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A Statistical Generalization of the Quantum Mechanics (I).

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Summary. — It is shown that the ordinary Schrödinger equation for a dynamical system Σ may be replaced by a more general equation, which has the form of the Schrödinger equation of a quantized Bose field whose quanta are the systems Σ. The linear wave functionals of the quantized field describe pure states of the system Σ and the non linear wave functionals describe, in general, mixtures of states of Σ. The representation in which the emission operators of the quantized Σ field are diagonal plays a central role in the present formalism. It is shown that the eigenfunctionals of the absorption operators of the quantized Σ field can be used to obtain a new description of the states of a system Σ, the expectation values of the field quantities in suitably chosen eigenstates of the emission operators coinciding with the expectation values of the corresponding quantities in the pure states of Σ, but the fluctuations being larger in the former case. The eigenfunctionals of the absorption operators have the remarkable property of being matrix elements of the unity operator of the field formalism, and seem to be of a more fundamental nature than the linear functionals which correspond to the ordinary description of the pure states by means of the wave functions of Σ.

1. – Introduction.

In the ordinary form of the quantum mechanics, the states of a system are described by wave functions $\Psi(t; x)$ depending on the time t and variables x. The time evolution of the wave function is determined by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t; x) = H\Psi(t; x) , \tag{1}$$

H being the hamiltonian operator of the system Σ. For the sake of simplicity we shall treat only the case of continuous variables x, so that the x will be coordinates of a point in a continuous space Ω. The case, in which some or all of the x are discrete variables, can be treated in a similar way, with minor modifications.

In general, we shall assume that the wave functions $\Psi(x)$ are normalizable and normalized as usual: $\int_\Omega |\Psi|^2 \mathrm{d}x = 1$. These normalizable wave functions may be considered as vectors of a Hilbert space, the ψ-space. The complex conjugate functions $\Psi^*(x)$ are vectors of another Hilbert space, the ψ^*-space. We have shown (1) that it is possible to describe the motion of the system Σ by means of the vectors of the Hilbert space dual to the ψ^*-space, the χ-space. The vectors of the χ-space correspond to the continuous linear functionals $\chi[\psi^*]$ defined in the ψ^*-space:

$$\chi[\psi^*] = \int_\Omega \Psi(x)\psi^*(x)\,\mathrm{d}x\,. \tag{2}$$

It follows from (1) and (2) that the linear functionals $\chi[\psi^*]$ built with the solutions of (1) satisfy the equation.

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\chi[t;\psi^*] = \mathscr{H}\chi[t;\psi^*] = \int_\Omega \mathrm{d}x\,\psi^*(x) H \frac{\delta}{\delta\psi^*(x)}\chi[t;\psi^*]\,, \tag{3}$$

which was already given in reference (1). Equation (3) may be considered as the Schrödinger equation of the dual χ-space. Equation (3) admits as solutions non linear functionals, which are no more equivalent to wave functions Ψ. It is preferable to consider (3) as the basic equation, instead of (1), because the functionals $\chi[\psi^*]$ allow us to describe, in an effective non symbolic way, states of Σ which can only be described by means of symbolic functions, such as the Dirac δ-function. It is well known from the Schwarz theory of distributions that, in order to avoid the use of symbolic functions, it is necessary to use functionals.

Once it is assumed that (3) is the fundamental equation of evolution, it becomes natural to consider also the non linear functionals which are solutions. In this paper we shall consider only the solutions of (3) which can be expanded in Volterra series:

$$\chi[\psi^*] = \chi[0] + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_\Omega \Psi_n(x_1,\dots,x_n)\psi^*(x_1)\dots\psi^*(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,. \tag{4}$$

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(1) M. SCHÖNBERG: *Nuovo Cimento*, **10**, 350 (1953).

The $\Psi_n(t; x)$ are solutions of the equations.

$$i\hbar \frac{\partial}{\partial t} \Psi_n(t; x_1, ..., x_n) = \sum_{l=1}^{n} H(x_l) \Psi(t; x_1, ..., x_n) . \tag{5}$$

These equations have the same form as the ordinary Schrödinger equations for assemblies of non interacting systems Σ. The Ψ_n are symmetrical with respect to the n sets of variables x.

