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Vortex Motions of the Madelung Fluid.

M. Schönberg

Faculdade de Filosofia, Ciencias e Letras da Universidade de São Paulo, Brasil

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Summary. — The general motions of the continuous medium (Madelung fluid), whose irrotational motions are described by the Schrödinger equation, are discussed. It is shown that many of the basic theorems of the vortex motions of the inviscid barotropic fluids are also valid for the Madelung fluid. A detailed discussion of the Clebsch parameters is given. It is shown that there is a special type of steady motions, similar to the Beltrami motions in which the streamlines coincide with the vortex-lines, which corresponds to a close generalization of the ordinary stationary states. The quantization of the vortex-tubes is discussed. Examples of Beltrami and discontinuous motions of the Madelung fluid are given. It is shown that the general motions of the Madelung fluid can also be physically interpreted in terms of the ordinary quantal states of a particle.

1. – Introduction.

Madelung (1) showed that the Schrödinger equation for a particle is equivalent to a set of equations that describe a flow in space. Let us denote by Ψ the wave function of a particle of mass m and charge e moving in the electromagnetic field described by the potentials A_0, $\boldsymbol{A}$. It follows from the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{x}} - \frac{e}{c}\boldsymbol{A}\right)^2 + eA_0\right\}\Psi, \tag{1}$$

(1) E. Madelung: *Zeits. f. Phys.*, **40**, 332 (1926).

that the amplitude R and the phase $S/\hbar$ of Ψ satisfy the equations

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial \boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}\right)^2+eA_0-\frac{\hbar^2}{2m}\frac{\Delta R}{R}=0\,, \tag{2}$$

$$\frac{\partial R^2}{\partial t}+\operatorname{div}\left\{\frac{R^2}{m}\left(\frac{\partial S}{\partial \boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}\right)\right\}=0\,, \tag{3}$$

$$\Psi=R\exp\left[\frac{i}{\hbar}S\right]. \tag{4}$$

Let us introduce the velocity $\boldsymbol{v}$ and the mass density $m\varrho$

$$\boldsymbol{v}=\frac{1}{m}\left(\frac{\partial S}{\partial \boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}\right),\qquad \varrho=R^2. \tag{5}$$

It follows from (2) and (3) that

$$m\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v}=e\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)+\frac{\hbar^2}{2m}\frac{\partial}{\partial \boldsymbol{x}}\frac{\Delta R}{R}\,, \tag{6}$$

($\boldsymbol{E}$ = electric field; $\boldsymbol{H}$ = magnetic field)

$$\frac{\partial \varrho}{\partial t}+\operatorname{div}(\varrho\boldsymbol{v})=0\,. \tag{7}$$

Equation (6) has the form of the Euler equation of motion of a special kind of charged fluid medium of charge density $e\varrho$, the velocity being $\boldsymbol{v}$ and the spatial components of the stress tensor having the values

$$T_{ab}=m\varrho v_a v_b+\frac{\hbar^2}{m}\frac{\partial R}{\partial x_a}\frac{\partial R}{\partial x_b}-\delta_{ab}\frac{\hbar^2}{4m}\Delta\varrho\,. \tag{8}$$

Indeed, since the density of mechanical momentum is

$$\boldsymbol{G}=m\varrho\boldsymbol{v} \tag{9}$$

the equation (6) is equivalent to the following

$$\frac{\partial \boldsymbol{G}}{\partial t}+\operatorname{div}T=e\varrho\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right), \tag{10}$$

div T denoting the divergence of the three-dimensional tensor defined by (8). Equation (7) is the continuity equation of the Madelung fluid.

It follows from the first equation (5) that the vorticity ζ of the motion of the Madelung fluid associated to the Schrödinger equation (1) satisfies the condition

$$\zeta = \operatorname{rot} \boldsymbol{v} = -\frac{e}{mc}\boldsymbol{H}\,. \tag{11}$$

In the absence of magnetic fields $\zeta = 0$ and the motions associated to the equation (1) are irrotational. We shall call the motions satisfying the condition (11) quasi-irrotational. TAKABAYASHI (²) called the attention to the fact that the equations (6) and (7), in the absence of magnetic fields, are valid both for the rotational and irrotational motions of the Madelung fluid and may be regarded as a generalization of the Schrödinger equation (1). Independently, we remarked that (6) and (7) generalize the Schrödinger equation (1) and introduced also the distinction between quasi-irrotational and general motions of the Madelung fluid. *The presence or absence of vorticity is not the fundamental fact, since the Schrödinger equation may be applicable even when there is vorticity, provided the motion be quasi-irrotational. This is shown even clearer by the extension of the Madelung hydrodynamical model to particles with spin* (³)*: in the case of a particle with spin there is vorticity, even in the absence of magnetic fields.*

We have shown (⁴) that it is possible to generalize the wave equations of the quantum mechanics by a suitable modification of the usual variational principles. We started from a generalized form of the classical Hamilton-Jacobi theory and from a new kind of classical variational principle associated to that generalization. In the particular case of the non relativistic wave equation for a spinless particle, our generalization can also be derived from the analysis of the general motions of the Madelung fluid, as will be shown in section **3**. The theory of reference (⁴) was further extended (⁵) to all the field formalisms admitting the gauge-invariance of the first kind. We showed in that paper that the generalization procedure amounts to replace certain integrable field quantities by non integrable quantities, whose differences of values at infinitesimally close points in space-time are definite. In the particular case of the non relativistic Schrödinger equation, the phase S becomes a non integrable quantity in the general vortex motion of the Madelung fluid.

We showed in reference (⁵) that the general theorems on the vortex motions

(²) T. TAKABAYASHI: *Prog. Theor. Phys.*, **9**, 187 (1953).

(³) M. SCHÖNBERG: *Nuovo Cimento*, **12**, 103 (1954); D. BOHM, R. SCHILLER and J. TIOMNO: *Suppl. Nuovo Cimento*, **1**, 48 (1955).

(⁴) M. SCHÖNBERG: *Nuovo Cimento*, **11**, 674 (1954).

(⁵) M. SCHÖNBERG: *Nuovo Cimento*, **12**, 649 (1954).

of the inviscid barotropic fluids under the action of conservative forces have their analogues in the generalization of the ordinary gauge-invariant field formalisms, by the introduction of non integrable quantities, the vorticity being replaced by a non-integrability vector $\boldsymbol{\eta}$. The lines of force of the $\boldsymbol{\eta}$-field play the part of the vortex-lines of the ordinary hydrodynamics. The η-tubes formed by those η-lines have a time independent strength, the strength being defined as the flux of the vector $\boldsymbol{\eta}$ through any section of the tube. In the case of the vortex motions of the Madelung fluid we have

$$(12) \qquad \boldsymbol{\eta} = \operatorname{rot} \boldsymbol{u}\,, \qquad \boldsymbol{u} = m\boldsymbol{v} + \frac{e}{c}\boldsymbol{A}\,.$$

There is a circulation theorem for the vector $\boldsymbol{u}$

$$(13) \qquad \oint_C \boldsymbol{u}\cdot\delta\boldsymbol{x} = \oint_{C_0} \boldsymbol{u}_0\cdot\delta\boldsymbol{x}_0\,,$$

C being a closed fluid line formed, at the time t, by the fluid elements lying at the time 0 on the closed curve C_0; $\boldsymbol{x}_0$ and $\boldsymbol{u}_0$ being the position and $\boldsymbol{u}$-vectors at the time 0. We shall extend the ordinary hydrodynamical vortex theorems to the Madelung fluid in section 2.

The importance of the Clebsch parameters for the generalization of the gauge-invariant field theories was shown in references ([4]) and ([5]). The properties of the Clebsch parameters will be discussed in detail in section 3. The first equation (5) is replaced by

$$(14) \qquad \boldsymbol{v} = \frac{1}{m}\left(\frac{\partial S}{\partial \boldsymbol{x}} + \lambda\frac{\partial \mu}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A}\right),$$

in the case of a general motion. S, λ and μ are the Clebsch parameters. It will be shown in section 4 that the same distribution of velocities is also described by the parameters S', λ' and μ' defined by the equations

$$(15) \qquad \begin{cases} \lambda\dfrac{\partial \mu}{\partial \boldsymbol{x}} - \lambda'\dfrac{\partial \mu'}{\partial \boldsymbol{x}} = \dfrac{\partial}{\partial \boldsymbol{x}}\Gamma(t, \mu, \mu')\,; \\ S' = S + \Gamma + F(t)\,. \end{cases}$$

Γ being an arbitrary function of t, μ, μ' and F an arbitrary function of t. The equations of motion in terms of the Clebsch parameters are

$$(16) \qquad \begin{cases} \dfrac{\partial \mu}{\partial t} + \boldsymbol{v}\cdot\dfrac{\partial \mu}{\partial \boldsymbol{x}} = \dfrac{\partial K}{\partial \lambda}\,, \qquad \dfrac{\partial \lambda}{\partial t} + \boldsymbol{v}\cdot\dfrac{\partial \lambda}{\partial \boldsymbol{x}} = -\dfrac{\partial K}{\partial \mu}\,, \\ \dfrac{\partial S}{\partial t} + \lambda\dfrac{\partial \mu}{\partial t} + \dfrac{1}{2m}\left(\dfrac{\partial S}{\partial \boldsymbol{x}} + \lambda\dfrac{\partial \mu}{\partial \boldsymbol{x}} - \dfrac{e}{c}\boldsymbol{A}\right)^2 + eA_0 - \dfrac{h^2}{2m}\dfrac{\Delta R}{R} = K\,, \end{cases}$$

$$(K = \text{function of } t,\ \lambda,\ \mu)\,.$$

$K(t, \lambda, \mu)$ being an arbitrary function. The simplest choice $K=0$ renders the parameters λ and μ constants of the motion. We shall always assume that λ and μ are chosen as constants of the motion and $K=0$. We shall extend the equation (4) to the case of the general motions of the fluid. Thus we get the generalized Schrödinger equation

$$(17)\quad \begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = \dfrac{1}{2m}\left(\dfrac{\hbar}{i}\dfrac{\partial}{\partial \boldsymbol{x}} - \dfrac{e}{c}\boldsymbol{A} + \lambda \dfrac{\partial \mu}{\partial \boldsymbol{x}}\right)^2 \Psi + \left(eA_0 + \lambda \dfrac{\partial \mu}{\partial t}\right)\Psi, \\ \dfrac{\partial \lambda}{\partial t} + \boldsymbol{v}\cdot\dfrac{\partial \lambda}{\partial \boldsymbol{x}} = 0, \qquad \dfrac{\partial \mu}{\partial t} + \boldsymbol{v}\cdot\dfrac{\partial \mu}{\partial \boldsymbol{x}} = 0. \end{cases}$$

When λ and μ are taken as constants of the motion, the function Γ in the transformation (15) is to be taken also as time independent. It will be shown in section **6** that

$$(18)\qquad \lambda' \frac{\partial \mu'}{\partial t} = \lambda \frac{\partial \mu}{\partial t} - \frac{\partial \Gamma}{\partial t}, \qquad \left(\frac{\partial \Gamma}{\partial t} = \frac{\lambda \Gamma}{\partial \lambda}\frac{\partial \lambda}{\partial t} + \frac{\partial \Gamma}{\partial \mu}\frac{\partial \mu}{\partial t}\right).$$

It is convenient to take $F=0$ in the second equation (15) in order to have

$$(19)\qquad \Psi' = \Psi \exp\left[\frac{i}{\hbar}\Gamma\right].$$

The form of the first equation (17) shows taht $-(c/e)\lambda(\partial\mu/\partial\boldsymbol{x})$ and $(1/e)\lambda(\partial\mu/\partial t)$ may be regarded as the potentials of a field. This point of view allows us to consider the change of the Clebsch parameters as a change of the gauge of those potentials. Equations (15), (18) and (19) correspond precisely to a change of the gauge of the potentials such that the new potentials be also expressible in terms of Clebsch parameters λ', μ'. This point will be discussed in more detail in section **6**.

At any instant of time t the Madelung fluid will have a quasi-irrotational part occupying a region Ω_t and a region where $\boldsymbol{\eta} \neq 0$, if the two parts do exist at a given instant of time t_0. In the region Ω_t, λ and μ can be taken as nil, but the function S will not be single-valued in general, unless Ω_t is simply connected. The circulation of the vector $\boldsymbol{u}$ along a closed path C within Ω_t is a linear combination with integer coefficients p_l of the cyclic constants K_l

$$(20)\qquad \int_C \boldsymbol{u}\cdot\delta\boldsymbol{x} = \sum_{l=1}^{n} p_l K_l,$$

$n+1$ being the order of connectivity of Ω_t. The order of connectivity and the cyclic constants are invariants of the motion, as a consequence of the circulation theorem (13). *In order that the wave function Ψ be single-valued the* *strengthes of the η-tubes must be integral multiples of h.* This point will be discussed in section **5**.

