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Mario Schönberg

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On the Grassmann and Clifford Algebras I

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INTRODUCTION

1. The basic aim of the vector calculus is the construction of an algorithm adequate to deal with a certain kind of geometric objects. The ordinary vector calculus is an algorithm for the metric euclidean geometry of the three-dimensional space. Its extension to the metric euclidian geometry of spaces of dimensionality higher than three requires already substantial modifications. Other well known kinds of vertor calculus are the quaternion algebra and the Grassmann algebras, which can both be extended to n-dimensional spaces. The extension of the quaternion algebra leads to the important Clifford algebras, which are a calculus for the metric geometry of n-dimensional euclidian and pseudo-euclidian spaces. The Grassmann algebras are of two different kinds, according to the commutative or anticommutative nature of the product of the vectors, and give algorithms for the affine geometry of n-dimensional spaces. Grassmann developed also a point-calculus, which contains the vector calculus and is applicable to the projective feometry of n-dimensional spaces.

In the present paper we shall discuss in general lines some ideas on the geometric calculus which will be developed in more detail in the followinf papers of this series.*) We shall examine different kinds of geometric calculi associated to different kinds of geometry: affine, projective, conformal, euclidian and pseudo-euclidian, unitary and hermitian, symplectic, non-euclidian. All those geometries are associated to linear groups. Our methods consist essentially in the association of algebras to some types of linear groups, algebras in the sense of systems of hypercomplex numbers. Those algebras have a finite number of linearly independent elements for some of the linear groups and infinite numbers of linearly independent elements for other groups, when the underlying spaces are of finite dimensionality. We shall also consider spaces of infinite dimensionality, the corresponding algebras having always an infinity of linearly independent elements. Most of the algebras we shall consider exist for spaces with n dimensions, both for odd an even n's, although some of them be associated with symplectic groups. The various algebras

*) The main results of this series of papers were given in a lecture at the Recife meeting of the *Sociedade Brasileira para o Progresso da Ciência* the 8 july 1955.

can be defined over fields of characteristic zero, but we shall consider them only over the real or complex numbers, because our main interest is the application of those geometric algebras to physical theories.

THE EXTENDED GRASSMANN ALGEBRAS G_n AND L_n

2. The contravariant and covariant vectors of the n-dimensional affine space A_n will be denoted by V's and U's, respectively, and their components with respect to the basis of the I_j's and I^j's by v^j's and U^j's: $V = V^j I_j$, $U = U_j I^j$. The I_j's and I^j's are two reciprocal systems of vectors. The invariant $U_j V^j$ will be denoted by $\langle U, V \rangle$. In the algebra G_n the generators are the elements (V), (U) associated to the V's and U's with the multiplication rules

$$(V)(V') + (V')(V) = 0\,,\ (U)(U') + (U')(U) = 0\,,\ (U)(V) + (V)(U) = \langle U,V \rangle 1_{G_n} \tag{1}$$

G_n is assumed to be an associative algebra with a unity 1_{G_n}. Similarly L_n is an associative algebra with a unity 1_{L_n} generated by the elements $\{V\}$ and $\{U\}$ with the multiplication rules

$$\{V\}\{V'\} - \{V'\}\{V\} = 0\,,\ \{U\}\{U'\} - \{U'\}\{U\} = 0\,,$$
$$\{U\}\{V\} - \{V\}\{U\} = \langle U,V \rangle 1_{L_n} \tag{2}$$

The numbers of the basic field will be denoted by c's. We shall assume that

$$(\Sigma\, c\, V) = \Sigma\, c\, (V),\ (\Sigma\, c\, U) = \Sigma\, c\, (U);\ \{\Sigma\, c\, V\} = \Sigma\, c\, \{V\},\ \{\Sigma\, c\, U\} = \Sigma\, c\, \{U\} \tag{3}$$

Hence

$$(V) = V^j (I_j)\,,\ (U) = U_j (I^j)\ ;\ \{V\} = V^j \{I_j\}\,,\ \{U\} = U_j \{I^j\} \tag{3}$$

