_n when translations are considered, because all the elements of G_n must be invariant for translations of the cartesian frame.*

Let (ω') denote the element of G'_n analogous to the element (ω) of G_n introduced in Sect. 2. (ω') does not depend on the choice of the $\mathcal{J}$'s and anticommutes with all the $(\mathcal{J})$'s. The elements

$$\tau_+ = (\mathcal{J}^0)(\omega'), \qquad \tau_- = (\omega')(\mathcal{J}_0), \qquad \tau_3 = 2(\mathcal{J}^0)(\mathcal{J}_0) - 1_{G'_n}, \tag{2}$$

generate a sub-algebra isomorphic to the Pauli spin algebra, whose elements are commutable with those of the sub-algebra generated by the $(\mathcal{J}_j)$'s and $(\mathcal{J}^j)$'s. *G'_n is isomorphic to the direct product of G_n by a Pauli spin algebra. By taking the $(\mathcal{J})$'s corresponding to homogeneous cartesian co-ordinates, the elements of the τ sub-algebra are invariant for central-affine transformations leaving fixed the origin of the cartesian co-ordinates.*

The existence of non-localized vectors is characteristic of flat spaces. In a curved space, the vectors are anchored to the points and we must consider the parallel displacement of the vector-body from a point to the neighbouring ones. There is a G_n associated to the flat tangent space at each point, the isomorphic G_n's as different points being connected by a kind of parallel displacement related to that of the vectors. We have also a G'_n associated to each point and there is no interest in displacing the origin of the co-ordinates from the point of contact in the flat tangent space; the above difficulty of identifying G_n to a sub-algebra of G'_n becomes therefore irrelevant. In the physical case, the flat space-time is an approximation to the real curved space-time.

When only central-affine transformations are considered, the linear space of the homogeneous cartesian Y_{μ}'s is simply the direct sum of the linear spaces of the scalars Y_0 and the covariant vectors of components Y_j. This linear space is interesting because of its association with the irreducible Duffin-Kemmer-Petiau algebra $D_{n,0}$ of Sect. 2, which is its total metric algebra. *In the case of the space-time,* $D_{4,0}$ *is the Duffin-Kemmer-Petiau algebra of the scalar particles of spin* 0; G_4' *is essentially the direct product of* G_4 *by the* G_1 *of the one-dimensional linear space of the scalars. We can now identify* G_4 *with a sub-algebra of* G_4' *and the* τ *sub-algebra with the isotopic spin algebra of the nucleon.* G_4 *may be considered as the direct product of the Dirac* γ*-algebras of the electron and the nucleon and* G_4' *as the algebra of the discrete variables of the electron and the nucleon.*

6. – Alternative approach to the geometric calculus.

The possibility of obtaining the isotopic spin from a G_5, suggests a new approach to the geometric calculus of the affine spaces. We started from the pair of dual linear spaces of the contravariant and covariant vectors. It is more satisfactory to start from the direct sum of the linear spaces of the contravariant vectors and scalars and its dual, which is the direct sum of the linear space of the covariant vectors and of a second linear space of scalars. The scalars are as fundamental as the vectors, their rôle does not appear clearly, because they are not distinguished from the numbers of the field underlying the linear spaces. The difference between the scalars and the numbers appears in the vector analysis, when scalar functions are considered. The basis in the space of the scalars and contravariant vectors is constituted by the $n+1$ elements I_μ; the basis in the dual space is constituted by the I^μ's such that $\langle I_\mu, I^\nu\rangle = \delta^\nu_\mu$. The generators of the G_{n+1} associated to our pair of $(n+1)$-dimensional dual linear spaces have the commutation rules $[(I_\mu), (I_\nu)]_+ = 0$, $[(I^\mu), (I^\nu)]_+ = 0$, $[(I_\mu), (I^\nu)] = \delta^\nu_\mu 1_{G_{n+1}}$. *Thus we get directly an algebra isomorphic to* G_n', *which can be identified with the algebra of the discrete variables of the electron and the nucleon, in the case of the space-time. The total metric algebras of our* $(n+1)$*-dimensional linear spaces are equivalent to the Duffin-Kemmer-Petiau algebra* $D_{n,0}$ *and its reciprocal algebra.*