The functionals (4) may be considered as vectors of a linear space which contains as a sub-space the dual of the ψ^*-space. This extended χ-space can be made into a Hilbert space by introducing the following definition of the inner product of two functionals χ:

$$(\chi, \chi') = \{\chi[0]\}^* \chi'[0] + \sum_{n=1}^{\infty} \int_{\Omega} \{\Psi_n(x_1, ..., x_n)\}^* \Psi'_n(x_1, ..., x_n) \, dx_1 ... dx_n . \tag{6}$$

In the special case of linear functionals

$$(\chi, \chi') = \int_{\Omega} \Psi^*(x) \Psi'(x) \, dx , \tag{6a}$$

so that the inner product of two linear functionals coincides with the inner product of the corresponding wave functions Ψ. The definition of the inner product leads to a definition of the lengh $\sqrt{(\chi, \chi)}$ of a vector χ and to the normalization condition:

$$(\chi, \chi) = |\chi[0]|^2 + \sum_{n=1}^{\infty} \int_{\Omega} |\Psi_n|^2 \, dx_1 ... dx_n = 1 . \tag{7}$$

Equation (3) coincides with the Schrödinger equation of the second quantization of the systems Σ, treated as bosons, in the representation in which the emission operators $\psi^*_{op}(x)$ are diagonal. From the commutation rules

$$[\psi_{op}(x), \psi^*_{op}(x')] = \delta(x - x') , \quad [\psi_{op}(x), \psi_{op}(x')] = [\psi^*_{op}(x), \psi^*_{op}(x')] = 0 , \tag{8}$$

it follows that it is possible to diagonalize simultaneously all the $\psi^*_{op}(x)$. They are satisfied by taking:

$$\psi_{op}(x) = \frac{\delta}{\delta \psi^*(x)} . \tag{9}$$

The Schrödinger equation of the second quantization

$$i\hbar \frac{d\chi}{dt} = \int_{\Omega} \psi^*_{op}(x) H \psi_{op}(x) \, dx \, \chi , \tag{10}$$

has the same form as (3), in the representation in which the $\psi^*_{op}(x)$ are diagonal. The physical content of the second quantization equation of evolution and of equation (3) are nevertheless different, because the second quantization is a treatment of the many body problem whereas equation (3) corresponds to a single system Σ.

We have shown ([2]) that the second quantization can be used as a formalism for the statistical quantum mechanics of a system Σ, treated as a quantum of the quantized field obtained by the second quantization of its Schrödinger equation. The Volterra series (4) becomes the Fock expansion in the second quantization and, by means of the functions Ψ_n, it is possible to build a density operator $R(t)$ for a single system Σ with the wave functional $\chi[t; \psi^*]$ of the second quantization:

$$(11)\qquad \langle x' | R(t) | x'' \rangle = \sum_{n=1}^{\infty} \int_{\Omega} \Psi_n(t; x', x_1, \dots, x_{n-1}) \Psi_n^*(t; x'', x_1, \dots, x_{n-1})\, \mathrm{d}x_1 \dots \mathrm{d}x_{n-1} \,.$$

It can be seen that $R(t)$ satisfies the von Neumann equation of motion of the density operators:

$$(12)\qquad i\hbar \frac{\mathrm{d}}{\mathrm{d}t} R(t) = [H, R(t)] \,.$$

In the case of a linear functional χ of the type (2), we have

$$(13)\qquad \langle x' | R(t) | x'' \rangle = \Psi(t; x') \Psi^*(t; x'')$$

and $R(t)$ describes the same pure state as the wave function Ψ. Since our formalism for the motion of the system Σ is the same as that of the second quantization, we can apply the same method of definition of density operators $R(t)$ to the solution of (3). *Thus we get a possible interpretation of the non linear solutions of* (3) *as describing, in general, mixtures of states of* Σ.

A non linear solution of (3) may sometimes correspond to a $R(t)$ which describes a pure state of Σ. Thus the denstiy matrix corresponding to the non linear functional $\Delta[\psi^*, \Psi]$

$$(14)\qquad \Delta[\psi^*, \Psi] = \exp\left[\int_{\Omega} \psi^*(x)\Psi(x)\,\mathrm{d}x\right] ,$$

is:

$$(15)\qquad \langle x' | R_\Delta | x'' \rangle = \Psi(x')\Psi^*(x'')\Big(\exp\left[\int_{\Omega} |\Psi|^2\,\mathrm{d}x\right] - 1\Big)\Big/\int |\Omega|^2\,\mathrm{d}x \,,$$

([2]) M. Schönberg: *Nuovo Cimento*, **10**, 697 (1953).