It follows from (1) and (2) that

$$(I_j)(I_k) + (I_k)(I_j) = 0,\ (I^j)(I^k) + (I^k)(I^j) = 0,\ (I^j)(I_k) + (I_k)(I^j) = \delta^j_k 1_{G_n} \tag{4}$$

$$\{I_j\}\{I_k\} - \{I_k\}\{I_j\} = 0,\ \{I^j\}\{I^k\} - \{I^k\}\{I^j\} = 0,$$
$$\{I^j\}\{I_k\} - \{I_k\}\{I^j\} = \delta^j_k 1_{L_n} \tag{5}$$

By taking

$$q_j = \{I_j\}\,,\ p_j = (2\pi i)^{-1} h \{I^j\} \tag{6}$$

equations (5) become the Heisenberg commutation rules for the coordinate and momentum operators of a quantum-mechanical system with n degrees of freedom. L_n *over the complex numbers is equivalent to the Heisenberg algebra of the p's and q's of a system with n degrees of freedom.*

A basis of G_n is constituted by the 2^{2n} elements $(I_1)^{r_1} \ldots (I_n)^{r_n} (I^n)^{s_n} \ldots (I^1)^{s_1}$ the exponents r,s taking the values 0 and 1. A basis of L_n is constituted by the elements $\{I_1\}^{r_1} \ldots \{I_n\}^{r_n} \{I^1\}^{s_1} \ldots \{I^n\}^{s_n}$, $r, s = 0, 1, 2, \ldots \infty$. G_n is of the order 2^{2n}, L_n of infinite order. The general forms of the elements Γ of G_n and Λ of L_n are

$$\Gamma = \sum_{p,q}^{0,\ldots,n} (p!\,q!)^{-1}\, C^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} (I_{j_1}) \ldots (I_{j_p}) (I^{k_q}) \ldots (I^{k_1}) \tag{7}$$

$$\Lambda = \sum_{p,q}^{0,\ldots,\infty} (p!q!)^{-1} (a_1! \ldots a_n!)(b_1! \ldots b_n!)\, C^{j_1,\ldots,j_p}_{k_1,\ldots,k_q} \{I_{j_1}\} \ldots \{I_{j_p}\} \{I^{k_1}\} \ldots \{I^{k_q}\} \tag{8}$$

a_r denoting the number of j's equal to the integer r and b_r the number of k's equal to r.

The coefficients C in equations (7) and (8) transforms as the components of tensors for a change of the basic vectors I_j. In the case of (7) the tensors are antisymmetric with respect to the j's and k's, separately, and symmetric in the case of (8). *G_n is a calculus for the affine geometric objects described by sets of antisymmetric tensors, L_n a calculus for the affine geometric objects described by sets of symmetric tensors. In order to get a calculus for the affine objects described by tensors of any kind, it suffices to take the direct product $G_n \times L_n$, which is an algebra generated by the (I)'s and $\{I\}$'s satisfying the multiplication rules (4) — (5) with the identification of the unities 1_{G_n} and 1_{L_n} with the unity of the product algebra, any (I) being taken as commutable with any $\{I\}$.*

The anticommutative Grassmann algebra of the contravariant vectors is the sub-algebra G_n^v of G_n generated by 1_{G_n} and the (I_j)'s. The comutative Grassmann algebra of the contravariant vectors is the sub-algebra L_n^v of L_n generated by 1_{L_n} and the $\{I_j\}$'s. Our algebras G_n and L_n are therefore extensions of the two kinds of Grassmann algebras of contravariant vectors. $G_n \times L_n$ may be considered as the complete affine vector algebra.

Let T_j^k be a tensor of determinant $|T| \neq 0$. T_j^k defines a central-affine transformation of A_n. We shall associate to T_j^k the element (T) of G_n and the element $\{T\}$ of L_n

$$(T) = T_k^j (I_j)(I^k) \qquad \{T\} = T_k^j \{I_j\}\{I^k\} \tag{9}$$

The central-affinity T transforms V into V^T and U into U^T: $V^{T,j} = T_k^j \ V^k, U_j^T = \mathcal{T}_j^k U_k$, $\mathcal{T}_j^h T_h^k = \mathcal{T}_h^k T_j^h = \delta_j^k \cdot \mathcal{T}$ is the inverse of T. We have

$$(V^T) = [(T),(V)], (U^T) = [(U),(\mathcal{T})]; \{V^T\} = [\{T\},\{V\}], \{U^T\} = [\{U\},\{\mathcal{T}\}] \tag{10}$$