The n-dimensional projective group comes in only as the group of the automorphisms of the metric algebra of the $(n+1)$-dimensional linear space, our $D_{n,0}$. G_{n+1} has a sub-algebra which can be identified to the G_n of our n-dimensional space: the sub-algebra generated by the (I_j)'s and (I^j)'s. We do not have now the difficulty with the translations and G_n presented by G_n', because the I_j's and I^j's are assumed to transform as components of a n-dimensional covariant vector and a n-dimensional contravariant vector, respectively, whereas in the case of G_n' the $\mathcal{J}^\mu$'s and $\mathcal{J}_\mu$'s must transform as the X^μ's and Y_μ's, respectively.

We can build a $\underline{L}_{n+1}$ associated to the above pair of dual linear sum-spaces, which allows to obtain functions and distributions of $n+1$ variables in the case of an n-dimensional case. These functions for the space-time would depend on the space and time variables and also on a fifth scalar variable, whose meaning is not clear. In connection with the new variable there is a new one-dimensional harmonic oscillator hamiltonian.

The direct sum of the linear spaces of the contravariant vectors and the scalars of the three-dimensional space is involved in the theory of the quaternions. Our formalism is not equivalent to the quaternion algebra: it gives a G_4 in the case of a three-dimensional space.

7. – Algebras of the conformal geometry.

We showed in Sect. **1** that G_4 is the direct product of two Clifford algebras equivalent to the Dirac algebra of the γ-matrices. Thus the affine geometry led to the consideration of two basic types of spin $\frac{1}{2}$ fermions. We should expect them to be the lepton (electron-neutrino) and the nucleon (proton-neutron), but this identification requires two new spin-like variables in order to distinguish the electron and neutrino states of the lepton and the neutron and proton states of the nucleon. The passage from G_4 to G_4' gave us one such variable, which was identified with the isotopic spin of the nucleon. We could get the missing variable by going over from G' to a broader algebra G_5', equivalent to a G_6. We shall now see that there are geometric reasons to introduce a G_5'.

It is well known that the conformal group of the space-time is important for the particles with zero-mass. This indicates that the algebras of the conformal group may give us the two-valued variable which distinguishes the electron and neutrino states of the lepton. In order to discuss the conformal geometry of an n-dimensional space, it is convenient to embed it in a $(n+1)$-dimensional space. Let us denote by $x^1, \ldots, x^{n+1}$ the cartesian co-ordinates of the $(n+1)$-dimensional space, chosen in such a way that our n-dimensional space be the hyperplane of equation $x^{n+1}=0$. The points of this hyperplane can be obtained by stereographic projection from those of the hyperquadric of equation $g_{jk}x^jx^k+(x^{n+1})^2=1$, the centre of projection being the point of co-ordinates $x^j=0$, $x^{n+1}=1$. Let the $n+2$ X^{ϱ}'s be homogeneous co-ordinates: $x^j=X^j/X^0$, $x^{n+1}=X^{n+1}/X^0$. The X's of the points of the above hyperquadric will be denoted by ξ's: $g_{jk}\xi^j\xi^k+(\xi^{n+1})^2=(\xi^0)^2$. The ξ's are hyperspherical co-ordinates of the points of the n-dimensional space. It is well known that the conformal transformations of this space are given by linear transformations of its ξ's leaving invariant the above quadratic equation, when $n>2$. *The Clifford algebra* C_{n+2} *corresponding to the quadratic form* $g_{jk}X^jX^k+(X^{n+1})^2-(X^0)^2$ *is obviously the geometric algebra of the conformal geometry of the n-di-*

mensional space with the metric g_{jk}. In the case of the space-time, we get a C_6, whose spinors have 8 components. We have thus added a new two-valued variable to the spin-like variables involved in the relativistic theory of the electron.