The generalized Schrödinger equation is discussed in section **6**. It is shown that it can be obtained from a variational principle of the same type as that for the ordinary Schrödinger equation. The new principle is obtained from the ordinary one by introducing the extra potentials $-(c/e)\lambda(\partial\mu/\partial\boldsymbol{x})$ and $(1/e)\lambda(\partial\mu/\partial t)$, λ and μ being variated as independent quantities. It is shown that, besides the gauge-transformations corresponding to the different possible choices of the Clebsch parameters, the equations (17) admit also the transformation

$$\Psi' = C\Psi\,, \quad \lambda' = \lambda\,, \quad \mu' = \mu \quad (C = \text{arbitrary constant}). \tag{21}$$

This invariance results from the fact that the quantum potential is invariant for the substitution of R by CR.

The steady motions of the Madelung fluid are discussed in section **7**. In those motions

$$W = \frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{h^2}{2m}\frac{\Delta R}{R}\,, \tag{22}$$

is a constant of the motion of the elements of the fluid (Bernoulli theorem for the Madelung fluid). The value of W is not the same for all the streamlines, unless

$$\boldsymbol{v} \wedge \boldsymbol{\eta} = 0 \qquad \text{(everywhere)}. \tag{23}$$

The steady motions satisfying the condition (23) correspond to those discussed by BELTRAMI [6] and STEKLOFF [7] in hydrodinamics. In those motions the energy per unit mass of the fluid is everywhere the same. It is proven in section **7** that (23) is a necessary and sufficient condition in order that it be possible to choose λ and μ as time independent constants of the motion, in a steady motion of the Madelung fluid. *The steady motions satisfying the condition (23) may be considered as the generalization of the ordinary stationary states for the generalized Schrödinger equation. The ordinary stationary states correspond to the steady Beltrami motions of the Madelung fluid in which $\boldsymbol{\eta} = 0$. Therefore the condition $\boldsymbol{\eta} = 0$ plays a fundamental part in the quantization of the values of the energy.*

A simple case of steady motion in which $\boldsymbol{v} \wedge \boldsymbol{\eta} = 0$ is discussed in section **8**. The trajectories of the elements of fluid within a circular cylinder are helices. The motion is irrotational outside the cylinder, where the trajectories are

(6) E. BELTRAMI: *Nuovo Cimento*, **25**, 212 (1889).

(7) W. STEKLOFF: *An. Fac. Sci. Univ. Toulouse*, **10**, 271 (1908).

circles lying on planes perpendicular to the axis of the cylinder. The azimuthal component of the vorticity on the cylinder limiting the vortex is infinite. This Beltrami motion is closely related to the two dimensional vortex motion discussed in reference [8]. The method used to derive the former motion from the latter consists in impressing to the elements of fluid velocities perpendicular to the plane of the motion and constant along each trajectory of the two-dimensional motion, in such a way that W be constant through the mass of the fluid. Thus the circular trajectories are replaced be helices within the cylinder. In the outer region no extra-velocity was impressed, because W was already constant in that region. This method can be applied to derive Beltrami motions from a large class of two-dimensional vortex motions in which the trajectories are circles centred at the origin and described in uniform motion.

Motions in which the velocity is discontinuous across a surface Σ are discussed in section 9. The discontinuity of the tangential component of the velocity is associated with the existence of a vortex sheet on Σ. A method is given to obtain simple discontinuous motions from stationary two-dimensional solutions of the ordinary Schrödinger equation. It is shown that with an R satisfying the condition

$$\frac{\partial R}{\partial n}\frac{\partial R}{\partial \boldsymbol{x}}\wedge \boldsymbol{n} = 0 , \tag{24}$$

$\boldsymbol{n}$ denoting the unit vector on the normal to a surface Σ, the stresses on Σ are normal, as in the case of the inviscid ordinary fluids. There is an essential difference between the discontinuous motions of the Madelung fluid and those of the inviscid ordinary fluids, arising from the fact that the effect of the pressure in the latter case is replaced by that of the quantum potential which depends on second order derivatives of the density.

The physical interpretation of the general motions of the Madelung fluid is not yet entirely clear. *It is however remarkable that any motion of the fluid corresponds to a quantal state of motion of a particle, because the first equation* (17) *may be regarded as an ordinary Schrödinger equation for a particle moving in an electromagnetic field described by the potentials* $A_0 + A_{0,\text{in}}$, $\boldsymbol{A} + \boldsymbol{A}_{\text{in}}$

$$A_{0,\text{in}} = \frac{\lambda}{e}\frac{\partial \mu}{\partial t} , \qquad \boldsymbol{A}_{\text{in}} = -\frac{c\lambda}{e}\frac{\partial \mu}{\partial \boldsymbol{x}} . \tag{25}$$

The « inner » electromagnetic field described by the $A_{0,\text{in}}$, $\boldsymbol{A}_{\text{in}}$ is

$$\boldsymbol{H}_{\text{in}} = -\frac{c}{e}\boldsymbol{\eta} , \qquad \boldsymbol{E}_{\text{in}} = -\frac{\boldsymbol{v}}{c}\wedge \boldsymbol{H}_{\text{in}} . \tag{26}$$

The Lorentz force due to the « inner » field vanishes: $\boldsymbol{E}_{\text{in}} + \boldsymbol{v}/c \wedge \boldsymbol{H}_{\text{in}} = 0$.

(8) M. Schönberg: *Nuovo Cimento*, 12, 300 (1954).

It will be shown in section **6** that the system (17) may be replaced by the following

$$(27)\quad \begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = \dfrac{1}{2m}\left(\dfrac{\hbar}{i}\dfrac{\partial}{\partial \boldsymbol{x}} - \dfrac{e}{c}(\boldsymbol{A} + \boldsymbol{A}_{\text{in}})\right)^2 \Psi + e(A_0 + A_{0,\text{in}})\Psi, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\boldsymbol{H}_{\text{in}}}{\varrho}\right) - \left(\dfrac{\boldsymbol{H}_{\text{in}}}{\varrho}\cdot\dfrac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v} = 0, \qquad \boldsymbol{E}_{\text{in}} + \dfrac{\boldsymbol{v}}{c}\wedge \boldsymbol{H} = 0, \\ \boldsymbol{H}_{\text{in}} = \operatorname{rot} \boldsymbol{A}_{\text{in}}, \qquad \boldsymbol{E}_{\text{in}} = -\dfrac{\partial}{\partial \boldsymbol{x}} A_{0,\text{in}} - \dfrac{1}{c}\dfrac{\partial}{\partial t}\boldsymbol{A}_{\text{in}}. \end{cases}$$

The second equation (27) gives the variation of η with time.

2. – Extension of the Helmholtz and Cauchy Vortex Theorems to the Madelung Fluid.

By solving equations (6) and (7) of section **1** we get $\boldsymbol{v}(t, \boldsymbol{x})$ and $\varrho(t, \boldsymbol{x})$. The trajectories of the elements of the fluid are determined by the differential equation

$$(1)\qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}(t, \boldsymbol{x}).$$

Along the trajectories we have

$$(2)\qquad \boldsymbol{x} = \boldsymbol{f}(t, \boldsymbol{x}_0).$$

Let us consider two neighbouring trajectories and denote by $\delta\boldsymbol{x}$ the displacement vector between the positions on the two trajectories. $\delta\boldsymbol{x}$ satisfies the linear equation

$$(3)\qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{x} = \left(\delta\boldsymbol{x}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v}(t, \boldsymbol{v}) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\,\delta\boldsymbol{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta = \delta\frac{\mathrm{d}}{\mathrm{d}t},$$

$\partial\boldsymbol{v}/\partial\boldsymbol{x}$ being a dyadic. Hence

$$(4)\qquad \delta\boldsymbol{x} = L(t, \boldsymbol{x}_0)\,\delta\boldsymbol{x}_0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{x} = \frac{\partial L}{\partial t}L^{-1}\,\delta\boldsymbol{x};$$

$L(t, \boldsymbol{x}_0)$ being a dyadic

$$(5)\qquad L(t, \boldsymbol{x}_0) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}_0}, \qquad \frac{\partial L}{\partial t} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} L.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\cdot\delta\boldsymbol{x}) = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\delta\boldsymbol{x} + \boldsymbol{v}\cdot\delta\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\delta\boldsymbol{x} + \delta\frac{\boldsymbol{v}^2}{2}, \tag{6}$$

we get from the Euler equation

$$m\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\cdot\delta\boldsymbol{x}) = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)\cdot\delta\boldsymbol{x} + \delta\left(\frac{m\boldsymbol{v}^2}{2} + \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right). \tag{7}$$

It is easily seen that

$$\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)\cdot\delta\boldsymbol{x} = \delta\left(\frac{\boldsymbol{v}}{c}\cdot\boldsymbol{A} - A_0\right) - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{A}\cdot\delta\boldsymbol{x}). \tag{8}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{u}\cdot\delta\boldsymbol{x}) = \delta\left(\frac{m\boldsymbol{v}^2}{2} - eA_0 + \frac{e}{c}\boldsymbol{v}\cdot\boldsymbol{A} + \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right). \tag{9}$$

By integrating both sides of (9) with respect to t we get

$$\boldsymbol{u}\cdot\delta\boldsymbol{x} - \boldsymbol{u}_0\cdot\delta\boldsymbol{x}_0 = \delta\chi, \tag{10}$$

$$\chi = \int_0^t\left\{\frac{m\boldsymbol{v}^2}{2} - e\left(A_0 - \frac{\boldsymbol{v}}{c}\cdot\boldsymbol{A}\right) + \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right\}\mathrm{d}t. \tag{11}$$

Equation (10) leads immediately to the circulation theorem for $\boldsymbol{u}$

$$\int_C \boldsymbol{u}\cdot\delta\boldsymbol{x} = \int_{C_0}\boldsymbol{u}_0\cdot\delta\boldsymbol{x}_0. \tag{12}$$

The circulation theorem means that $\int_C \boldsymbol{u}\cdot\delta\boldsymbol{x}$ is an integral invariant of order 1 of the motion of the elements of the fluid. Equation (12) can be transformed by means of the Stokes theorem as follows

$$\int_S \boldsymbol{\eta}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_0}\boldsymbol{\eta}_0\cdot\boldsymbol{n}_0\,\mathrm{d}S_0, \tag{13}$$

S and S_0 being formed by the same elements of fluid at the times t and 0, respectively, and limited by the contours C and C_0. $\boldsymbol{n}$ and $\boldsymbol{n}_0$ denote unit vectors on the normals to S and S_0, respectively. The choice of S being arbitrary in (13), we must have

$$\boldsymbol{\eta}\cdot\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x}=\boldsymbol{\eta}_0\cdot\delta_1\boldsymbol{x}_0\wedge\delta_2\boldsymbol{x}_0\,, \tag{14}$$

for arbitrary $\delta_1\boldsymbol{x}$, $\delta_2\boldsymbol{x}$. Equation (14) is the differential form of (13).