The square bracket denotes the commutator, as usual. It is well known that there are always τ's such that $T = e^{\tau}$, $\mathcal{T} = e^{-\tau}$. It follows from (10) that

$$(V^T) = \exp(\tau)(V)\exp(-\tau), (U^T) = \exp(\tau)(U)\exp(-\tau) \tag{11}$$

$$\{V^T\} = \exp\{\tau\}\{V\}\exp\{-\tau\}, \{U^T\} = \exp\{\tau\}\{U\}\exp\{-\tau\} \tag{12}$$

(τ) and $\{\tau\}$ being defined in the same way as (T) and $\{T\}$, respectively. The (V^T)'s and (U^T)'s can be taken as generators of G_n, instead of the corresponding (V)'s and (U)'s. Thus we get an automorphism $\Gamma \longrightarrow \Gamma^T$ of G_n. In a similar way we define an automorphism $\Lambda \longrightarrow \Lambda^T$ of L_n. It follows from equations (11)-(12) that

$$\Gamma^T = \exp(\tau)\,\Gamma\,\exp(-\tau) \quad , \quad \Lambda^T = \exp\{\tau\}\,\Lambda\,\exp\{-\tau\} \tag{13}$$

The central-affinities of A_n induce inner automorphisms in G_n and L_n.

The metric geometry is subordinated to the affine geometry. We may expect the affine algebra G_n to contain metric algebras as sub-algebras.

It is indeed so, as we shall now see. By means of the metric tensor g_{jk} we build the units $\gamma_j^{(\pm)}$ of two Clifford algebras C_n^g and C_n^{-g}: $\gamma_j^{(\pm)} = (I_j) \pm g_{jk} (I_k)$

$$\gamma_j^{(\pm)} \gamma_k^{(\pm)} + \gamma_k^{(\pm)} \gamma_j^{(\pm)} = \pm\, 2\, g_{jk}\, 1_{G_n} \quad , \quad \gamma_j^{(+)} \gamma_k^{(-)} + \gamma_k^{(-)} \gamma_j^{(+)} = 0 \tag{14}$$

The $\gamma_j^{(+)}$*'s generate a Clifford algebra* C_n^g *corresponding to the metric* $g_{jk}\, x^j x^k$. *The* $\gamma_j^{(-)}$*'s generate a Clifford algebra* C_n^{-g} *corresponding to the metric* — $g_{jk}\, x^j x^k$. *Since the* $\gamma_j^{(+)}$*'s and* $\gamma_j^{(-)}$*'s are a set of linearly independent generators of* G_n, *it follows from (14) that* G_n *is the Clifford algebra of the 2n-dimensional pseudo-euclidian space with the metric* $g_{jk}\, (x^j\, x^k - x^{n+j}\, x^{n+k})$, *whose canonical form has n positive and n negative squares for any choise of* g_{jk}.

The algebra of the Dirac γ-matrices is isomorphic to the C_4 of the space-time. The algebra $C_4 \times L_4$ of the quantities of the relativistic quantum mechanics of the electron is therefore a sub-algebra of the affine algebra $G_4 \times L_4$ of the space-time. We shall prove in the paper II of this series that $G_4 \times L_4$ contains also as sub-algebras the algebras of the relativistic quantum mechanics of the particles with spins 0 and 1, because G_4 has sub-algebras of the Duffin-Kemmer type.

In the case of even n, there are antisymmetric tensors f_{jk} of determinant $|f| \neq 0$. We shall introduce an algebra K_n with a unity 1_{K_n} generated by the elements λ_j with the multiplication rule $[\lambda_j, \lambda_k] = 2 f_{jk}\, 1_{K_n}$, $|f| \neq 0$. K_n is the analogue of C_n for the group of linear transformations T_f leaving invariant the bilinear form $f_{jk}\, x_1^j\, x_2^k$, the symplectic group associated to f_{jk}, in the same way as the g-orthogonal group of the linear transformations T_g to the symmetric bilinear form $g_{jk}\, x_1^j\, x_2^k$. The 2n elements of L_n $\lambda_j^{(\pm)} = \{I_j\} \pm f_{jk} \{I^k\}$ are linearly independent and

$$[\lambda_j^{(\pm)}, \lambda_k^{(\pm)}] = \pm\, 2\, f_{jk}\, 1_{k_n}, \quad [\lambda_j^{(\pm)}, \lambda_k^{(-)}] = 0 \tag{15}$$