The above C_{n+2} is a sub-algebra of the G'_{n+1} of the projective geometry of the $(n+1)$-dimensional space. All the collineations of this space induce inner automorphisms in G'_{n+1}, even if they do not leave invariant the hyperquadric associated to the conformal geometry of the n-dimensional space. The collineations leaving invariant the hyperplane $X^{n+1}=0$ induce in it the projective transformations of our n-dimensional space; those leaving invariant the hyperquadric of equation $g_{jk}X^jX^k+(X^{n+1})^2 = (X^0)^2$ correspond to the n-dimensional conformal transformations. We may therefore identify the projective algebra G'_n with the sub-algebra of G'_{n+1} generated by the $(\mathcal{J}_\mu)$, $(\mathcal{J}^\mu)$ with $\mu = 0, 1, ..., n$, the $(\mathcal{J}_\varrho)$, $(\mathcal{J}^\varrho)$ being the $2(n+2)$ $(\boldsymbol{I})$-generators of G'_{n+1}. *The algebra G'_{n+1} is associated to both the projective and conformal geometries of our n-dimensional space. The G'_n of the space-time allows us to include the neutrino in our geometrical picture.*

We shall now choose the hyperspherical co-ordinates in a somewhat different way, in order that $x^{n+1}=g_{jk}x^jx^k$, $X^0X^{n+1}=g_{jk}X^jX^k$. The new x^5 of the space-time is a physically interesting variable, because it represents the square of the interval between the origin and the point. The equations of motion of a classical relativistic charged particle can be written as follows [3]

$$\frac{dx^j}{ds}=\frac{\partial K}{\partial p_j}, \qquad \frac{dp_j}{ds}=-\frac{\partial K}{\partial x^j}, \qquad K=\{g^{jh}(p_j-eA_j)(p_h-eA_h)\}^{\frac{1}{2}},$$

s is the interval along the world-line, treated as a fifth independent variable. *K is clearly a constant of the motion, which must be identified with the rest-mass, the formalism allowing the rest-mass to take different values. The rest-mass appears now as a kind of conjugate momentum of s.* The variable x^5 is related to s. In the Dirac theory K corresponds to the operator $\gamma^j(p_j-eA_j)$. The above classical equations correspond to a kind of generalized Dirac equation involving a fifth variable: $i(\partial\psi/\partial s)+\gamma^j(p_j-eA_j)\psi=0$. The Dirac equation for the rest-mass m is obtained by taking $\psi=\exp[ims]\varphi$, φ depending only on the space-time co-ordinates x^j. The conformal geometry of the space-time seems to be involved in the theory of the mass-spectrum of the elementary particles.

The algebra G'_5 is associated to the geometry of the hyperspheres of the pseudo-euclidean space-time. Indeed, the equation of the hyperspheres of the n-dimensional space is $Y_\varrho X^\varrho=0$, the Y's being the so-called homogeneous co-ordinates of the hypersphere. The Y's are not restricted by any condition. The hypersphere geometry leads to the introduction of the G_{n+2} of the dual linear spaces of vectors with components X^ϱ and Y_ϱ. The algebra G'_{n+1} is this G_{n+2}. The hyperplanes are also included in the present geometry as a special kind of hypersphere. Our special choice of the hyperspherical co-ordinates corresponds to take as hyperspheres of indices 1, ..., n the co-ordinate hyperplanes of the cartesian frame in the n-dimensional space.

The full development of the hypersphere geometry is given by the Lie geometry, in which the hyperspheres of the n-dimensional space are described by $n+3$ homogeneous co-ordinates Z satisfying a quadratic equation $\varphi(Z) = 0$. The basic group is now constituted by the linear transformations of the Z's leaving invariant the equation $\varphi(Z) = 0$. We can take $Z_\varrho = Y_\varrho$: $(Z_{n+2})^2 = g^{jk}Z_jZ_k + (Z_{n+1})^2 - (Z_0)^2$ is the quadratic equation. We have now a Clifford algebra C_{n+3} associated to the quadratic form $\varphi(Z)$ and a G_{n+3} containing as a sub-algebra the above G'_{n+1}. *The transformations of the Lie geometry are not always point transformations. The Lie geometry may give a new two-valued variable in the case of the space-time, since it is associated to a* G_7.

We have considered only algebras and quasi-algebras associated to groups depending on a finite number of parameters. The geometric groups depending on arbitrary functions, such as the group of the contact transformations, may also be important for the physics of the elementary particles, perhaps in connection with the theory of fields.