It was shown in reference (1) that a normalized linear functional χ defines a special kind of probability distribution of the rays of the ψ-space, in which a probability $|\chi[\psi^*]|^2$, with a normalized ψ, is assigned to the ray determined by the vector ψ. The ordinary interpretation rule of the quantum mechanics for discrete non degenerate eigenvalues was obtained by assuming that the probability of finding a value A' of the physical quantity (A) in a measurement in the state of Σ described by the normalized linear functional χ is given by the probability $|\chi[\psi^*_{A'}]|^2$ of the ray determined by the eigenfunction $\psi_{A'}$. We shall see in section **7** that the above result can be easily extended to degenerate discrete eigenvalues or to continuous eigenvalues, with suitable modifications. Thus the probability of obtaining a degenerate discrete value A' is given by the probability of the linear manifold of the eigenvectors of A corresponding to the eigenvalue A', the probability of a linear manifold being taken as the sum of the probabilities of the rays corresponding to a complete set of orthogonal vectors of the manifold.

In the case of non linear functionals, there are difficulties in the introduction of a probability distribution by means of $|\chi[\psi^*]|^2$. The normalization of the vector ψ does not determine completely ψ, since there remains always an undetermined constant phase factor. The value of $|\chi[\psi^*]|^2$ depends in general on the choice of the constant phase factor. Nevertheless in the case of the homogeneous functionals $\chi_n[\psi^*]$

(16) $$\chi_n[\psi^*] = \frac{1}{\sqrt{n!}}\int_\Omega \Psi_n(x_1,\dots, x_n)\psi^*(x_1)\dots\psi^*(x_n)\,dx_1\dots dx_n\,,$$

$|\chi_n[\Psi^*]|^2$ does not depend on the choice of the constant phase factor in the normalized ψ. In the second quantization interpretation of the χ-formalism, we are led to take $|\chi[\Psi^*]|^2$ as the probability of finding the values $\Psi(x)$ of the $\psi_{op}(x)$, by applying to the non hermitian operators ψ_{op} the ordinary rule for the computation of expectation values. In this case there is no reason to take only normalized functions $\Psi(x)$ and the value of $|\chi[\Psi^*]|^2$ may be larger than 1. This point will be discussed in section **8**.

As long as we consider only linear functionals χ, the replacement of the wave function $\Psi(x)$ by the corresponding linear functional (2) may be considered as a change of representation:

(17) $$\chi[\psi^*] = \int_\Omega \langle\psi^*|x\rangle\Psi(x)\,dx\,, \qquad \langle\psi^*|x\rangle = \psi^*(x)\,.$$

Notwithstanding the fact that now we have a transformation functional $\langle\psi^*|x\rangle$, instead of the ordinary transformation functions, the change of representation defined by (17) has properties which correspond to those of the ordinary changes of representation. In order to use the new representation, it is necessary to introduce an integration with respect to the functional variable ψ^*. We shall take:

$$\int \{\chi[\psi^*]\}^* \chi'[\psi^*] d\psi^* = (\chi[\psi^*], \chi'[\psi^*]), \tag{18}$$

$\{\chi[\psi^*]\}^*$ must be considered as a functional of the function ψ, instead of ψ^*, in the same way as the complex conjugate of an analytic function $f(z)$ of the complex variable z is an analytic function of z^*. We shall write:

$$\chi^*[\psi] = \{[\psi^*]\}^* . \tag{19}$$

In our integration with respect to ψ^*, the integrated quantity is to be considered as a functional of both ψ and ψ^*, the integration being analogous to a contraction of an upper index with an equal lower index in the tensor calculus. Since

$$\Psi(x) = (\langle \psi^* | x \rangle, \chi[\psi^*]) , \tag{20}$$

we have:

$$\Psi(x) = \int \langle x | \psi \rangle \chi[\psi^*] d\psi^* , \tag{21}$$

$$\langle x | \psi \rangle = \{\langle \psi^* | x \rangle\}^* = \psi(x) . \tag{22}$$

We may say that the quantity (ψ) has the value ψ when the system Σ is in the state described by the wave function ψ, ψ being taken normalized. With this convention, the ordinary rule of interpretation of the transformation functions is still applicable: $|\langle \psi^* | x \rangle|^2 dx$ *is the probability of obtaining values of the* (x) *in the ranges* $x - x + dx$, *in the state in which the quantity* (ψ) *has the value* ψ.