The $\lambda_j^{(+)}$'s generate a symplectic algebra K_n^f associated to f_{jk}. The $\lambda_j^{(-)}$'s generate a symplectic algebra K_n^{-f} associated to $-f_{jk}$. *It follows from the commutability of the two kinds of* λ*'s that* L_n *is the direct product of two* K_n*'s, one associated to* f_{jk} *and the other to* $-f_{jk}$, *when n is even.* There is a corresponding theorem for G_n and the Clifford algebras. Let us choose the I_j's in such a way that $g_{jk} = \varepsilon_j\, \delta_{j,k}$, $\varepsilon_j = \pm 1$, as it is always possible. The element $\omega = \gamma_1^{(+)} \cdots \gamma_n^{(+)} \gamma_1^{(-)} \cdots \gamma_n^{(-)}$ anticommutes with all the $\gamma_j^{(\pm)}$'s. The $\omega \gamma_j^{(-)}$'s generate a Clifford algebra C_n^g associated to g_{jk} *Since all the elements of* C_n^g *and* C_n^g *are commutable,* G_n *is the direct product of two* C_n*'s associated to the same* g_{jk} *both for even and odd n's.*

The symplectic groups have only one family of transformations, all of determinant 1. The g-orthogonal groups have transformations of determinant $\pm$ 1, each of the two families containing two different sub-families when $g_{jk} x^j x^k$ is not definite. All the T_f's can be expressed as exponentials $\exp \tau_f$, τ_f being of the nature of an infinitesimal transformation: $\tau^h_{f,j} f_{hk} = = \tau^h_{f,k} f_{hj}$. The T_g's can also be expressed as exponentials $\exp \tau_g$, but τ_g can only be chosen of the nature of an infinitesima transformation for the T_g's of determinant 1 of the subfamily containing the unity transformation of A_n: $\tau^h_{g,j} g_{hk} = - \tau^h_{g,k} g_{h,j}$. Therefore we can define a symmetric tensor t^{jk}_f by the condition $t^{jh}_f f_{hk} = \tau^j_{f,k}$ for all T_f's and an antisymmetric tensor t^{jk}_g by the condition $t^{jh}_g g_{hk} = \tau^j_{g,k}$ for the above mentioned T_g's, the proper g-rotations. It is easily seen that $\{\tau_f\} = \frac{1}{4} t^{jk}_f (\lambda^{(+)}_j \lambda^{(+)}_k - \lambda^{(-)}_j \lambda^{(-)}_k)$ for all T_f's and for the proper g-rotations$(\tau_g) = \frac{1}{4} t^{jk}_g (\gamma^{(+)}_j \gamma^{(+)}_k - \gamma^{(-)}_j \gamma^{(-)}_k)$. Hence

$$S^{(\pm)}_f \lambda^{(\pm)}_j S^{(\pm)}_f = T^k_{f,j} \lambda^{(\pm)}_k \quad , \quad S^{(\pm)}_f = \exp\left(\pm \frac{1}{4} t^{jk}_f \lambda^{(\pm)}_j \lambda^{(\pm)}_k \right) \tag{16}$$

$$S^{(\pm)}_g \gamma^{(\pm)}_j S^{(\pm)}_g = T^k_{g,j} \gamma^{(\pm)}_k \quad , \quad S^{(\pm)}_g = \exp\left(\pm \frac{1}{4} t^{jk}_g \gamma^{(\pm)}_j \gamma^{(\pm)}_k \right) \tag{17}$$

Equations (17) are valid for proper g-rotations of spaces of even and odd dimensionality. It is well known that in the case of spaces of even dimensionality all the g-orthogonal transformations induce inner automorphisms in the corresponding Clifford algebra. This can be shown by the consideration of the g-reflections, i.e. g-orthogonal transformations leavinf invariant the vectors of a hyperplane. We shall discuss this point in the next paper of the series. The above results show that the behaviour of G_n with respect to the g-orthogonal transformations is similar to that of L_n with respect to the sympletic transformations. We shall prove in the next section that L_n is the symplectic algebra of a 2n-dimensional space, as G_n is the Clifford algebra of this space.