8. – Geometric algebras of complex spaces.

The geometric algebras we have been considering correspond to real spaces and may be taken over the real numbers. The quantum mechanics requires however algebras over the complex numbers. Thereby we are obliged to take our algebras over the complex numbers, so that we go over to complex spaces. *The geometry of the complex spaces requires both linear and anti-linear transformations, as well known from the projective geometry. The same happens in the affine geometry. The necessity of considering anti-linear transformations leads to an extension of the geometric algebras, as we shall now see.*

In a complex affine space we have, besides the $\boldsymbol{V}$'s and $\boldsymbol{U}$'s, conjugate contravariant and covariant vectors $\boldsymbol{V}^*$'s and $\boldsymbol{U}^*$'s ,whose components transform by the conjugate linear transformations of those of the V^j's and U_j's for a change of the complex cartesian reference frame. We must therefore consider the basis of the $\boldsymbol{I}^*_j$'s in the linear space whose elements are the $\boldsymbol{V}^*$'s and the dual basis of the $\boldsymbol{I}^{*j}$'s in the dual linear space of the $\boldsymbol{U}^*$'s, with $\langle \boldsymbol{I}^*_j, \boldsymbol{I}^{*k} \rangle = \delta^k_j$. We shall denote by G^*_n the G-algebra associated to this pair of dual linear spaces. In order to treat conveniently the antilinear transformations of $\boldsymbol{V}$'s into $\boldsymbol{V}^*$'s and $\boldsymbol{U}$'s into $\boldsymbol{U}^*$'s, the algebras G_n and G^*_n may be fused into an algebra $\overline{G}_n$, by assuming that the generators $(\boldsymbol{I})$ of G_n anticommute with the generators $(\boldsymbol{I}^*)$ of G^*_n and taking the $(\boldsymbol{I}_j)$, $(\boldsymbol{I}^j)$, $(\boldsymbol{I}^*_j)$, $(\boldsymbol{I}^{*j})$ as generators of $\overline{G}_n$. $\overline{G}_n$ is the G_{2n} associated to the direct sum of the linear spaces of the V's and V^*'s and to the direct sum of the U and U^* spaces. The $\underline{L}$ quasi-algebra associated to those spaces will be denoted by $\overline{\underline{L}}_n$.

It is important to notice that the above distinction of the V's and V^*'s as vectors of different linear spaces is rather different from the treatment of points in a complex space in which the conjugate points belong to the same space. The introduction of the direct sum of two conjugate linear spaces is familiar from the theory of the spinors of the Lorentz group: the spinor space is the direct sum of the two conjugate spaces of the half-spinors. The two kinds of half-spinors are exchanged by a spatial reflection, but transform by conjugate linear transformations for a proper Lorentz transformation. The C_4 of the space-time may be considered as the $\overline{G}_1$ of a one dimensional complex space. This corresponds to the well known relations between the Lorentz group and the one-dimensional complex projective group.

The metric of a hermitian complex space is defined by a non-degenerate hermitian invariant form $h_{jk}V_1^jV_2^{*k} = (V_1, V_2)$. The unitary metric corresponds to $h_{jk} = \delta_{j,k}$. By means of h_{jk} we can associate to V the U^* of components $h_{jk}V^j$ and to V^* the U of components $h_{jk}V^{*k}$. *In the hermitian geometry it suffices to introduce the linear spaces of the V's and V^*'s, because we can build the U's and U^*'s with those two kinds of vectors, by means of the available h_{jk}.* The metric of a real space defined by the symmetric tensor g_{jk} can be extended into a hermitian metric of the complex space of the same number of complex dimensions, in which the former is embedded, by assuming that g_{jk} transforms as $U_jU_k^*$ for a complex linear transformation of the co-ordinates x^j. *The unitary metric is obviously the extension of the euclidean metric to the complex domain.*