In the representation in which the wave function of Σ becomes the linear functional $\chi[\psi^*]$, the physical quantity (A) of Σ is described by the operator $\mathcal{A}$

$$\mathcal{A} = \int_\Omega dx\, \psi^*(x) A \frac{\delta}{\delta \psi^*(x)} = \int_\Omega dx\, \psi^*_{op}(x) A \psi_{op}(x) , \tag{23}$$

since we must have

$$\mathcal{A} \chi[\psi^*] = \int_\Omega \langle \psi^* | x \rangle A \Psi(x) dx = \int_\Omega \psi^*(x) A \Psi(x) dx , \tag{24}$$

and it results from (23) that:

$$\mathcal{A} \int_\Omega \psi^*(x) \Psi(x) dx = \int_\Omega \psi^*(x) A \Psi(x) dx . \tag{25}$$

$\mathcal{A}$ is the well known operator for the quantity (A) in the second quantization. Let us introduce the projection operator on the direction of the normalized vector φ in the ψ-space:

(26) $$p_\varphi \psi(x) = \varphi(x) \int_\Omega \varphi^*(x')\psi(x')\, dx' .$$

The χ-operator corresponding to p_φ will be denoted by N_φ:

(27) $$N_\varphi = \int_\Omega \psi^*_{op}(x) p_\varphi \psi_{op}(x)\, dx = \int_\Omega \psi^*_{op}(x)\varphi(x)\, dx \int_\Omega \varphi^*(x')\psi_{op}(x')\, dx' .$$

We have

(27a) $$N_\varphi = a^*_\varphi a_\varphi , \qquad a_\varphi = \int_\Omega \varphi^*(x)\psi_{op}(x)\, dx ,$$

and since

(28) $$[a_\varphi, a^*_\varphi] = \int_\Omega \varphi^*(x)\varphi(x') [\psi_{op}(x), \psi^*_{op}(x')]\, dx\, dx' = \int_\Omega |\varphi|^2\, dx = 1 ,$$

the eigenvalues of N_φ are the non negative integers: 0, 1, 2,... . In the second quantization, N_φ is the operator for the number of systems Σ in the state φ. The definition of a_φ can be applied even to a non normalized φ and we have quite generally:

(29) $$[a_\varphi, a_{\varphi'}] = [a^*_\varphi, a^*_{\varphi'}] = 0 , \qquad [a_\varphi, a^*_{\varphi'}] = \int_\Omega \varphi^*(x)\varphi'(x)\, dx .$$

The simplest of the operators A is the unity operator of the ψ-space, the corresponding χ-operator will be denoted by N_{op}:

(30) $$N_{op} = \int_\Omega \psi^*_{op}(x)\psi_{op}(x)\, dx .$$

N_{op} is the operator for the total number of systems Σ in the second quantization. Let the φ_λ be a complete set of orthonormal functions and the p_λ the corresponding projection operators: $p_\lambda = p_{\varphi_\lambda}$. We shall denote the N_{φ_λ} by N_λ. Since

(31) $$\sum_\lambda p_\lambda = 1 ,$$

we have:

(32) $$N_{op} = \sum_\lambda N_\lambda .$$

It follows from the commutation rules (29) that the N_λ are commutable, therefore the eigenvalues of N_{op} are the non negative integers. The linear functionals χ are eigenfunctionals of N_{op} corresponding to the eigenvalue 1. More generally, the homogeneous functionals χ_n defined by (16) are eigenfunctionals of N_{op} corresponding to the eigenvalue n:

$$(33)\qquad N_{op}\chi_n[\psi^*] = \int_\Omega \mathrm{d}x\, \psi^*(x)\frac{\delta\chi_n[\psi^*]}{\delta\psi^*(x)} = n\chi_n[\psi^*]\,.$$

The operators $\mathcal{A}$ were derived from the A by the standard methods of the transformation theory, for linear functionals. In the case of non linear functionals, it is often convenient to describe the physical quantity (A) by an operator $\mathcal{A}_{nor}$ which we shall now define. Let us consider the projection operators of the eigenvalues of N_{op}:

$$(34)\qquad N_{op}P_n = nP_n\,,\qquad P_nP_{n'} = \delta_{n,n'}P_n\,,\qquad \sum_{n=0}^{\infty} P_n = 1\,.$$

The spectral decomposition of N_{op} is:

$$(35)\qquad N_{op} = \sum_{n=1}^{\infty} nP_n\,.$$

Let us introduce the operator L:

$$(36)\qquad L = \sum_{n=1}^{\infty} n^{-1}P_n\,,\qquad LN_{op} = N_{op}L = 1 - P_0\,.$$