The relation between L_n and the Heisenberg commutation rules shows that the symplectic group plays a fundamental role in nature. In recent years the importance of the symplectic group in the classical mechanics has been more clearly understood. The above results show its significance for the quantum kinematics. We shall see in the following papers of the series its importance in the electromagnetism. In the case of the space-time, a uniform electromagnetic field such that $E \cdot H \neq 0$ defines a f_{jk} with $|f| \neq 0$. *There is a* K_4 *associated to any such field.*

THE 2_n-DIMENSIONAL SPACE ASSOCIATED TO A_n

3. We shall now consider the 2n-dimensional linear space S_{2n} which is the direct sum of the linear spaces of the v's and u's of A_n. The vectors of S_{2n} will be denoted by W's. We shall assume that the components $W^1, \ldots, W^n$ are V^j-like and the components $W^{n-1}, \ldots, W^{2n}$

U_j-like. Thus the W's whose components W^{n+j} are nil can be identified with the V's and the W's whose components W^j are nil with the U's. Any W is then a sum $V + U$. When A_n is the configuration space of a dynamical system with n degrees of freedom, its phase-space is analogous to S_{2n}. We shall call S_{2n} the phase-space associated to A_n.

There is a natural definition of the inner product of two W's, which gives a pseudo-euclidian metric in $S_{2n} : (W_1, W_2)_+ = \frac{1}{2}(\langle U_1, V_2\rangle + \langle U_2, V_1\rangle)$, with $W = U + V$. There is also a natural definition of the symplectic product of two W's, which gives a sympletic geometry in S_{2n}: $(W_1, W_2)_- = \frac{1}{2}(\langle U_1, V_2\rangle - \langle U_2, V_1\rangle)$ *We shall now prove that* G_n *is the* C_{2n} *of the symplectic geometry of* S_{2n} *defined by* $(W_1, W_2)_+$ *and* L_n *the* K_{2n} *of the symplectic geometry of* S_{2n} *defined by* $(W_1, W_2)_-$. It is well known that in a C_n there is an element γ_V associated to each contravariant vector $V : \gamma_V = V^j \gamma_j$. The γ_j's are the elements of C_n associated to the I_j's and their commutation rule is a particular case of the general commutation rule $\gamma_{V_1}\gamma_{V_2} + \gamma_{V_2}\gamma_{V_1} = 2 g_{jk} V_1^j V_2^k 1_{C_n}$. In a similar way in K_n we shall associate to each V the element $\lambda_V = V^j \lambda_j$, the commutation rule of the λ_V's being $[\lambda_{V_1}, \lambda_{V_2}] = 2 f_{jk} V_1^j V_2^k 1_{K_n}$. In the case of S_{2n}, we have for the above defined inner products and symplectic products $\Gamma_{W_1}\Gamma_{W_2} + \Gamma_{W_2}\Gamma_{W_1} = (\langle U_1, V_2\rangle + \langle U_2, V_1\rangle) 1_{C_{2n}}$ and $[\Lambda_{W_1}, \Lambda_{W_2}] = (\langle U_1, V_2\rangle - \langle U_2, V_1\rangle) 1_{K_{2n}}$, the first order elements of C_{2n} and K_{2n} being denoted by Γ_W's and Λ_W's, respectively. *By taking* $1_{C_{2n}} = 1_{G_n}$ and $1_{K_{2n}} = 1_{L_n}$, $\Gamma_W = (U) + (V)$ *and* $\Lambda_W = \{U\} + \{V\}$ *we can identify* C_{2n} *with* G_n *and* K_{2n} *with* L_n.

We can now apply to G_n and L_n the fundamental theorems on the inner automorphisms induced in the Clifford and symplectic algebras by the g-orthogonal and symplectic linear transformations, because they are a C_{2n} and a K_{2n}, respectively. *The linear transformations of the 2n variables* U_j, V^j *leaving invariant the quadratic form* $U_j V^j = \frac{1}{4}\sum_j^{1,\dots,n} \{(U_j + V^j)^2 - (U_j - V^j)^2\}$ *induce inner automorphisms in* G_n. *The linear transformations of the 2n variables* U_j, V^j *leaving invariant the alternate bilinear form* $U_{1,j} V_2^j - U_{2,j} V_1^j$ *induce inner automorphisms in* L_n. *Thus we get a group of automorphisms of* G_n *depending on* $n(2n-1)$ *parameters and a group of automorphisms of* L_n *depending on* $n(2n+1)$ *parameters. Each of those groups contains as a sub-group the full linear group in n variables, i.e. the central-affine group of* A_n, *because these central-affine transformations leave invariant* $\langle U_1, V_2\rangle$ *and* $\langle U_2, V_1\rangle$.