We can now understand why G_n over the complex numbers is a Jordan-Wigner algebra. Let us write $I_k^\dagger = I^jh_{jk}$. The $I_k^\dagger$'s transform as the I_k^* for the linear transformations leaving invariant the h_{jk}'s, which constitute the h-unitary group. We can take the (I_j)'s and the $(I_j^\dagger)$'s as generators of G_n over the complex numbers and we have the commutation rules $[(I_j), (I_k)]_+ = 0$, $[(I_j^\dagger), (I_k^\dagger)]_+ = 0$, $[(I_j), (I_k^\dagger)]_+ = h_{jk}1_{G_n}$. *In the case of the unitary metric $h_{jk} = \delta_{j,k}$, $(I_j^\dagger) = (I^j)$ can be taken as the adjoint of (I_j). The adjunction is an involution of G_n corresponding to the anti-polar transformation $V \to \sum (V^j)^* I_j^\dagger$ associated to the unitary metric.* In the case of G_n over the real numbers the adjunction becomes the transposition, which is an involution corresponding to the polar transformation $V \to \sum V^jI^j$, $U \to \sum U_jI_j$. Similar considerations can be applied to $\underline{L}_n$.

When the affine geometry of a complex space is considered, we cannot restrict $\overline{G}_n$ and $\overline{\underline{L}}_n$ to G_n and $\underline{L}_n$. It may be necessary to extend the theory of the separable Hilbert spaces of the quantum mechanics into that of the associated complex affine spaces, at least in some cases. This leads to a duplication of the numbers of available states for the quanta in second quantization, since the second quantization of Jordan-Wigner and Dirac-Jordan-

Klein quasi-algebras will be replaced by the corresponding $\overline{G}_\infty$ and $\underline{L}_\infty$. This duplication of the states corresponds to the introduction of a new two-valued variable of the quanta of the field.

The passage $G \to \overline{G}$ can be tentatively done in the case of the various G-algebras considered in the preceding sections. The G_1 of the scalars of Sect. **6** goes over into a $\overline{G}_1$ equivalent to the direct product of two Pauli spin algebras. *Thus we get the possibility of describing isotopic spin* 0, $\frac{1}{2}$ *and* 1. The G_4 of Sect. **6** goes over into a $\overline{G}_4$ equivalent to the direct product of four Dirac γ-algebras, which can be associated to four kinds of spin $\frac{1}{2}$ particles. *Thus we can introduce the electron, the μ-meson, the nucleon and the hyperon in our geometric-algebraic theory by replacing the* G_5 *of Sect.* **6** *by the corresponding* $\overline{G}_5$. The passage from the G_6 of Sect. **7** to its $\overline{G}_6$ allows the inclusion of the neutrino and gives us a new two-valued variable. *The extension of the G-algebras into $\overline{G}$-algebras leads to an extension of the associated Duffin-Kemmer-Petiau algebras and gives room for the introduction of several types of bosons.*

The above considerations indicate that the elementary particles reflect not only the geometric properties of the real space-time but also those of a complex hermitian four-dimensional space-time. This is not surprising, because of the essential part played by the complex numbers in the quantum mechanics. The property of symmetry of the field of the complex numbers corresponding to the conjugation seems to matter not only in connection with the adjunction but also with the symmetry of the $\overline{G}$'s and $\underline{L}$'s.

9. – Vector analysis and wave equations.

We have discussed in this paper several kinds of vector calculus corresponding to G, C and D algebras. *There is a kind of vector analysis associated to each kind of vector calculus.* The C_4 vector analysis is characterized by the association of the symbolic covariant vector $\boldsymbol{\partial}$ of components ∂_j to the symbol $\gamma_{\boldsymbol{\partial}} = \gamma^j \partial_j$, which lies at the core of the Dirac theory of the spin $\frac{1}{2}$ particles. In a Duffin-Kemmer-Petiau vector analysis, we have the association of $\boldsymbol{\partial}$ to $\beta_{\boldsymbol{\partial}} = \beta^j \partial_j$. *The free-particle Dirac and Duffin-Kemmer-Petiau equations* $(\gamma_{\boldsymbol{\partial}} - im)\psi = 0$ and $(\beta_{\boldsymbol{\partial}} - im)\psi = 0$ *are obviously important geometric equations defining eigen-*ψ's *of* $\gamma_{\boldsymbol{\partial}}$ *and* $\beta_{\boldsymbol{\partial}}$, *respectively.* Those equations are actually generalizations of the Dirac and Duffin-Kemmer-Petiau equations, because ψ depends on a number of functions which may be different from the numbers of components involved in the ordinary wave equations.