It is well known that the adjoint $K^\dagger$ of a dyadic K is defined by the equation

$$\boldsymbol{a}\cdot K\boldsymbol{b}=\boldsymbol{b}\cdot K^\dagger\boldsymbol{a}\,, \tag{15}$$

$\boldsymbol{a}$ and $\boldsymbol{b}$ being arbitrary vectors. The determinant $|K|$ of the matrix of K satisfies the equation

$$|K|\,\boldsymbol{a}\wedge\boldsymbol{b}\cdot\boldsymbol{c}=K\boldsymbol{a}\wedge K\boldsymbol{b}\cdot K\boldsymbol{c}=\boldsymbol{c}\cdot\{K^\dagger(K\boldsymbol{a}\wedge K\boldsymbol{b})\}\,, \tag{16}$$

for any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Therefore we have

$$|K|\,\boldsymbol{a}\wedge\boldsymbol{b}=K^\dagger(K\boldsymbol{a}\wedge K\boldsymbol{b})\,. \tag{17}$$

By taking into account that

$$(K^\dagger)^{-1}=(K^{-1})^\dagger \tag{18}$$

we get

$$|L|\,\boldsymbol{\eta}_0\cdot\delta_1\boldsymbol{x}_0\wedge\delta_2\boldsymbol{x}_0=\boldsymbol{\eta}_0\cdot L^\dagger(L\,\delta_1\boldsymbol{x}_0\wedge L\,\delta_2\boldsymbol{x}_0)=(L\boldsymbol{\eta}_0)\cdot\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x}\,. \tag{19}$$

It follows from (14) and (19) that there is an equation for $\boldsymbol{\eta}$ similar to that of Cauchy for the vorticity in hydrodynamics

$$|L|\,\boldsymbol{\eta}=L\boldsymbol{\eta}_0=\left(\boldsymbol{\eta}_0\cdot\frac{\partial}{\partial\boldsymbol{x}_0}\right)\boldsymbol{x}\,. \tag{20}$$

Since

$$|L|\,\delta_1\boldsymbol{x}_0\wedge\delta_2\boldsymbol{x}_0\cdot\delta_3\boldsymbol{x}_0=\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x}\cdot\delta_3\boldsymbol{x}=\frac{\varrho_0}{\varrho}\,\delta_1\boldsymbol{x}_0\wedge\delta_2\boldsymbol{x}_0\cdot\delta_3\boldsymbol{x}_0\,, \tag{21}$$

because of the conservation of the mass of a fluid element during the motion. Hence

$$\varrho\,|L|=\varrho_0\,. \tag{22}$$

It follows from (20) *and* (22) *that the vector* $\boldsymbol{\eta}/\varrho$ *changes during the motion in the same way as a* $\delta\boldsymbol{x}$

$$\frac{\boldsymbol{\eta}}{\varrho} = L\frac{\boldsymbol{\eta}_0}{\varrho_0}. \tag{23}$$

Let $\delta\boldsymbol{x}_0$ be an element of a η-line at the time 0

$$\delta\boldsymbol{x}_0 = \alpha\frac{\boldsymbol{\eta}_0}{\varrho_0}, \qquad (\alpha = \text{number}). \tag{24}$$

It follows from (4) and (23) that the corresponding $\delta\boldsymbol{x}$ will also be an element of a η-line at the time t: $\delta\boldsymbol{x} = \alpha\boldsymbol{\eta}/\varrho$. Hence

The elements of the Madelung fluid lying on a η*-line at the time* 0 *will also lie on a* η*-line at any other instant of time.*

The η*-surfaces being formed by* η*-lines, the elements of the fluid lying on a* η*-surface at the time* 0 *will also lie on a* η*-surface at any other instant of time. This holds in particular for the* η*-tubes. Equation* (13) *expresses the invariance of the strength of a* η*-tube during the motion.*

The above reasoning shows that equations (23) and (13) are equivalent.

Let $\boldsymbol{\gamma}$ be a vector such that $\int_S \boldsymbol{\gamma}\cdot\boldsymbol{n}\,\mathrm{d}S$ be an integral invariant

$$\int_S \boldsymbol{\gamma}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_0} \boldsymbol{\gamma}_0\cdot\boldsymbol{n}_0\,\mathrm{d}S_0; \tag{25}$$

we have

$$\frac{\boldsymbol{\gamma}}{\varrho} = L\frac{\boldsymbol{\gamma}_0}{\varrho_0}. \tag{26}$$

Therefore the Helmholtz theorems hold for any such vector $\boldsymbol{\gamma}$, except the theorem on the invariance of the strength of a tube, because the strength of a γ-tube has only a meaning when the flux of $\boldsymbol{\gamma}$ is the same through all the sections of the tube and this does not happen for any tube, unless $\operatorname{div}\boldsymbol{\gamma} = 0$.

It follows from (12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \boldsymbol{v}\cdot\delta\boldsymbol{x} = -\frac{e}{mc}\frac{\mathrm{d}}{\mathrm{d}t}\int_C \boldsymbol{A}\cdot\delta\boldsymbol{x} = -\frac{e}{mc}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{H}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{27}$$

The rate of variation of the circulation of the velocity is proportional to the rate of variation of the magnetic flux through the contour C.

It follows from (23) and the second equation (5) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\eta}}{\varrho}\right) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\,\frac{\boldsymbol{\eta}}{\varrho} = \left(\frac{\boldsymbol{\eta}}{\varrho}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v}\,. \tag{28}$$

This equation corresponds to that of Helmholtz in hydrodynamics. It is of the same form as the first equation (3). The Helmholtz equation shows that in any infinitesimal time interval $\boldsymbol{\eta}/\varrho$ varies as a $\delta\boldsymbol{x}$, it is the differential form of the Cauchy equation. The equation (23) follows immediately from (28) by taking into account the second equation (5). We shall now prove that the Helmholtz equation expresses the law of variation of the angular momentum of the elements of the fluid.

Let us consider an element of fluid Ω whose barycentric ellipsoid of inertia at the time t is a sphere. Denoting by G the center of gravity of the fluid element and by $\boldsymbol{r}$ the barycentric position vector of a generic point, the moment of the forces $\boldsymbol{F}$ acting on the element with respect to G is

$$\begin{aligned}\mathfrak{M} &= \int_\Omega \boldsymbol{r}\wedge\left\{\boldsymbol{F}_G + \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\right\}\varrho\,\mathrm{d}\boldsymbol{r} = \int_\Omega \boldsymbol{r}\wedge\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\varrho\,\mathrm{d}\boldsymbol{r} = \\ &= \sum_{k,l} \boldsymbol{i}_k\wedge\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{i}_l\int_\Omega r_k r_l\varrho\,\mathrm{d}\boldsymbol{r} = \frac{I}{3m}\sum_k \boldsymbol{i}_k\wedge\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{i}_k = \frac{I}{3m}(\mathrm{rot}\,\boldsymbol{F})_G\,.\end{aligned} \tag{29}$$

(I = moment of inertia with respect to G)

the $\boldsymbol{i}_k$ being the unit vectors of the coördinate axes and the r_k the components of $\boldsymbol{r}$. The angular momentum of Ω with respect to G is

$$\boldsymbol{M} = m\int_\Omega \boldsymbol{r}\wedge\left\{\boldsymbol{v}_G + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\right\}\varrho\,\mathrm{d}\boldsymbol{r} = m\int_\Omega \boldsymbol{r}\wedge\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\varrho\,\mathrm{d}\boldsymbol{r}\,. \tag{30}$$

Since

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{v} - \boldsymbol{v}_G = \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\,, \tag{31}$$

we have

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} &= m\int_\Omega \boldsymbol{r}\wedge\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{r}\varrho\,\mathrm{d}\boldsymbol{r} + m\int_\Omega \boldsymbol{r}\wedge\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G^2 \boldsymbol{r}\varrho\,\mathrm{d}\boldsymbol{r} = \\ &= \frac{I}{3}\sum_k \boldsymbol{i}_k\wedge\left\{\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G^2\right\}\boldsymbol{i}_k = \frac{I}{3}\,\frac{\mathrm{d}}{\mathrm{d}t}\sum_k \boldsymbol{i}_k\wedge\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G \boldsymbol{i}_k + \frac{I}{3}\sum_k \boldsymbol{i}_k\wedge\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)_G^2 \boldsymbol{i}_k = \\ &= \frac{I}{3}\left(\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} + \boldsymbol{\zeta}\,\mathrm{div}\,\boldsymbol{v} - \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\boldsymbol{\zeta}\right)_G,\end{aligned} \tag{32}$$

$$\boldsymbol{\zeta} = \mathrm{rot}\,\boldsymbol{v}\,. \tag{33}$$

By taking into account the equation of continuity, we get

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}=\frac{I\varrho_G}{3}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\zeta}}{\varrho}\right)-\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\,\frac{\boldsymbol{\zeta}}{\varrho}\right\}_G . \tag{34}$$

Therefore the variation of the angular momentum of Ω is given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\zeta}}{\varrho}\right)-\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\,\frac{\boldsymbol{\zeta}}{\varrho}=\frac{1}{\varrho m}\operatorname{rot}\boldsymbol{F} . \tag{35}$$

It is easily seen that

$$\frac{1}{\varrho}\operatorname{rot}\boldsymbol{F}=\frac{e}{\varrho}\operatorname{rot}\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)=\frac{e}{c}\,\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\,\frac{\boldsymbol{H}}{\varrho}-\frac{e}{c}\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{H}}{\varrho}\right) . \tag{36}$$

The Helmholtz equation (28) follows immediately from (35) and (36).

The way in which we derived equation (20) shows that it is equivalent to the circulation theorem (12). *The Cauchy equation and the circulation theorem for* $\boldsymbol{u}$ *are therefore equivalent.*

It follows from the identity (6) that for any closed curve C

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C m\boldsymbol{v}\cdot\delta\boldsymbol{x}=\int_C m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\cdot\delta\boldsymbol{x} . \tag{37}$$

The identity (8) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \frac{e}{c}\boldsymbol{A}\cdot\delta\boldsymbol{x}=-\int_C e\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)\cdot\delta\boldsymbol{x} . \tag{38}$$

From (37) and (38) we get the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_C \boldsymbol{u}\cdot\delta\boldsymbol{x}=\int_C\left\{m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}-e\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)\right\}\cdot\delta\boldsymbol{x} . \tag{39}$$

The left hand side vanishes as a consequence of (12). Therefore the circulation theorem is equivalent to the equation

$$\operatorname{rot}\left\{m\,\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}-e\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}\right)\right\}=0 . \tag{40}$$

3. – Introduction of the Clebsch Parameters.

It is well known from the theory of the Pfaff expressions that the differential form $\omega=\sum_{i=1}^{n}X_i(x)\,\delta x_i$ can be reduced to one of the canonical forms

(1) $$\omega = \sum_{r=1}^{s} y_r \, \delta z_r \,,$$

(2) $$\omega = \sum_{r=1}^{s} y_r \, \delta z_r + \delta z_{s+1} \,;$$

$2s$ or $2s+1$ being the minimum number of independent variables in terms of which ω can be expressed, the y and z being independent functions of the x. This minimum number is called the class of ω. It is at most equal to the number n.

In the particular case of a differential form $\sum_{l=1}^{3} X_l \, \delta x_l$ in a three dimensional space, there are only three canonical forms: δz, $y\,\delta z$, $y\,\delta z + \delta z_1$. We are interested in the differential form $\boldsymbol{u}\cdot\delta\boldsymbol{x}$, the time t being considered as a parameter. When $\boldsymbol{u}\cdot\delta\boldsymbol{x}$ is of class 1, $\boldsymbol{u}$ is the gradient of a function S, i.e. the motion of the Madelung fluid is quasi-irrotational. $\boldsymbol{u}\cdot\delta\boldsymbol{x}$ is of class 2 when it admits an integrating factor, without being an exact differential. It is well known that the condition for the existence of an integrating factor is

(3) $$\boldsymbol{u}\cdot\boldsymbol{\eta} = \boldsymbol{u}\cdot \mathrm{rot}\, \boldsymbol{u} = 0 \,,$$

$\boldsymbol{u}\cdot\delta\boldsymbol{x}$ is of class 3 when $\boldsymbol{u}\cdot\boldsymbol{\eta} \neq 0$.

When $\boldsymbol{u}\cdot\delta\boldsymbol{x}$ is reduced to the canonical form, we have

(4) $$\boldsymbol{u}\cdot\delta\boldsymbol{x} = \delta S + \lambda\,\delta\mu \,, \qquad \boldsymbol{u} = \frac{\partial S}{\partial \boldsymbol{x}} + \lambda \frac{\partial \mu}{\partial \boldsymbol{x}} \,.$$

In the case of class 1 we can take $\lambda = \mu = 0$, more generally, λ as a function of t and μ. In the case of class 2, we can take $S = 0$, λ and μ being independent. More generally, we can take λ and μ independent and S as a function of t and μ. In the case of class 3, S, λ and μ are independent functions of x_1, x_2 and x_3. Thus we have proven the existence of the Clebsch parameters S, λ, μ by means of the theory of the reduction to the canonical forms.