The theory of the geometric algebras G_n and L_n leads to a remarkable extension of the affine geometry. The linear spaces of the U's and V's are totally isotropic sub-spaces of S_{2n} with respect to the metric defined by the inner product $(W_1,W_2)_+$, because $(V_1,V_2)_+=0$ and $(U_1,U_2)_+=0$; these totally isotropic spaces are of the maximal dimension. The linear spaces of the U's and V's are also totally isotropic sub-spaces of the maximal dimension with respect the symplectic product $(W_1, W_2)_-$. Let us introduce the linear operator of S_{2n} defined by the conditions $\theta V=V, \theta U=-U$. We have $(W_1, W_2)_- = (W_1, \theta W_2)_+$. In order that a linear transformation τ of S_{2n} leave invariant both the inner and symplectic products it must be commutable with θ, because $(W_1, \theta\tau W_2)_+ = (W_1, \tau W_2)_- = = (\tau^{-1} W_1, W_2)_- = (\tau^{-1} W_1, \theta W_2)_+ = (W_1, \tau\theta W_2)_+$ for any W's. Therefore τ transforms U's into U's and V's into V's and, since it leaves invariant the inner product $(W, W)_+ = \langle U, V \rangle$ it is equivalent to a central-affine transformation of A_n. *The only transformations common to the abofe pseudo-orthogonal and symplectic groups of* S_{2n} *are the central-affine transformations of* A_n *acting on the* U *and* V *parts of the* W's. *Although we have found more general groups inducing inner automorphisms in* G_n *and* L_n, *they do not lead to a group larger than the central-affine of* A_n *inducing inner automorphisms in the algebra* $G_n \times L_n$.

The Clifford algebra C_{2n} over the real numbers associated to a quadratic form with n positive and n negative squares is a total metric algebra. In the next paper of this series we shall give the expressions of the units of the total matric algebra C_{2n} equivalent to $G_n \cdot G_n$ *over a field of characteristic* $\neq 2$ *is equivalent to the algebra of all the* $2^n \times 2^n$ *matrices over the same field. The group of the inner automorphisms of* G_n *is isomorphic to the projective group of the* $(2^n - 1)$ *dimensional over the same field.*

It is now generally accepted that in the natural system of units of physics the Planck constant h must be taken dimensionless with the value 2π. Thus the momentum has the dimension of the inverse of a length, i.e. the natural dimension of the covariant vectors of space-time. The S_8 associated to the space-time can be identified with the relativistic phase-space of a particle. The pseudo-orthogonal and symplectic groups of this S_8 are likely to be important physical groups. *It is interesting to notice that the coefficients of the general transformations of those groups mixing the* U_j's *and* V^j's *must involve some constant with the dimension of a length, because the* V^j's *have the dimension of a length and the* U^j's *that of the inverse of a length. Those groups seem therefore to be related to the theory of the elementary length.*

The S_{2n} associated to the configuration space of a dynamical system with n degrees of freedom can be identified with its non-relativistic phase-space, by giving to the momentum the dimension of the inverse of a length. In this case the symplectic group of S_{2n} associated to L_n is simply that associated to the differential element of the Poincaré integral-invariant of the second order $\int \sum_{j}^{1,\ldots,n} (d_1 p_j d_2 q_j - d_2 p_j d_1 q_j)$, as a bilinear form in the vectors $(d_1 p_j, d_1 q_j)$ and $(d_2 p_j, d_2 q_j)$. *The Heisenberg algebra of the coordinate and momentum operators of a dynamical system with n degrees of freedom is the symplectic algebra associated to the Poincaré bilinear alternate form* $\sum_{j}^{1,\ldots,n} (d_1 p_j d_2 q_j - d_2 p_j d_1 q_j)$. The motion of the quantal system corresponds to the transformation $\Lambda \longrightarrow \exp(-itH)\ \Lambda\ \exp(itH)$ of the quantities Λ of the system, H being the quantal time-independent hamiltonian and t the time. The theory of the motion of the quantal system is therefore included in the theory of the inner automorphisms of L_n. The group of the inner automorphisms of L_n is the projective group of a space of infinite dimensionality, the points of this space being the states of the dynamical system.