In the case of the generalized Dirac equation, ψ denotes an element of C_4 depending on the co-ordinates x^j and involves in general 16 functions. *The generalized Dirac equation contains as particular cases, not only the ordinary Dirac equation, but also the Klein-Gordon and Proca equations for particles of* *mass m, the latter corresponding to the choices* $\psi=\gamma^j B_j + B1_{C_4}$ *and* $\psi=\frac{1}{2}\gamma^j\gamma^k B_{jk}+\gamma^j B_j$ (TIOMNO, private communication), B_{jk} *denoting an antisymmetric tensor. When* $m=0$ and $\psi=\frac{1}{2}\gamma^j\gamma^k B_{jk}$, *the generalized Dirac equation is equivalent to the Maxwell equations,* B_{jk} *being antisymmetric.*

The above results indicate that both the quantum kinematics and the quantum dynamics deal with geometric properties of the space-time, the situation being, in principle, simpler in the relativistic quantum mechanics. The unity of matter, space and time in the macroscopic scale, discovered by EINSTEIN, seems to exist also in the microscopic scale. The fundamental symbol γ_∂ combines the generators γ of the metric algebra of the space-time with the operators ∂ for the infinitesimal translations in an exceedingly simple way.

Let us consider now the vector analysis corresponding to G_n. $\boldsymbol{\partial}$ is now associated to the symbol $(\boldsymbol{\partial}) = (\boldsymbol{I}^j)\partial_j$. Let (Ψ) be any element of $G_n(P)$ depending on the x^j's: $(\Psi) = \sum^{0,\dots,n} (p!)^{-1} A_{j_1,\dots,j_p}(P^{j_1,\dots,j_p})$, the A's being antisymmetric tensors. $(\boldsymbol{\partial})(\Psi)$ may be denoted by *rot* (Ψ), because the left-multiplication of (Ψ) by $(\boldsymbol{\partial})$ corresponds to replace the tensors A by their rotationals. By means of the metric tensor, we can build the symbol $g^{jk}(\boldsymbol{I}_j)\partial_k$, $g^{jk}(\boldsymbol{I}_j)\partial_k(\Psi)$ may be denoted by *div* (Ψ), because the A's are replaced by their divergences. The divergence is of course a metrical operator and the rotational an affine operator. We have the fundamental properties $rot^2=0$ and $div^2=0$. The laplacian operator for the (Ψ)'s is $\underline{\Delta} = g^{jk}\partial_j\partial_k 1_{G_n} = [rot, div]_+$. The Dirac operator $\gamma_\partial = div + rot$. We see that the G_n vector analysis is more flexible than the C_n one, and richer too.

The generalized Dirac equation $(\gamma_\partial - im)\psi = 0$ can be conveniently included in the G_n vector analysis. Let us take $g_{jk}=\varepsilon_j\delta_{j,k}$, as always possible by a suitable choice of the basic vectors. It is easily seen that $(P^{j_1,\dots,j_p}) = \gamma^{j_1}\dots\gamma^{j_p}(P)$, when all the j's are different. We have therefore $(\Psi)=\psi(P)$ for any (Ψ), ψ denoting an element of the C_n generated by the γ^j's, The equation $(\gamma_\partial - im)\psi=0$ is equivalent to *rot* $(\Psi) = (im - div)(\Psi)$. When $m=0$, the generalized Dirac equation becomes simply *rot* $(\Psi) = -$ *div* (Ψ). *In the case of the space-time, the general solution of* $(\gamma_\partial - im)(\Psi)=0$ *with* $m\neq 0$ *is obtained by means of the general solutions of the wave equations for scalar and pseudo-scalar, vector and pseudo-vector particles of mass m. The general solution of the ordinary Dirac equation is obtained from that of* $(\gamma_\partial - im)\psi = 0$ *by the right-multiplication of* ψ *by a suitable constant element of* C_4, *which can be taken as* $(1_{C_4}+i\gamma^4)(1_{C_4}+i\gamma^1\gamma^2)$, *with* $g_{jk}dx^j dx^k = (dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2$.