The existence of the Clebsch parameters can be easily established by the consideration of the η-lines. Let us assume that the η-lines at the time t are defined by the equations

(5) $$\lambda_1(t, \boldsymbol{x}) = \mathrm{const.} \,, \qquad \mu(t, \boldsymbol{x}) = \mathrm{const.}$$

The η-line passing through a point is orthogonal to the vectors $\partial\lambda_1/\partial\boldsymbol{x}$ and $\partial\mu/\partial\boldsymbol{x}$, since it lies on the two surfaces $\lambda_1 = $ constant and $\mu = $ constant passing through the point. Hence

$$\boldsymbol{\eta} = \gamma \frac{\partial \lambda_1}{\partial \boldsymbol{x}} \wedge \frac{\delta \mu}{\delta \boldsymbol{x}} , \tag{6}$$

γ being a function of t and $\boldsymbol{x}$. Since

$$\frac{\partial \gamma}{\partial \boldsymbol{x}} \cdot \frac{\partial \lambda_1}{\partial \boldsymbol{x}} \wedge \frac{\partial \mu}{\partial \boldsymbol{x}} = \operatorname{div} \boldsymbol{\eta} = 0 , \tag{7}$$

γ, λ_1 and μ are not independent functions of $\boldsymbol{x}$

$$\gamma = F(t, \lambda_1, \mu) . \tag{8}$$

Let us introduce the function $\lambda(t, \boldsymbol{x})$

$$\lambda = \int \gamma \, \mathrm{d}\lambda_1 . \tag{9}$$

It is easily seen that

$$\boldsymbol{\eta} = \frac{\partial \lambda}{\partial \boldsymbol{x}} \wedge \frac{\partial \mu}{\partial \boldsymbol{x}} , \qquad \operatorname{rot} \boldsymbol{u} = \operatorname{rot} \left(\lambda \frac{\partial \mu}{\partial \boldsymbol{x}} \right) . \tag{10}$$

It follows from (10) that there are functions S such that

$$\boldsymbol{u} - \lambda \frac{\partial \mu}{\partial \boldsymbol{x}} = \frac{\partial S}{\partial \boldsymbol{x}} . \tag{11}$$

We shall now prove that it is possible to choose the Clebsch parameters λ and μ as constants of the motion. It follows from the equation (10) of section 2 that

$$\delta S(t, \boldsymbol{x}) + \lambda(t, \boldsymbol{x}) \, \delta \mu(t, \boldsymbol{x}) = \delta S(0, \boldsymbol{x}_0) + \lambda(0, \boldsymbol{x}_0) \, \delta \mu(0, \boldsymbol{x}_0) + \delta \chi . \tag{12}$$

Since (12) is equivalent to the equations of mot'on for S, λ and μ, we can take

$$\lambda(t, \boldsymbol{x}) = \lambda(0, \boldsymbol{x}_0) , \qquad \mu(t, \boldsymbol{x}) = \mu(0, \boldsymbol{x}_0) , \qquad S(t, \boldsymbol{x}) = S(0, \boldsymbol{x}_0) + \chi , \tag{13}$$

in order to get the solution of the Euler equations that corresponds to the initial distribution of velocities given by the Clebsch parameters $S(0, \boldsymbol{x}_0)$, $\lambda(0, \boldsymbol{x}_0)$, $\mu(0, \boldsymbol{x}_0)$. Any motion can be obtained in this way, because any initial distribution of the velocities can be obtained by a suitable choice of the initial values of S, λ and μ.

It follows from the third equation (13) and the equation (11) of section 2 that

$$\frac{\partial S}{\partial t} + \boldsymbol{v}\cdot\left(\frac{\partial S}{\partial \boldsymbol{x}} - \boldsymbol{u}\right) + \frac{1}{2m}\left(\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A} + \lambda\frac{\partial \mu}{\partial \boldsymbol{x}}\right)^2 + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0 . \tag{14}$$

Since

$$\frac{\partial \lambda}{\partial t} + \boldsymbol{v}\cdot\frac{\partial \lambda}{\partial \boldsymbol{x}} = 0 , \qquad \frac{\partial \mu}{\partial t} + \boldsymbol{v}\cdot\frac{\partial \mu}{\partial \boldsymbol{x}} = 0 , \tag{15}$$

as a consequence of the two first equations (13), we have

$$\frac{\partial S}{\partial t} + \lambda\frac{\partial \mu}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A} + \lambda\frac{\partial \mu}{\partial \boldsymbol{x}}\right)^2 + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0 . \tag{16}$$

Equations (15) and (16) associated to the continuity equation

$$\frac{\partial R^2}{\partial t} + \operatorname{div}\left\{\frac{R^2}{m}\left(\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A} + \lambda\frac{\partial \mu}{\partial \boldsymbol{x}}\right)\right\} = 0 , \tag{17}$$

describe completely the motions of the Madelung fluid.

In order to get the most general choice of the Clebsch parameters let us take the substantial derivatives of both sides of (12)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta S + \lambda\,\delta\mu) = -\,\delta\left(\frac{mv^2}{2} + eA_0 - \boldsymbol{u}\cdot\boldsymbol{v} - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right); \tag{18}$$

δ corresponds to the passage from a fluid element to a neighbouring one, therefore δ and $\mathrm{d}/\mathrm{d}t$ do commute. Equation (18) can be written as follows

$$\delta\left(\frac{\partial S}{\partial t} + \lambda\frac{\partial \mu}{\partial t} + \frac{mv^2}{2} + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right) = \delta\lambda\frac{\mathrm{d}\mu}{\mathrm{d}t} - \delta\mu\frac{\mathrm{d}\lambda}{\mathrm{d}t} , \tag{19}$$

since

$$\frac{\mathrm{d}S}{\mathrm{d}t} - \boldsymbol{u}\cdot\boldsymbol{v} = \frac{\partial S}{\partial t} + \lambda\frac{\partial \mu}{\partial t} - \lambda\frac{\mathrm{d}\mu}{\mathrm{d}t} . \tag{20}$$

The form of the right-hand side of (19) shows that the left-hand side is the

δ differential of a function of t, λ and μ, so that

$$\frac{\partial S}{\partial t} + \lambda\frac{\partial \mu}{\partial t} + \frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m}\,\frac{\Delta R}{R} = K(t, \lambda, \mu)\,. \tag{21}$$

Furthermore we have the analogue of the Stuart equations [9]

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial K}{\partial \lambda}\,, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\,\frac{\partial K}{\partial \mu}\,. \tag{22}$$

Equation (19) is satisfied with any choice of the function K. The conditions (15) correspond to the choice $K = 0$.

The above results can be obtained in a more interesting form by using the basic equation (12). Let us introduce the notations

$$\lambda_0(\boldsymbol{x}_0) = \lambda(0, \boldsymbol{x}_0), \qquad \mu_0(\boldsymbol{x}_0) = \mu(0, \boldsymbol{x}_0), \qquad S_0(\boldsymbol{x}_0) = S(0, \boldsymbol{x}_0)\,. \tag{23}$$

It follows from (12) that

$$\delta(S - S_0 - \chi - \lambda_0\mu_0) = -\,\lambda\,\delta\mu - \mu_0\,\delta\lambda_0\,; \tag{24}$$

$S - S_0 - \chi - \lambda_0\mu_0$ is therefore a function of t, μ and λ_0

$$S - S_0 - \chi - \lambda_0\mu_0 = \Lambda(t, \mu, \lambda_0) \tag{25}$$

and we have

$$\lambda\,\delta\mu + \mu_0\,\delta\lambda_0 = -\,\delta\Lambda\,. \tag{26}$$

Hence

$$\lambda = -\,\frac{\partial \Lambda}{\partial \mu}\,, \qquad \mu_0 = -\,\frac{\partial \Lambda}{\partial \lambda_0}\,. \tag{27}$$

Equation (26) shows that λ, μ are related to λ_0, μ_0 by a contact transformation whose generating function is $-\Lambda$. Λ can be taken arbitrarily, as long as it defines a contact transformation. By taking $\Lambda = -\,\mu\lambda_0$ we get the equations (13). Equations (27) give the finite contact transformation corresponding to the solution of the Stuart equations (22). By taking the substantial time derivatives of both sides of equation (25) we get an equation

(9) T. STUART: *Dublin Dissertation* (1900).

equivalent to (21)

$$\frac{\partial S}{\partial t} + \lambda \frac{\partial \mu}{\partial t} + \frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m} \frac{\Delta R}{R} = \frac{\partial \Lambda}{\partial t}\,. \tag{28}$$

We get from (28) and (21) the following Hamilton-Jacobi equation

$$K\left(t, -\frac{\partial \Lambda}{\partial \mu}, \mu\right) - \frac{\partial \Lambda}{\partial t} = 0\,. \tag{29}$$

Let $\int_c \boldsymbol{w}\cdot\delta\boldsymbol{x}$ be any relative integral-invariant of order 1. We have

$$\boldsymbol{w}\cdot\delta\boldsymbol{x} - \boldsymbol{w}_0\cdot\delta\boldsymbol{x}_0 = \delta\Theta\,. \tag{30}$$

$\Theta(t, \boldsymbol{x}_0)$ being some function. By introducing Clebsch parameters for the vector field $\boldsymbol{w}$

$$\boldsymbol{w} = \frac{\partial T}{\partial \boldsymbol{x}} + \nu \frac{\partial \sigma}{\partial \boldsymbol{x}}\,, \tag{31}$$

we get

$$\delta T + \nu\,\delta\sigma = \delta T_0 + \nu_0\,\delta\sigma_0 + \delta\Theta\,, \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta T - \delta\Theta + \nu\,\delta\sigma) = 0. \tag{33}$$

Hence

$$\delta\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}\Theta}{\mathrm{d}t} + \nu\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}\,\delta\nu - \frac{\mathrm{d}\nu}{\mathrm{d}t}\,\delta\sigma, \tag{34}$$

and we must have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\partial G}{\partial \nu}\,, \qquad \frac{\mathrm{d}\nu}{\mathrm{d}t} = -\frac{\partial G}{\partial \sigma}\,, \qquad (G = G(t, \nu, \sigma))\,, \tag{35}$$

with

$$G = \frac{\mathrm{d}}{\mathrm{d}t}(T - \Theta) + \nu\frac{\mathrm{d}\sigma}{\mathrm{d}t}\,. \tag{36}$$

The function G can be taken arbitrarily, since with any choice of G the equation (34) is satisfied. Different choices of G correspond to different choices of the Clebsch parameters T, ν, σ. The choice $G = 0$ leads to ν and σ that are constants of the motion.

The above reasoning shows that the circulation theorem for $\boldsymbol{w}$ leads to equations of the Stuart type for ν and σ. We shall now prove that the circulation theorem follows from the equations (35). Indeed, we have

$$(37)\qquad \frac{\mathrm{d}}{\mathrm{d}t}(\delta_1\nu\,\delta_2\sigma - \delta_2\nu\,\delta_1\sigma) = \delta_2\left(\frac{\partial G}{\partial\nu}\delta_1\nu + \frac{\partial G}{\partial\sigma}\delta_1\sigma\right) - \delta_1\left(\frac{\partial G}{\partial\nu}\delta_2\nu + \frac{\partial G}{\partial\sigma}\delta_2\sigma\right) = $$
$$= \delta_2(\delta_1 G) - \delta_1(\delta_2 G) = 0\,.$$

Since

$$(38)\qquad \delta_1\nu\,\delta_2\sigma - \delta_2\nu\,\delta_1\sigma = \left(\frac{\partial\nu}{\partial\boldsymbol{x}}\cdot\delta_1\boldsymbol{x}\right)\left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\cdot\delta_2\boldsymbol{x}\right) - \left(\frac{\partial\nu}{\partial\boldsymbol{x}}\cdot\delta_2\boldsymbol{x}\right)\left(\frac{\partial\sigma}{\partial\boldsymbol{x}}\cdot\delta_1\boldsymbol{x}\right) =$$
$$= \left(\frac{\partial\nu}{\partial\boldsymbol{x}}\wedge\frac{\partial\sigma}{\partial\boldsymbol{x}}\right)\cdot(\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x}) = \operatorname{rot}\boldsymbol{w}\cdot\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x}\,,$$

equation (37) shows that

$$(39)\qquad \operatorname{rot}\boldsymbol{w}\cdot\delta_1\boldsymbol{x}\wedge\delta_2\boldsymbol{x} = \operatorname{rot}_{x_0}\boldsymbol{w}_0\cdot\delta_1\boldsymbol{x}_0\wedge\delta_2\boldsymbol{x}_0\,.$$

Hence

$$(40)\qquad \int_S \operatorname{rot}\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{S_0}\operatorname{rot}_{x_0}\boldsymbol{w}_0\cdot\boldsymbol{n}_0\,\mathrm{d}S_0\,,$$

S denoting an open surface limited by a contour C. Equation (40) can be transformed into the circulation theorem for $\boldsymbol{w}$

$$\oint_C \boldsymbol{w}\cdot\delta\boldsymbol{x} = \oint_{C_0}\boldsymbol{w}_0\cdot\delta\boldsymbol{x}_0\,.$$

The circulation theorem for $\boldsymbol{w}$ is equivalent to the proposition that the Clebsch parameters ν, σ satisfy equations of the Stuart type.