The operator $\mathcal{A}_{nor}$ is:

$$(37)\qquad \mathcal{A}_{nor} = L\mathcal{A} = \mathcal{A}L\,,\qquad\qquad (\mathcal{A}P_n = P_n\mathcal{A})\,,$$

$$(38)\qquad \mathcal{A}\chi[\psi^*] = \int_\Omega \psi^*(x)A\frac{\delta\chi[\psi^*]}{\delta\psi^*(x)}\mathrm{d}x =$$

$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}\int_\Omega \psi^*(x_1)\ldots\psi^*(x_n)\{A_{x_1}+\ldots+A_{x_n}\}\Psi_n(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\,,$$

$$(39)\qquad \mathcal{A}_{nor}\,\chi[\psi^*] =$$

$$= \sum_{n=1}^{\infty}\frac{n^{-1}}{\sqrt{n!}}\int_\Omega \psi^*(x_1)\ldots\psi^*(x_n)\{A_{x_1}+\ldots+A_{x_n}\}\,\Psi_n(x_1,\ldots,x_n)\,\mathrm{d}x_1\ldots\mathrm{d}x_n\,.$$

It results from (38) and (39) that for any linear functional χ_1 we have $\mathcal{A}\chi_1 = \mathcal{A}_{nor}\chi_1$.

We have seen that in the χ-formalism the ψ behave as values of a physical quantity, even when only the linear functionals are considered. When the non linear functionals are taken into account, there are eigenfunctionals of the $\psi_{op}(x)$ corresponding to the eigenvalues $\Psi(x)$. Indeed, we have

$$(40) \qquad \psi_{op}(x)\,\Delta[\psi^*, \Psi] = \frac{\delta}{\delta\psi^*(x)} \exp\left[\left|\int_\Omega \psi^*(x)\Psi(x)\,\mathrm{d}x\right| = \Psi(x)\,\Delta[\psi^*, \Psi]\right],$$

Δ being the functional defined by (14). We have seen that, with the R density matrix interpretation of the χ-formalism, the functional $\Delta[\psi^*, \Psi]$ describes the pure state of Σ of wave function Ψ. We shall see in section 8 that, in the second quantization, $\Delta[\psi^*, \Psi]$ with a normalized Ψ describes a state of the quantized ψ-field in which the expectation value of N_{op} is 1, the expectation value of N_φ being $\left|\int_\Omega \varphi^*(x)\Psi(x)\,\mathrm{d}x\right|^2$ and that of any operator $\mathfrak{A}$ being $\int_\Omega \Psi^*(x)A\Psi(x)\,\mathrm{d}x$, i.e. the expectation value of A in the state Ψ of Σ. It follows from (14) and (18) that for any functional $\chi[\psi^*]$ we have:

$$(41) \qquad \chi[\psi^*] = \int \Delta[\psi^*, \Psi]\,\chi[\Psi^*]\,\mathrm{d}\Psi^* .$$

Equation (41) shows that any state χ of the quantized ψ-field can be obtained by the superposition of states $\Delta[\psi^*, \Psi]$ in which the $\psi_{op}(x)$ have the values Ψ.

The properties of the states of the quantized ψ-field described by the wave functionals $\Delta[\psi^*, \Psi]$ are discussed in section 8. The $\Delta[\psi^*, \Psi]$ with normalized Ψ do not describe one quantum state, because $\Delta[\psi^*, \Psi]$ is not an eigenfunctional of N_{op}, but the expectation values of both N_{op} and N_ψ are 1. In the field state $\Delta[\psi^*, \Psi]$, the probabilities of the eigenvalues of N_φ are given by a Poisson law with average value equal to the expectation value $\left|\int_\Omega \Psi^*\varphi\,\mathrm{d}x\right|^2$ of N_φ. This expectation value of N_φ for a normalized Ψ coincides with the probability of the value 1 of p_φ in the Σ-state Ψ. More generally, the expectation value of $\mathfrak{A}$ in the field state $\Delta[\psi^*, \Psi]$ coincides with the expectation value $\int_\Omega \Psi^* A\Psi\,\mathrm{d}x$ of the quantity (A) in the Σ-state Ψ, for a normalized Ψ. The probabilities of the eigenvalues of N_{op} are given by a Poisson law with an average value 1. *The above results show that the state of the quantized ψ-field in which the $\psi_{op}(x)$ have the values $\Psi(x)$, i.e. the field state $\Delta[\psi^*, \Psi]$, when Ψ is normalized has an average behaviour coinciding with that of the Σ-state Ψ or of the one quantum field state* $\int_\Omega \Psi(x)\psi^*(x)\mathrm{d}x$.