The vector analysis consists essentially in the theory of the direct product of C_n or G_n by $\underline{L}_n$. G_n is a geometric calculus for affine objects described by antisymmetric tensors and $\underline{L}_n$ a geometric calculus for affine objects described by symmetric tensors. $G_n \times \underline{L}_n$ is a more complete geometric calculus, because it allows to deal with objects described by general tensors. *This calculus leads to a generalization of the vector analysis, involving ordinary and symbolic functions.* In order to obtain a representation of $G_n \times \underline{L}_n$, it is convenient to consider the linear space $G_n \times \underline{L}_n(P)\{P\}$, whose elements χ are of the form $\chi = \sum_p (p!)^{-1} \cdot A_{j_1,\dots,j_p}(q)(I^{j_1}) \dots (I^{j_p})(P)\{\underline{P}\}$, the $A(q)$'s denoting formal series of Hermite functions. *χ can be associated to the generalized external differential expression* $\sum_p (p!)^{-1} A_{j_1,\dots,j_p}(x)[\mathrm{d}x^{j_1} \dots \mathrm{d}x^{j_p}]$. $G_n \times \underline{L}_n$ is therefore a formalism for the theory of the external differentials, which allows the coefficients to be taken as ordinary or symbolic functions. Generalized external differential forms play an important part in the modern topology of differentiable manifolds (De Rham's currents). *The algebra of the relativistic wave equations is therefore related to the topology of the space-time as a differentiable manifold. The fact that the theory of the relativistic wave equations requires the passage to the quantum field theory suggests that the quasi-algebra of the quantum field theory is of topological nature.*

The generalized neutrino equation $\gamma_\partial(\Psi) = 0$ is particularly interesting. When $(p!)^{-1} A_{j_1,\dots,j_p}(P^{j_1,\dots,j_p}) = (\Psi)$, $\gamma_\partial(\Psi) = 0$ is equivalent to *div* $(\Psi) = 0$ and *rot* $(\Psi) = 0$. These equations, with $g_{jk} = \delta_{j,k}$, are well known from the theory of the harmonic integrals, where they characterize the harmonic differential forms. In the physical case, the laplacian is a dalambertian, because of the indefinite metric of the space-time. The Maxwell equations correspond to $p = 2$. *In this case the equation $\gamma_\partial \psi = 0$ involves only the sub-algebra of C_4 generated by the Dirac α-matrices, $\boldsymbol{\alpha} = \gamma^4 \boldsymbol{\gamma}$, in three-dimensional vector notation. This sub-algebra is a C_3, a reducible algebra isomorphic to the direct sum of two Pauli spin algebras.*

The above discussion of the generalized Dirac equation shows clearly how spinors can be obtained from sets of antisymmetric tensors. *In order to build spinors with antisymmetric tensors, it is necessary to have a formalism allowing the « addition » of tensors of different orders, such as the C and G algebras. The conventional tensor calculus does not allow such « additions ». Here lies the essential advantage of the geometric algebras with respect to the tensor calculus.* It is interesting to notice that the anticommutative Grassmann algebra gives already the possibility of « adding » antisymmetric tensors of different orders. The (Ψ)-space is actually $G_n^U(P)$, G_n^U denoting the Grassmann algebra of the covariant vectors generated by 1_{G_n} and the (I^j)'s.

We used the relation $(P^{j_1,\dots,j_p}) = \gamma^{j_1} \dots \gamma^{j_p}(P)$ for $j_1 < \dots < j_p$, with $g_{jk} = \varepsilon_j \delta_{j,k}$. We have also $(P^{j_1,\dots,j_p}_{k_1,\dots,k_q}) = \gamma^{j_1} \dots \gamma^{j_p}(P)\gamma_{k_q} \dots \gamma_{k_1}$, with $j_1 < \dots < j_p$ and $k_1 < \dots < k_q$. *G_n is generated by the γ_j's and (P).* It suffices to introduce the extra generator (P) to pass from the metric algebra C_n to the affine algebra G_n.

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M. SCHÖNBERG
1957
N. 1 del *Supplemento* al Vol. 6, Serie X,
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