Let us consider the differential form Ω_d

$$(42)\qquad \Omega_d = \boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x} - \left(\frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right)\mathrm{d}t\,.$$

It follows from (16) that

$$(43)\qquad \Omega_d = \mathrm{d}S + \lambda\,\mathrm{d}\mu\,.$$

The class of Ω_d is less than 4, since it can be expressed in terms of the three functions S, λ, μ. Conversely, to assume that the class of Ω_d is less than 4 is equivalent to assume the existence of three functions S, λ, μ satisfying (43), so that they are Clebsch parameters for $\boldsymbol{u}$ satisfying (16). Hence

The Euler equations imply that Ω_d is of class less than 4 and also that it is possible to choose λ and μ as constants of the motion. Conversely, to assume that the class of Ω_d is less than 4 and that it is possible to write $\Omega_d = \mathrm{d}S + \lambda\,\mathrm{d}\mu$ with λ, μ being constants of the motion leads to the Euler equations.

The quasi-irrotational motions are determined by the condition that the class of Ω_d is 1, i.e. that Ω_d is an exact differential. Indeed, Ω_d being an exact differential, there are functions S such that for any $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}t$

$$\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x} - \left(\frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right)\mathrm{d}t = \mathrm{d}S\,. \tag{44}$$

Hence

$$\boldsymbol{u} = \frac{\partial S}{\partial \boldsymbol{x}}\,, \qquad \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A}\right)^2 + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0\,. \tag{45}$$

By taking the gradients of both sides of the second equation (45) we get the Euler equation for quasi-irrotational motions

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{\partial}{\partial \boldsymbol{x}}\left(\frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\right). \tag{46}$$

4. – Transformations of the Clebsch Parameters.

We shall now discuss the different possible choices of the Clebsch parameters for the same state of motion of the fluid. Since

$$\boldsymbol{\eta} = \frac{\partial \lambda}{\partial \boldsymbol{x}} \wedge \frac{\partial \mu}{\partial \boldsymbol{x}}\,, \tag{1}$$

λ and μ are both solutions of the partial differential equation

$$\boldsymbol{\eta}\cdot\frac{\partial f}{\partial \boldsymbol{x}} = 0\,, \tag{2}$$

which has only two independent solutions. Therefore the parameters λ' and μ' of another set must be functions of t, λ and μ

$$\lambda' = f_1(t, \lambda, \mu)\,, \qquad \mu' = f_2(t, \lambda, \mu)\,, \tag{3}$$

provided λ and μ be independent functions of $\boldsymbol{x}$, as we shall assume.

Since

$$\frac{\partial\lambda'}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu'}{\partial\boldsymbol{x}}=\frac{D(f_1,f_2)}{D(\lambda,\mu)}\,\frac{\partial\lambda}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}\,; \tag{4}$$

we have the necessary condition

$$\frac{D(f_1,f_2)}{D(\lambda,\mu)}=1\,. \tag{5}$$

Let us regard μ as a « coordinate » q and λ as its « conjugate momentum ». Equation (5) shows that the Poisson bracket of μ' and λ' is 1

$$\frac{\partial\mu'}{\partial\mu}\,\frac{\partial\lambda'}{\partial\lambda}-\frac{\partial\mu'}{\partial\lambda}\,\frac{\partial\lambda'}{\partial\mu}=1\,. \tag{6}$$

It is well known that (6) is a necessary and sufficient condition for the existence of a function $\Gamma(t,\mu,\mu')$ such that

$$\lambda\,\delta\mu-\lambda'\,\delta\mu'=\delta\Gamma \tag{7}$$

or

$$\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}-\lambda'\,\frac{\partial\mu'}{\partial\boldsymbol{x}}=\frac{\partial\Gamma}{\partial\boldsymbol{x}}\,. \tag{8}$$

It follows from (8) that equation (5) is also a sufficient condition for λ' and μ' to be Clebsch parameters, because

$$\boldsymbol{u}=\frac{\partial}{\partial\boldsymbol{x}}(S+\Gamma)+\lambda'\frac{\partial\mu'}{\partial\boldsymbol{x}}\,. \tag{9}$$

Hence

$$S'=S+\Gamma+\text{arbitrary function of }t=S+\Gamma+F(t)\,. \tag{10}$$

The function Γ may be taken arbitrarily, provided the equations

$$\lambda=\frac{\partial\Gamma}{\partial\mu}\,,\qquad \lambda'=-\frac{\partial\Gamma}{\partial\mu'}\,, \tag{11}$$

do define λ', μ' in terms of t, λ and μ.

In order that λ' and μ' be constants of the motion

$$\frac{d\lambda'}{dt} = 0\,, \qquad \frac{d\mu'}{dt} = 0\,, \tag{12}$$

we must have

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial f_1}{\partial \mu}\frac{d\mu}{dt} = 0\,, \qquad \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial \lambda}\frac{d\lambda}{dt} + \frac{\partial f_2}{\partial \mu}\frac{d\mu}{dt} = 0\,. \tag{13}$$

When λ and μ are constants of the motion, these equations become

$$\frac{\partial f_1}{\partial t} = 0\,, \qquad \frac{\partial f_2}{\partial t} = 0\,, \tag{14}$$

so that

$$\lambda' = f_1(\lambda, \mu)\,, \qquad \mu' = f_2(\lambda, \mu)\,. \tag{15}$$

Therefore, in this case, we can get the transformations of the Clebsch parameters by means of functions $\Gamma(\mu, \mu')$ not involving t.

It follows from (11) that

$$\lambda \frac{\partial \mu}{\partial t} - \lambda' \frac{\partial \mu'}{\partial t} = \frac{\partial \Gamma}{\partial \mu}\frac{\partial \mu}{\partial t} + \frac{\partial \Gamma}{\partial \mu'}\frac{\partial \mu'}{\partial t}\,. \tag{16}$$

Denoting by $(\partial \Gamma/\partial t)_{tot}$ the total derivative of Γ with respect to t, equation (16) can be written as follows

$$\lambda \frac{\partial \mu}{\partial t} - \lambda' \frac{\partial \mu'}{\partial t'} = \left(\frac{\partial \Gamma}{\partial t}\right)_{tot} - \frac{\partial \Gamma}{\partial t}\,. \tag{17}$$

When λ, μ and λ', μ' are constants of the motion, Γ can be taken as a function of μ and μ' only, so that

$$\lambda \frac{\partial \mu}{\partial t} - \lambda' \frac{\partial \mu'}{\partial t'} = \left(\frac{\partial \Gamma}{\partial t}\right)_{tot}\,. \tag{18}$$

It follows from (8) and (18) that:

A transformation of the Clebsch parameters λ, μ, which are constants of the motion, into another set λ', μ' with the same property corresponds to a gauge transformation of the second kind of the quantities $-\lambda\, \partial\mu/\partial \boldsymbol{x}$, $(\lambda/c)\, \partial\mu/\partial t$.

By choosing Γ as a function of μ and μ' only and taking $F(t) = 0$ in equation (10), *the system of equations* (15)-(16)-(17) *of section* **3** *becomes invariant for the transformation of the Clebsch parameters* $(S, \lambda, \mu) \to (S', \lambda', \mu')$.

We shall now discuss the behaviour of the equations (15-(16)-(17) of section **3** for a change of gauge of the electromagnetic potentials

$$(19) \qquad \boldsymbol{A}' = \boldsymbol{A} + \frac{\partial \varphi}{\partial \boldsymbol{x}}, \qquad A_0' = A_0 - \frac{1}{c}\frac{\partial \varphi}{\partial t}.$$

The velocity $\boldsymbol{v}$ must be invariant. Hence $\boldsymbol{u}$ is transformed as follows

$$(20) \qquad \boldsymbol{u}' = \boldsymbol{u} + \frac{e}{c}\frac{\partial \varphi}{\partial \boldsymbol{x}},$$

and the gauge transformation induces a transformation of the Clebsch parameters

$$(21) \qquad \delta S' + \lambda' \delta\mu' = \delta S + \lambda\, \delta\mu + \frac{e}{c}\,\delta\varphi.$$

The general solution of (21) can be obtained in the same way, as that of equation (12) of section **3**

$$(22) \qquad \lambda = \frac{\partial \Gamma}{\partial \mu}, \qquad \lambda' = -\frac{\partial \Gamma}{\partial \mu'}, \qquad S' = S + \Gamma + \frac{e}{c}\varphi + F(t),$$

in terms of an arbitrary function $\Gamma(t, \mu, \mu')$ and of another arbitrary function $F(t)$. By taking $F(t) = 0$ and a Γ depending only on μ and μ', the equations (15)-(16)-(17) of section **3** remain invariant. We can take simply

$$(23) \qquad S' = S + \frac{e}{c}\varphi, \qquad \lambda' = \lambda, \qquad \mu' = \mu,$$

in order to get a gauge transformation of the first kind of the wave function Ψ

$$(24) \qquad \Psi = R \exp\left[\frac{i}{\hbar} S\right],$$

$$(25) \qquad \Psi' = \Psi \exp\left[\frac{ie}{\hbar c}\varphi\right].$$

5. – Quasi-Irrotational and η-Regions of the Madelung Fluid.

At any instant of time the fluid is formed by a quasi-irrotational part and a part in which $\eta \neq 0$. We shall denote by Ω_t the region of space where the motion is quasi-irrotational and its boundary surface by Σ_t. The region where $\eta \neq 0$ will be denoted by $\bar{\Omega}_t$.

Σ_t is in general a surface of discontinuity for the partial derivatives of the components of the velocity $\boldsymbol{v}$. This surface of discontinuity has special properties of propagation, because it is always formed by the same elements of the fluid. Thus its velocity of propagation at any point coincides with the component of the velocity of the fluid at the point in the direction of the normal to Σ_t.

Within Ω_t the Clebsch parameters λ and μ can be taken null and the first formula (5) of section **1** is still valid; the equations (16) and (17) of section **3** go over into the equations (2) and (3) of section **1**. Hence

By taking $\lambda = \mu = 0$ in Ω_t, the Schrödinger equation still holds, the wave function being $\Psi = R \exp[(i/\hbar)S]$.

There is an important circumstance to be taken into account: *the function S will in general not be single-valued in Ω_t, when Ω_t is not a simply-connected region.* This corresponds to the well known fact that the potential of velocities of an ordinary fluid moving irrotationally in a multiply-connected region is cyclic. Let $n+1$ be the order of connectivity of Ω_t. The circulation of $\boldsymbol{u}$ along any closed path C within Ω_t is a linear combination of the cyclic constants $K_1, K_2, \ldots, K_n$

$$\int_C \boldsymbol{u} \cdot \delta \boldsymbol{x} = \sum_{l=1}^{n} p_l K_l \,, \tag{1}$$

the coefficients p being integers. The cyclic constants K, as well as n, are invariants of the motion.

In order that Ψ be single valued in the case of a multiply-connected region Ω_t, it is necessary and sufficient that the cyclic constants K_l be integral multiples of $2\pi\hbar = h$

$$K_l = m_l h \,, \qquad m_l = \text{integer} \,, \tag{2}$$

since by starting at a point of Ω_t and describing a closed path in Ω_t, the value of S will be increased by $\sum_l p_l K_l$. The equations (2) are restrictions on the initial conditions. Let us consider the case in which the multiple connection of Ω_t is due to the existence of n η-rings. The ciclic constants K_l are simply the values of the strengthes of the rings. Therefore the condition (2) lead to the quantization of the strengthes of the η-rings:

In order that Ψ be single-valued in Ω_t, it is necessary that the strengthes of the existing η-rings be integral multiples of h.

It is not necessary to take λ and μ null in Ω_t. In order that $\boldsymbol{\eta}$ be zero it is sufficient that $\partial\lambda/\partial\boldsymbol{x}\wedge\partial\mu/\partial\boldsymbol{x}$ be 0, i.e. that λ be a function of t and μ

(3) $$\lambda = f(t, \mu)\,.$$

When λ and μ are chosen as constants of the motion, the condition (3) requires that λ be a function of μ alone

(3a) $$\lambda = f(\mu)\,,$$

so that

(4) $$\lambda\frac{\partial\mu}{\partial\boldsymbol{x}} = \frac{\partial}{\partial\boldsymbol{x}}\int\lambda\,\mathrm{d}\mu\,, \qquad \lambda\frac{\partial\mu}{\partial t} = \frac{\partial}{\partial t}\int\lambda\,\mathrm{d}\mu\,.$$

By taking

(5) $$S' = S + \int\lambda\,\mathrm{d}\mu = S + \int f(\mu)\,\mathrm{d}\mu\,,$$

and μ as any constant of the motion

(6) $$\frac{\partial\mu}{\partial t} + \boldsymbol{v}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}} = 0\,,$$

the equations (15) of section **3** will be satisfied and (16)-(17) take the form of equations (2) and (3) of section **1**. Thus we get again the Schrödinger equation for $\Psi' = R\exp[(i/\hbar)S']$.