In the Schrödinger formalism, the momenta p_j are taken as the differential operators $-iD_j, D_j$ denoting the partial derivative with respect to the numerical variable q_j. This formalism is based on a representation of L_n in a linear space of indefinitely derivable functions $\psi(q_1, \ldots, q_n)$, in which $\{I_j\}$ corresponds to the linear transformation $\psi \longrightarrow q_j \psi$ and $\{I^j\}$ to the linear transformation $\psi \longrightarrow -iD_j \psi$. *Thus we see that* L_n *is closely related to the differential calculus of the functions of domain* A_n. L_n *is an algebra underlying differential calculus of the functions of n variables. We get in this way a remarkable relation between the differential calculus and the geometric calculus of the affine objects described by symmetric tensors.*

The above results show that there are deep relations between the symplectic geometry of a 2n-dimensional space and the differential calculus of functions of n-variables. The theory of K_{2n} or L_n does not depend on the use of any special representation. Moreover L_n can be taken over fields not possessing properties of continuity, over the rational numbers for instance. *Thus we see that the theory of* K_{2n} *leads to a generalization of the differential calculus.* The relations of the commutative Grassmann algebra with analysis were indicated by Grassmann in the Ausdehnungslehre of 1862. The introduction of L_n and of the modern algebric methods allows a considerable extension of the scope of Grassmann's ideas and shows new relations between geometry and analysis. Thus the very simple symplectic transformation of S_{2n} $U \longrightarrow V, V \longrightarrow -U$ corresponds to the inner automorphism of L_n defined by the relations $\{I^j\} \longrightarrow \{I_j\}, \{I_j\} \longrightarrow -\{I^j\}$. *The theory of this inner automorphism of* L_n *is obviously an algebrization and geometrization of the theory of the Fourier and Laplace transformations.*

It is interesting to remark that the symmetry of the inner product $(W_1, W_2)_+$ leads to the sign $+$ in the commutation rules of the $(W)'$s and the antisymmetry of the symplectic product to the sign — in the commutation rules of the $\{W\}$'s The symmetric nature of the metric bilinear form of S_{2n} renders its C_{2n} the algebra of the geometric objects of A_n described by antisymmetric tensors and the antisymmetric nature of the symplectic bilinear form of S_{2n} renders its K_{2n} the algebra of the geometric objects of A_n described by symmetric tensors.

The relations between the extension of the Grassmann geometric calculus and the quantum mechanics are remarkable and show that the properties of matter correspond very closely to the geometry of the space-time continuum, even in the atomic domain. This will be shown in more detail in the following papers of the series. Our considerations refer to flat spaces, which are tangent spaces of the curved space-time. The extension of the present theory to curved spaces allows to include the quantum dynamics in a geometric picture, the present results refering only to the quantum kinematics.

SUMMARY

An extension of the vector algebras of affine spaces is discussed. The extension of the Grassmann algebra of contravariant vectors with anticommutative product by the introduction of the covariant vectors of the affine n-dimensional space leads to an algebra G_n isomorphic to a Clifford algebra of a space with twice the number of dimensions. A similar extension of the Grassmann algebra with commutative product of contravariant vectors leads to an algebra L_n closely related to the differential calculus. An algebra K_n associated to the sympletic group of a space of even dimensionality and analogous to the Clifford algebra is introduced. L_n is a symplectic algebra K_{2n} of a space with twice the number of dimensions of the basic affine space. G_n contains as sub-algebras the Clifford algebras corresponding to all the euclidian and pseudo-euclidian metrics. L_n over the complex numbers is equivalent to the algebra of the position and momentum operators of a quantum-mechanical system with n degrees of freedom. Remarkable relations between the quantum kinematics and the geometry of the affine spaces are discussed. The spin-like quantal variables are associated to geometric objects described by anti-symmetric tensors and to the metric geometry. The position and momentum quantal variables are associated to geometric objects described by symmetric tensors and to the symplectic geometry.