6. – Generalization of the Schrödinger Equation for the General Motions of the Fluid.

The Euler equations and the continuity equation are valid for any motion of the Madelung fluid and, by replacing the Schrödinger equation by them, we obtain a generalization of the Schrödinger theory. This remark was made independently by TAKABAYASHI [2] and by us [3]. TAKABAYASHI introduced the Clebsch parameters and the equations (15), (16) and (17) of section **3**. We have obtained the same equations starting from a generalized form of the classical Hamilton-Jacobi equation [4]. Those equations are not a good analogue of the Schrödinger equation, because they separate the amplitude and the phase of the wave function. It is precisely interesting to see that the Schrödinger equation can be generalized by the introduction of an extra scalar potential and an extra vector potential, without losing the usual form

$$i\hbar\frac{\partial\Psi}{\partial t}=\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}+\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)^2\Psi+\left(eA_0+\lambda\frac{\partial\mu}{\partial t}\right)\Psi. \tag{1}$$

This equation was given by us in reference [4]. It is equivalent to the set (16)-(17) of section **3**, with

$$\Psi=R\exp\left[\frac{i}{\hbar}S\right]. \tag{2}$$

It is obvious that (1) leads to a continuity equation, no matter how λ and μ be taken

$$\frac{\partial\varrho}{\partial t}+\operatorname{div}\boldsymbol{j}=0,\qquad \varrho=\Psi^*\Psi,\qquad \boldsymbol{j}=\varrho\boldsymbol{v}, \tag{3}$$

$$\boldsymbol{j}=\frac{1}{2m}\left\{\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}+\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\Psi-\Psi\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{x}}+\frac{e}{c}\boldsymbol{A}-\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\Psi^*\right\}. \tag{4}$$

It follows from (3) and (4) that

$$m\boldsymbol{v}+\frac{e}{c}\boldsymbol{A}=\frac{\partial S}{\partial\boldsymbol{x}}+\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}. \tag{5}$$

It is remarkable that the equations (15) of section **3** can be obtained together with (1) from the ordinary Schrödinger variational principle

$$\delta\int\Psi^*\left\{i\hbar\frac{\partial\Psi}{\partial t}-\frac{1}{2m}\left(\frac{\hbar}{i}\frac{\partial}{\partial\boldsymbol{x}}-\frac{e}{c}\boldsymbol{A}+\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)^2\Psi-\left(eA_0+\lambda\frac{\partial\mu}{\partial t}\right)\Psi\right\}\mathrm{d}t\,\mathrm{d}\boldsymbol{x}=0, \tag{6}$$

by giving arbitrary variations to the wave function Ψ and the functions λ and μ, as we showed in reference [4]. Equation (1) follows from the variation of Ψ^* and the equations (15) of section **3** from the variation of λ and μ.

We have seen in section **4** that the most general change of the Clebsch parameters S, λ, μ, such that the new parameters λ' and μ' be constants of the motion, corresponds to a gauge transformation of the second kind of $-\lambda\,\partial\mu/\partial\boldsymbol{x}$ and $\lambda\,\partial\mu/\partial t$

$$\left|\begin{aligned}\lambda'\frac{\partial\mu'}{\partial\boldsymbol{x}}&=\lambda\frac{\partial\mu}{\partial\boldsymbol{x}}-\frac{\partial\Gamma}{\partial\boldsymbol{x}},\\ \lambda'\frac{\partial\mu'}{\partial t}&=\lambda\frac{\partial\mu}{\partial t}-\frac{\partial\Gamma}{\partial t},\end{aligned}\right. \tag{7}$$

Γ being an arbitrary real function $\Gamma(\lambda, \mu)$ of λ and μ not involving explicitly the time t, provided λ and μ be independent functions of $\boldsymbol{x}$. $\partial\Gamma/\partial t$ denotes now the $(\partial\Gamma/\partial t)_{\text{tot}}$ of section **4**. To the transformation of the λ and μ we can associate the following transformation of S

(8) $$S' = S + \Gamma ,$$

which leads to a gauge transformation of the first kind of Ψ

(9) $$\Psi' = \Psi \exp\left[\frac{i}{\hbar}\,\Gamma\right] .$$

The invariance of the generalized Schrödinger equation for the transformation (7)-(9) follows immediately from the variational principle (6), since for any real function Γ we have

(10) $$\begin{cases} \Psi'^*\left\{i\hbar\dfrac{\partial}{\partial t} - \left(eA_0 + \lambda'\dfrac{\partial\mu'}{\partial t}\right)\right\}\Psi' = \Psi^*\left\{i\hbar\dfrac{\partial}{\partial t} - \left(eA_0 + \lambda\dfrac{\partial\mu}{\partial t}\right)\right\}\Psi , \\ \Psi'^*\left(\dfrac{\hbar}{i}\,\dfrac{\partial}{\partial \boldsymbol{x}} - \dfrac{e}{c}\boldsymbol{A} + \lambda'\dfrac{\partial\mu'}{\partial \boldsymbol{x}}\right)^2\Psi' = \Psi^*\left(\dfrac{\hbar}{i}\,\dfrac{\partial}{\partial \boldsymbol{x}} - \dfrac{e}{c}\boldsymbol{A} + \lambda\dfrac{\partial\mu}{\partial \boldsymbol{x}}\right)^2\Psi . \end{cases}$$

Therefore it is not necessary to impose on Γ the condition of being a function of λ and μ only. When λ and μ are independent, it follows from (7) that Γ must be a function of λ and μ. Indeed, the equations (7) are equivalent to the following equation

(11) $$\lambda'\,\mathrm{d}\mu' = \lambda\,\mathrm{d}\mu - \mathrm{d}\Gamma .$$

The class of the Pfaff expression $\lambda'\,\mathrm{d}\mu'$ is at most 2, so that the class of $\lambda\,\mathrm{d}\mu - \mathrm{d}\Gamma$ cannot be larger than 2. When λ and μ are independent the class of $\lambda\,\mathrm{d}\mu - \mathrm{d}\Gamma$ is 3, unless Γ is a function of λ and μ.

When λ and μ are not independent, the generalized Schrödinger equation goes over into the ordinary Schrödinger equation. Indeed, by taking

(12) $$\mathrm{d}\Gamma = \lambda\,\mathrm{d}\mu ,$$

we get

(13) $$\lambda'\frac{\partial\mu'}{\partial \boldsymbol{x}} = 0 , \qquad \lambda'\frac{\partial\mu'}{\partial t} = 0 ,$$

and Ψ' satisfies the ordinary Schrödinger equation.

Let us consider now the transformation

$$\Psi' = C\Psi\,, \qquad \Psi'^* = C^*\Psi'\,, \tag{14}$$

C being an arbitrary real or complex constant. Since

$$R^2\boldsymbol{v} = \frac{1}{2m}\left\{\Psi^*\left(\frac{\hbar}{i}\,\frac{\partial}{\partial\boldsymbol{x}} - \frac{e}{c}\boldsymbol{A} + \lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\Psi - \Psi\left(\frac{\hbar}{i}\,\frac{\partial}{\partial\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A} - \lambda\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\Psi^*\right\}, \tag{15}$$

the velocity $\boldsymbol{v}$ is not changed by the transformation (14), λ and μ being kept unchanged. *Thus from any solution Ψ, λ, μ of the generalized Schrödinger equation we can derive other solutions $C\Psi^*$, λ, μ corresponding to the same distribution of velocities $\boldsymbol{v}(t, \boldsymbol{x})$, the density being everywhere multiplied by the constant factor $|C|^2$.*

The gauge-invariance of the generalized Schrödinger equation can be easily checked. The gauge transformation (19) of section **4** corresponds to the transformation of the wave function

$$\Psi' = \Psi \exp\left[\frac{ie}{\hbar c}\varphi\right], \qquad \lambda' = \lambda\,, \qquad \mu' = \mu\,, \tag{16}$$

which can also be combined with a transformation of the parameters S, λ and μ of the type (7).

Since the generalized Schrödinger equation has the same form as an ordinary Schrödinger equation for a particle moving in the electromagnetic field described by the potentials $A_{0,\,\text{in}} = (\lambda/e)\,\partial\mu/\partial t$, $A_{\text{in}} = -(c\lambda/e)\,\partial\mu/\partial\boldsymbol{x}$, we may associate to any motion of the Madelung fluid a quantal state of motion of a single particle described by the wave functions Ψ. This association requires the condition of single-valuedness of the wave function which leads to the quantization of the strengthes of the η-rings, as shown in section **5**.

Let us denote by $\boldsymbol{E}_{\text{in}}$ and $\boldsymbol{H}_{\text{in}}$ the electric and magnetic fields corresponding to the potentials $(\lambda/e)\,\partial\mu/\partial t$ and $-(c\lambda/e)\,\partial\mu/\partial\boldsymbol{x}$

$$\boldsymbol{H}_{\text{in}} = -\frac{c}{e}\,\frac{\partial\lambda}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}} = -\frac{c}{e}\boldsymbol{\eta}\,, \qquad \boldsymbol{E}_{\text{in}} = -\frac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}_{\text{in}} = \frac{\boldsymbol{v}}{e}\wedge\boldsymbol{\eta}\,. \tag{17}$$

The second equation (17) *shows that the Lorentz force due to the «inner» field vanishes.* The Helmholtz equation (28) of section **2** gives the equation of motion of the «inner» field

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\boldsymbol{H}_{\text{in}}}{\varrho} - \left(\frac{\boldsymbol{H}_{\text{in}}}{\varrho}\cdot\frac{\partial}{\partial\boldsymbol{x}}\right)\boldsymbol{v} = 0\,. \tag{18}$$

It is obvious that the equations of motion of the Madelung fluid can be written as follows

$$(19)\qquad \begin{cases} i\hbar\dfrac{\partial\Psi}{\partial t} = \dfrac{1}{2m}\left(\dfrac{\hbar}{i}\dfrac{\partial}{\partial \boldsymbol{x}} - \dfrac{e}{c}(\boldsymbol{A}+\boldsymbol{A}_{\text{in}})\right)^2\Psi + e(A_0 + A_{0,\text{in}})\Psi, \\[2ex] \boldsymbol{H}_{\text{in}} = \operatorname{rot}\boldsymbol{A}_{\text{in}}\,, \qquad \boldsymbol{E}_{\text{in}} = -\dfrac{\partial \boldsymbol{A}_{0,\text{in}}}{\partial \boldsymbol{x}} - \dfrac{1}{c}\dfrac{\partial \boldsymbol{A}_{\text{in}}}{\partial t}\,, \\[2ex] \left(\dfrac{\mathrm{d}}{\mathrm{d}t} - \dfrac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)\dfrac{\boldsymbol{H}_{\text{in}}}{\varrho} = 0\,, \qquad \boldsymbol{E}_{\text{in}} = -\dfrac{\boldsymbol{v}}{c}\wedge\boldsymbol{H}_{\text{in}}\,. \end{cases}$$

These equations do not involve any more the Clebsch parameters. The equations of motion of the Madelung fluid are equivalent to the system formed by the generalized Schrödinger equation, the Helmholtz equation in the form (18) *and the condition on the Lorentz force due to the « inner » field to vanish* $\boldsymbol{E}_{\text{in}} + \boldsymbol{v}/c\wedge\boldsymbol{H}_{\text{in}} = 0$.

7. – Steady Motions of the Madelung Fluid.

In the steady motions, the velocity and the density at any point are time independent

$$(1)\qquad \frac{\partial\varrho}{\partial t} = 0\,, \qquad \frac{\partial \boldsymbol{v}}{\partial t} = 0\,.$$

We shall assume that the external electromagnetic field is time independent and that the potentials do not depend on t. In this case the Euler equation becomes

$$(2)\qquad \frac{\partial W}{\partial \boldsymbol{x}} = \boldsymbol{v}\wedge\boldsymbol{\eta}\,,$$

$$(3)\qquad W = \frac{m\boldsymbol{v}^2}{2} + eA_0 - \frac{\hbar^2}{2m}\frac{\Delta R}{R}\,,$$

as is easily seen by taking into account the identity $(\partial/\partial\boldsymbol{x})\boldsymbol{v}^2/2 = (\boldsymbol{v}\cdot\partial/\partial\boldsymbol{x})\boldsymbol{v} + \boldsymbol{v}\wedge\operatorname{rot}\boldsymbol{v}$. Since

$$(4)\qquad \frac{\partial W}{\partial \boldsymbol{x}}\cdot\boldsymbol{v} = 0\,,$$

W is a constant of the motion. In the present case the trajectories coincide with the streamlines and W has the same value at all the points of a streamline. When

$$(5)\qquad \boldsymbol{v}\wedge\boldsymbol{\eta}=0 \qquad \text{(everywhere)}\,.$$

W has the same value at all the points of the fluid. Condition (5) is satisfied in the quasi-irrotational motions.

When $\boldsymbol{v}\wedge\boldsymbol{\eta}\neq 0$, *the value of* W *is not constant in the whole fluid. This is an essential difference between the general and the quasi-irrotational steady motions of the Madelung fluid.*

In the steady motions of the Madelung fluid

$$(6)\qquad \frac{\partial W}{\partial \boldsymbol{x}}\cdot\boldsymbol{\eta}=0\,,$$

so that W can be taken as a Clebsch parameter, provided $\boldsymbol{v}\wedge\boldsymbol{\eta}\neq 0$. Let us take

$$(7)\qquad \lambda=W\,.$$

We must determine the parameter μ by the conditions

$$(8)\qquad \boldsymbol{\eta}=\frac{\partial\lambda}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}\,,\qquad \frac{\partial\mu}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}}=0\,.$$

It follows from (7) and (2) that

$$(9)\qquad \frac{\partial\lambda}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}=\frac{\partial W}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}=(\boldsymbol{v}\wedge\boldsymbol{\eta})\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}=\left(\boldsymbol{v}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\boldsymbol{\eta}-\left(\boldsymbol{\eta}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\boldsymbol{v}\,.$$

The vector $\boldsymbol{\eta}$ is orthogonal to $\partial\mu/\partial\boldsymbol{x}$, as a consequence of the first equation (8). Hence

$$(10)\qquad \frac{\partial\lambda}{\partial\boldsymbol{x}}\wedge\frac{\partial\mu}{\partial\boldsymbol{x}}=\left(\boldsymbol{v}\cdot\frac{\partial\mu}{\partial\boldsymbol{x}}\right)\boldsymbol{\eta}=-\frac{\partial\mu}{\partial t}\boldsymbol{\eta}\,,$$

and we get

$$(11)\qquad \frac{\partial\mu}{\partial t}=-1\,,\qquad \mu=-t+\nu(\boldsymbol{x})\,.$$

When $\boldsymbol{v}\wedge\boldsymbol{\eta}\neq 0$ the two equations

$$(12)\qquad \boldsymbol{v}\cdot\frac{\partial f}{\partial\boldsymbol{x}}=0\,,\qquad \boldsymbol{\eta}\cdot\frac{\partial f}{\partial\boldsymbol{x}}=0\,,$$

admit only common solutions of the form $F(t, W)$, F denoting an arbitrary function. Therefore, in that case, a time independent Clebsch parameter λ or μ which is also a constant of the motion must be a function of W. When $\boldsymbol{v}\wedge\boldsymbol{\eta}=0$, the two equations (12) are not distinct and it is possible to choose λ and μ as time independent constants of the motion. Let us assume that $\boldsymbol{v}\wedge\boldsymbol{\eta}=0$ and $\boldsymbol{\eta}\neq 0$. The second equation (12) has two independent solutions f_1 and f_2, that are also time independent constants of the motion. $\boldsymbol{\eta}$, being orthogonal to $\partial f_1/\partial \boldsymbol{x}$ and $\partial f_2/\partial \boldsymbol{x}$, is parallel to $\partial f_1/\partial \lambda \wedge \partial f_2/\partial \boldsymbol{x}$

$$\boldsymbol{\eta}=\alpha\frac{\partial f_1}{\partial \boldsymbol{x}}\wedge\frac{\partial f_2}{\partial \boldsymbol{x}}\,. \tag{13}$$

Since

$$\operatorname{div}\boldsymbol{\eta}=0=\frac{\partial \alpha}{\partial \boldsymbol{x}}\cdot\frac{\partial f_1}{\partial \boldsymbol{x}}\wedge\frac{\partial f_2}{\partial \boldsymbol{x}}\,, \tag{14}$$

we have

$$\alpha=\Phi(f_1, f_2)\,. \tag{15}$$

Let F_1 and F_2 be two independent functions of f_1, f_2. Since

$$\frac{\partial F_1}{\partial \boldsymbol{x}}\wedge\frac{\partial F_2}{\partial \boldsymbol{x}}=\frac{D(F_1, F_2)}{D(f_1, f_2)}\cdot\frac{\partial f_1}{\partial \boldsymbol{x}}\wedge\frac{\partial f_2}{\partial \boldsymbol{x}}=\frac{\boldsymbol{\eta}}{\alpha}\,\frac{D(F_1, F_2)}{D(f_1, f_2)}\,, \tag{16}$$

by choosing the F in order that

$$\frac{D(F_1, F_2)}{D(f_1, f_2)}=\alpha\,, \tag{17}$$

we can take

$$\lambda=F_1\,,\qquad \mu=F_2\,. \tag{18}$$

In order that time independent constants of the motion may be taken as Clebsch parameters in a steady motion of the Madelung fluid, it is necessary and sufficient that $\boldsymbol{v}\wedge\boldsymbol{\eta}=0$.

The flux through any section of a stream tube has a value independent of the choice of the section, in a steady motion. Hence

$$\int_S \varrho\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S=\int_{S_0}\varrho_0\boldsymbol{v}_0\cdot\boldsymbol{n}_0\,\mathrm{d}S_0\,. \tag{19}$$

In a steady motion $\varrho \boldsymbol{v}$ *is a* $\boldsymbol{\gamma}$*-vector.* By the application of the formula (26) of section 2, we get

$$\boldsymbol{v} = \left(\boldsymbol{v}_0 \cdot \frac{\partial}{\partial \boldsymbol{x}_0}\right) \boldsymbol{x} . \tag{20}$$

The dyadic that transforms $\boldsymbol{v}_0$ *into* $\boldsymbol{v}$ *transforms also* $\boldsymbol{\eta}_0/\varrho_0$ *into* $\boldsymbol{\eta}/\varrho$*, in any steady motion of the Madelung fluid.*

It follows from the above theorem that $\boldsymbol{v}_0 \wedge \boldsymbol{\eta}_0 = 0$ leads to $\boldsymbol{v} \wedge \boldsymbol{\eta} = 0$ at the points of the streamline passing through $\boldsymbol{x}_0$ and to the constancy of $\eta/\varrho v$

$$\frac{\eta}{\varrho v} = \frac{\eta_0}{\varrho_0 v_0} . \tag{21}$$

$\eta/\varrho v$ *is a constant of the motion in the steady motions in which the streamlines coincide with the* η*-lines, i.e. in the steady motions in which* W *is constant through the whole fluid.* This theorem corresponds to that of Beltrami for ordinary inviscid fluids. We shall call Beltrami motions those satisfying the condition $\boldsymbol{v} \wedge \boldsymbol{\eta} = 0$. *In the Beltrami motions the «inner» electric field* $\boldsymbol{E}_{\text{in}}$*, defined in section* 6 *is null.*

We are led to the Beltrami motions by considering the motions of the Madelung fluid with wave functions of the stationary quantum mechanical type

$$\Psi(t, \boldsymbol{x}) = \Phi(\boldsymbol{x}) \exp\left[-\frac{i}{\hbar} Et\right] , \qquad E = \text{constant}, \tag{22}$$

with time independent λ and μ. In this case the generalized Schrödinger equation becomes

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial \boldsymbol{x}} - \frac{e}{c}\boldsymbol{A} + \lambda \frac{\partial \mu}{\partial \boldsymbol{x}}\right)^2 \Phi + eA_0 \Phi = E\Phi , \tag{23}$$

$$\boldsymbol{v} \cdot \frac{\partial \lambda}{\partial \boldsymbol{x}} = 0 , \qquad \boldsymbol{v} \cdot \frac{\partial \mu}{\partial \boldsymbol{x}} = 0 . \tag{24}$$

The existence of the constant W in the steady motions of the Madelung fluid corresponds to the Bernoulli theorem, which expresses the conservation of the total energy of any element of the fluid during its motion. The total energy per mass m is the sum of the kinetic energy $mv^2/2$, the external potential energy eA_0 and the internal energy $-(\hbar^2/2m)\Delta R/R$. In the steady Beltrami motions, the energy per unit mass is the same for all the elements of the fluid. This happens in particular in the quasi-irrotational steady motions described by the ordinary Schrödinger equation.

It is interesting to notice that the steady motions corresponding to discrete eigenvalues of the hamiltonian are characterized by the conditions of having wave functions Ψ that are regular everywhere and tend to zero at infinity with sufficient rapidity to render $\int_{-\infty}^{+\infty} |\Psi|^2 \mathrm{d}\boldsymbol{x}$ *finite and of having* $\boldsymbol{\eta} = 0$ *everywhere.*

8. – Simple Case of Beltrami Motion.

Beltrami discussed, in hydrodynamics, the motion of an incompressible fluid with the following distribution of velocities

$$\boldsymbol{v} = \boldsymbol{\omega} \wedge \boldsymbol{x} + \sqrt{2}\sqrt{a^2 - r^2}\,\boldsymbol{\omega} \qquad (r \leqslant a)(r^2 = x_1^2 + x_2^2), \tag{1}$$

$\boldsymbol{\omega}$ = constant vector parallel to the x_3-axis.

It is easily seen that

$$\operatorname{rot} \boldsymbol{v} = \sqrt{\frac{2}{a^2 - r^2}}\,\boldsymbol{v}, \qquad \operatorname{div} \boldsymbol{v} = 0 \qquad (r \leqslant a). \tag{2}$$

We shall assume the distribution of velocities (1) for $r \leqslant a$ and take

$$\boldsymbol{v} = \boldsymbol{\omega} \wedge \frac{a^2 \boldsymbol{x}}{r^2}, \qquad (r > a). \tag{3}$$

It is easily seen that

$$\operatorname{rot} \boldsymbol{v} = 0, \qquad \operatorname{div} \boldsymbol{v} = 0. \qquad (r > a). \tag{4}$$

Equations (1) and (3) define a distribution of velocities satisfying everywhere the condition $\boldsymbol{v} \wedge \boldsymbol{\eta} = 0$, in the absence of magnetic fields. We shall assume that there are no electromagnetic fields. Since $\mathrm{d}r/\mathrm{d}t = 0$, we have

$$\begin{cases} m \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{\omega} \wedge \boldsymbol{v} = -m\omega^2 \boldsymbol{r}, & r \leqslant a, \\[2ex] m \dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \dfrac{ma^2}{r^2}\boldsymbol{\omega} \wedge \boldsymbol{v} = -\dfrac{m\omega^2 a^4}{r^4}\boldsymbol{r}, & r > a, \\[2ex] \boldsymbol{r} = x_1 \boldsymbol{i}_1 + x_2 \boldsymbol{i}_2 . \end{cases} \tag{5}$$

The density mR^2 is now a constant of the motion, since $\operatorname{div} \boldsymbol{v} = 0$. R can be taken as a function of r satisfying the equations

$$(6)\qquad \frac{\hbar^2}{2m}\frac{\mathrm{d}}{\mathrm{d}r}\frac{\Delta R}{R} = \begin{cases} -\,m\omega^2 r\,, & r \leqslant a,\\ -\,\dfrac{m\omega^2 a^4}{r^3}\,, & r > a.\end{cases}$$

Hence

$$(7)\qquad \frac{\Delta R}{R} = \begin{cases} -\,\dfrac{m^2\omega^2}{\hbar^2}(r^2 - 2a^2) - k^2\,, & r \leqslant a,\\ \dfrac{m^2\omega^2 a^4}{\hbar^2 r^2} - k^2\,, & r > a.\end{cases}$$

(k^2 = constant of integration).

It follows from the equations (1) and (2) of section **5** that we must take

$$(8)\qquad m\omega a^2 = l\hbar \qquad (l = \text{positive integer}).$$

since

$$(9)\qquad 2\pi\int_0^a \boldsymbol{i}_3\cdot \mathrm{rot}\, \boldsymbol{v}\, r\, \mathrm{d}r = 2\pi\omega a^2\,.$$

We get from (7)

$$(10)\qquad \begin{cases} \dfrac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}R}{\mathrm{d}r} + \left\{\dfrac{l^2}{a^4}(r^2 - 2a^2) + k^2\right\} R = 0\,, & r < a,\\ \dfrac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \dfrac{1}{r}\dfrac{\mathrm{d}R}{\mathrm{d}r} + \left(k^2 - \dfrac{l^2}{r^2}\right) R = 0\,, & r > a.\end{cases}$$

Hence

$$(11)\qquad R = \alpha J_l(kr) + \beta N_l(kr)\,, \qquad r > a \quad (\alpha \text{ and } \beta = \text{constants}),$$

J_l and N_l denoting the Bessel and Neumann functions of order l, respectively. The first equation (10) has only one solution regular at $r = 0$, we shall denote it by $G(r)$, with the normalization $G(0) = 1$. The fitting conditions at the boundary of the vortex tube

$$(12)\qquad \begin{cases} G\,(a) = \alpha J_l(ka) + \beta N_l(ka)\,,\\ G'(a) = k\{\alpha J_l'(ka) + \beta N_l'(ka)\}\,,\end{cases}$$

determine α and β

$$\text{(13)} \qquad \left\{ \begin{aligned} \alpha &= \frac{\pi a}{2} \{kG(a)N'_l(ka) - G'(a)N_l(ka)\}, \\ \beta &= -\frac{\pi a}{2} \{kG(a)J'_l(ka) - G'(a)J_l(ka)\}. \end{aligned} \right.$$

The trajectories of the elements of the fluid are helices in the vortex tube and circles outside the tube. It is easily seen that the value of W is everywhere the same

$$\text{(14)} \qquad W = \frac{m\boldsymbol{v}^2}{2} - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = \frac{\hbar^2 k^2}{2m}.$$

We can take for $r < a$

$$\text{(15}a\text{)} \qquad S = -Wt + l\hbar\varphi\left(2 - \frac{r^2}{a^2}\right), \quad \lambda = \frac{l\hbar}{a^2}\sqrt{2(a^2 - r^2)}, \quad \mu = x_3 - \sqrt{2(a^2 - r^2)}\varphi,$$

and for $r > a$

$$\text{(15}b\text{)} \qquad S = -Wt + l\hbar\varphi, \qquad \lambda = 0, \qquad \mu = 0,$$

φ denoting the azimuthal angle around the x_3 axis

$$\text{(16)} \qquad \operatorname{tg}\varphi = \frac{x_2}{x_1}.$$

The above choice of the Clebsch parameters leads to a multiple valued Ψ for $r < a$. A single valued wave function can be obtained by taking for $r < a$

$$\text{(17)} \quad S = -Wt + \frac{l\hbar x_3}{\sqrt{2a^2}}\sqrt{a^2 - r^2} + l\hbar\varphi, \; \lambda = \frac{l\hbar}{\sqrt{2a^2}}(a^2 - r^2), \; \mu = \frac{x_3}{\sqrt{a^2 - r^2}} - \sqrt{2}\varphi.$$

Equation (2) shows that the component of the vorticity tangent to the cirles $x_3 = \text{const}$ $r = a$ is infinite. Situations in which the vorticity becomes infinite on a surface are well known in hydrodynamics.

9. – Discontinuous Motions and Vortex Sheets.

We shall now examine some cases of discontinuous motion associated to vortex sheets. Let us discuss firstly a simple case of steady motion, in the absence of electromagnetic fields, whose velocity distribution is

$$\text{(1)}\qquad \boldsymbol{v} = \begin{cases} v_{\mathrm{I}}\,\boldsymbol{i}_3\,, & x_1 > 0\,,\\ v_{\mathrm{II}}\boldsymbol{i}_3\,, & x_1 < 0\,,\end{cases}$$

$$\text{(2)}\qquad R = \text{constant} \qquad v_{\mathrm{I}},\ v_{\mathrm{II}} = \text{constants}$$

The Euler and continuity equations are obviously satisfied for $x_1 \neq 0$. Although $\boldsymbol{v}$ is discontinuous, we can give a meaning to the equations of motion even when $x_1 = 0$, since

$$\text{(3)}\qquad \lim_{x_1\to 0+}\left\{m\frac{\partial \boldsymbol{v}}{\partial t} + m\left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v} - \frac{\hbar^2}{2m}\frac{\partial}{\partial \boldsymbol{x}}\frac{\Delta R}{R}\right\} = \lim_{x_1\to 0-}\left\{m\frac{\partial \boldsymbol{v}}{\partial t} + m\left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v} - \frac{\hbar^2}{2m}\frac{\partial}{\partial \boldsymbol{x}}\frac{\Delta R}{R}\right\} = 0\,,$$

$$\text{(4)}\qquad \lim_{x_1\to 0+}\left\{\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v})\right\} = \lim_{x_1\to 0-}\left\{\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v})\right\} = 0\,.$$

Denoting by $\delta(x)$ the Dirac singular function, we have

$$\text{(5)}\qquad \operatorname{rot}\boldsymbol{v} = (v_{\mathrm{II}} - v_{\mathrm{I}})\,\delta(x_1)\boldsymbol{i}_2\,.$$

The discontinuity surface $x_1 = 0$ is a vortex sheet. The distribution of velocities (1) can be described by means of discontinuous Clebsch parameters

$$\text{(6)}\qquad S = m|\boldsymbol{v}|x_3\,, \qquad \lambda = -x_3\,, \qquad \mu = m|\boldsymbol{v}|\,.$$

The above results can be easily extended to more general cases. Let Σ be a surface dividing the space into two regions in which the vector field $\boldsymbol{w}(\boldsymbol{x})$ is continuous and has continuous derivatives, there being a jump of the tangential component of $\boldsymbol{w}$ across Σ. Let $\boldsymbol{n}(\boldsymbol{x})$, $\boldsymbol{\tau}_1(\boldsymbol{x})$, $\boldsymbol{\tau}_2(\boldsymbol{x})$ be three mutually orthogonal continuous unit vectors such that $\boldsymbol{n} = \boldsymbol{\tau}_1 \wedge \boldsymbol{\tau}_2$ and $\boldsymbol{n}(\boldsymbol{x})$ be normal to Σ when $\boldsymbol{x}$ lies on Σ. Since

$$\text{(7)}\qquad \operatorname{div}\boldsymbol{\omega} = \left(\boldsymbol{\tau}_1\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega}\cdot\boldsymbol{\tau}_1 + \left(\boldsymbol{\tau}_2\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega}\cdot\boldsymbol{\tau}_2 + \left(\boldsymbol{n}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega}\cdot\boldsymbol{n},$$

we can define $\operatorname{div}\boldsymbol{w}$ on Σ when $(\boldsymbol{\tau}_1\cdot\partial/\partial\boldsymbol{x})\,\boldsymbol{w}\cdot\boldsymbol{\tau}_1 + (\boldsymbol{\tau}_2\cdot\partial/\partial\boldsymbol{x})\,\boldsymbol{w}\cdot\boldsymbol{\tau}_2 + (\boldsymbol{n}\cdot\partial/\partial\boldsymbol{x})\boldsymbol{w}\cdot\boldsymbol{n}$ tends to a defined value, independent of the path, as $\boldsymbol{x}$ approaches any point of Σ. $\operatorname{div}\boldsymbol{w}$ can be finite on Σ, notwithstanding the discontinuity of the tangential component of $\boldsymbol{w}$. The situation with $\operatorname{rot}\boldsymbol{w}$ is essentially different. Since

$$\operatorname{rot}\boldsymbol{\omega} = \boldsymbol{\tau}_1 \wedge \left(\boldsymbol{\tau}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega} + \boldsymbol{\tau}_2 \wedge \left(\boldsymbol{\tau}_2 \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega} + \boldsymbol{n} \wedge \left(\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{\omega}\,, \tag{8}$$

and

$$\boldsymbol{n} \wedge \left(\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{w} = \boldsymbol{n} \wedge \frac{\partial \boldsymbol{w}}{\partial n} = \frac{\partial}{\partial n}(\boldsymbol{n} \wedge \boldsymbol{w}) + \boldsymbol{w} \wedge \left(\boldsymbol{n} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{n}\,, \tag{9}$$

$\operatorname{rot}\boldsymbol{w}$ has a delta-like singularity arising from the derivation of the discontinuous tangential vector $\boldsymbol{n} \wedge \boldsymbol{w}$. By the application of the above results to the velocity, we see that:

The discontinuity of the tangential component of the velocity across a surface Σ is associated to the existence of a vortex sheet on Σ corresponding to a delta-like singularity of the tangential component of $\operatorname{rot}\boldsymbol{v}$. *It may nevertheless be possible to define the divergence of $\boldsymbol{v}$ on Σ and the value of* $\operatorname{div}\boldsymbol{v}$ *may be finite on Σ.*

The surface Σ may either contain always the same elements of the fluid or propagate in the mass of the fluid as a wave of discontinuity. In the case of a fixed Σ containing always the same elements of the fluid, their trajectories lie on Σ, so that the normal component of the velocity is continuous and null on Σ. In order that the equations of motion be satisfied on Σ, it is necessary that

$$m\frac{\partial \boldsymbol{v}}{\partial t} + m\left(\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}}\right)\boldsymbol{v} - \frac{\hbar^2}{2m}\frac{\partial}{\partial \boldsymbol{x}}\frac{\Delta R}{R} - \boldsymbol{E}\,,$$

and

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \varrho}{\partial \boldsymbol{x}} + \varrho \operatorname{div}\boldsymbol{v}\,,$$

tend to zero as $\boldsymbol{x}$ approaches Σ, independently of the path. There is no difficulty with the terms involving the derivatives of $\boldsymbol{v}$, as we have shown. With respect to this point the situation is similar to that of the dynamics of ordinary compressible fluids, but the requirements on the behaviour of the density are essentially different, because the force derived from the quantum potential involves derivatives of the third order of the density.

We shall now discuss another example of steady motions with vortex sheets, corresponding to a potential $V(r)$ depending only on the distance to the x_3 - axis. Let $R(r)\exp[il\varphi - (i/\hbar)Et]$ be a stationary solution of the ordinary Schrödinger equation for the potential $V(r)$, corresponding to the eigenvalue E of the energy. The trajectories of the elements are circles $r = \text{const.}$ $x_3 = \text{const.}$

described in uniform motion, the velocity being $(l\hbar/mr^2)\boldsymbol{i}_3\wedge\boldsymbol{r}$. Let us consider the motion in which the density distribution is the same as in that stationary state, the distribution of the velocities being

$$\boldsymbol{v} = \begin{cases} \dfrac{l\hbar}{mr^2}\,\boldsymbol{i}_3\wedge\boldsymbol{r} + v_{\mathrm{I}}\,\boldsymbol{i}_3\,, & r < a, \\ \dfrac{l\hbar}{mr^2}\,\boldsymbol{i}_3\wedge\boldsymbol{r} + v_{\mathrm{II}}\boldsymbol{i}_3\,, & r > a, \end{cases} \tag{10}$$

$$(v_{\mathrm{I}},\, v_{\mathrm{II}} = \text{constants})\,.$$

In the present case Σ is the cylinder $r = a$. It is easily seen that the Euler equation and the equation of continuity are satisfied everywhere. The motion is irrotational for $r \neq a$ but the cylinder $r = a$ is a vortex sheet

$$\operatorname{rot}\boldsymbol{v} = \delta(r-a)(v_{\mathrm{I}} - v_{\mathrm{II}})\boldsymbol{i}_3\wedge\frac{\boldsymbol{r}}{r}\,. \tag{11}$$

It follows from the equation (8) of section **1** that the stress $\boldsymbol{T}$ acting on a surface element $d\sigma$ of the fluid is in general not directed along the normal $\boldsymbol{n}$

$$\boldsymbol{T} = \frac{\hbar^2}{m}\,\frac{\partial R}{\partial n}\,\frac{\partial R}{\partial \boldsymbol{x}} - \frac{\hbar^2}{4m}\,\Delta\varrho\,\boldsymbol{n}\,. \tag{12}$$

A necessary and sufficient condition for $\boldsymbol{T}$ to be normal is

$$\frac{\partial R}{\partial n}\,\frac{\partial R}{\partial \boldsymbol{x}}\wedge\boldsymbol{n} = 0\,, \tag{13}$$

This condition is satisfied in the motion defined by (10) on Σ.

RIASSUNTO (*)

Si discutono i moti generici del mezzo continuo (fluido di Madelung) i cui moti irrotazionali sono descritti dall'equazione di Schrödinger. Si dimostra che molti dei teoremi fondamentali del moto vorticoso dei fluidi barotropici non viscosi sono validi anche per il fluido di Madelung. Si dà una discussione dettagliata dei parametri di Clebsch. Si dimostra l'esistenza di uno speciale tipo di moti permanenti, simili ai moti di Beltrami, in cui le linee di flusso coincidono con le linee di vorticosità, che corrisponde a una stretta generalizzazione degli ordinari stati stazionari. Si discute la quantizzazione dei tubi di vorticosità. Si dànno esempi di moti di Beltrami e di moti discontinui del fluido di Madelung. Si dimostra che i moti generici del fluido di Madelung possono essere interpretati anche fisicamente in termini degli ordinari stati quantici di una particella.

(*) *Traduzione a cura della Redazione*