The χ-formalism for a single system Σ can be related to the theory of the quantized ψ-field. Thus the linear wave functional (2) coincides with the wave functional of the one quantum state of the ψ-field in which the quantum Σ is in the state Ψ. The relation is more complicated in the case of non linear functionals. It is discussed in section **9**. The R density operator interpretation of the χ-formalism amounts to consider only the field states without a vacuum part, i.e. field states described by $\chi[\psi^*]$ such that $\chi[0] = 0$, and to represent the physical quantities (A) by the field operators $\mathcal{A}_{\text{nor}}$. It is interesting to notice that the spectrum of $\mathcal{A}_{\text{nor}}$ is dense, with the possible exception of the eigenvalue 0 which may be isolated, and covers densely the interval between the minimum eigenvalue A'_{min} and the maximum eigenvalue A'_{max} of A. The interpretation of the χ-formalism by means of the density operators R amounts to apply the ψ-field formalism to the computation of the expectation values of the operators $\mathcal{A}_{\text{nor}}$. This interpretation is consistent, but does not seem to be the most fundamental, because different wave functionals describe the same mixed or pure Σ-state. It is shown in section **5** that any pure or mixed state of Σ can be described by a quadratic functional $\chi_2[\psi^*]$.

It seems likely that the χ-formalism for a single system Σ be a more complete theory than the ordinary form of the quantum mechanics. The different wave functionals which correspond to the same density operator R may describe different conditions of the system Σ which appear as identical in the degree of approximation in which the description given by the ordinary quantum mechanics is correct. We may think that the ordinary description corresponds to some kind of time average over very small intervals of time, and perhaps over small regions of space also. With a higher degree of accuracy than is now experimentally possible, the description given by the present quantum mechanics may appear as incomplete, different physical conditions being lumped into a single quantum state.

Although the χ-formalism be applicable to any system, it seems likely that its true significance will appear in the cases in which the systems Σ are the elementary particles of the present quantum mechanics. In a more refined description, the present elementary particles will probably appear as complex structures which have an average behaviour of a particle, but whose detailed behaviour in very short intervals of time may be rather different from that of a particle. The functionals $\Delta[\psi^*, \Psi]$ are obviously a more fundamental element of the χ-formalism than the linear functionals and, with a normalized Ψ, they describe quasi-one quantum states, which are quasi-particle states in the cases in which the system Σ is a particle of the ordinary quantum mechanics. The expectation values of the operators $\mathcal{A}$ in a state $\Delta[\psi^*, \Psi]$ coincide, with the expectation values of the ordinary quantum mechanics, when Ψ is normalized, but there is a considerable mean square deviation in a state $\Delta[\psi^*, \Psi]$. It seems reasonable to assume that the expectation values correspond to some kind of time average of the instantaneous values and that the fluctuations correspond to deviations from the particle picture. The large fluctuations corresponding to the Poisson distributions for the values of N_{op} and the N_φ are particularly remarkable. The spread of the values of the position variables would give rise to a kind of spatial extension of the point particles.

The dense spectrum of the $\mathcal{A}_{nor}$ is also a remarkable property. It may indicate that the quantization of the values of the physical quantities arises from the averaging which introduces the particles.

The representation of the second quantization in which the ψ^*_{op} are diagonal was already discussed by FOCK (3). In Fock's treatment $\psi^*(x)$ is expanded in a series of orthonormal functions φ^*_λ

$$\psi^*(x) = \sum_\lambda a^*_\lambda \varphi^*_\lambda , \tag{42}$$

and the wave functional $\chi[\psi^*]$ is considered as a function of a countable infinity of complex variables a^*. DIRAC (4) has developped the Fock treatment by introducing a definition of the inner product involving contour integration with respect to the complex variables a^*. It is more convenient to define the inner product of two functions $f_1(z_1,..., z_r)$ and $f_2(z_1,..., z_r)$ as follows:

$$(f_1(z_1,..., z_r), f_2(z_1,..., z_r)) = \tag{43}$$

$$= \pi^{-r} \int_{-\infty}^{+\infty} \{f_1(z_1,..., z_r)\}^* f_2(z_1,..., z_r) \exp\left[-\sum_{l=1}^{r} |z_l|^2\right] d\alpha_1 ... d\alpha_r d\beta_1 ... d\beta_r ,$$

$$\alpha_l = \mathcal{R}e z_l , \qquad \beta_l = \mathcal{I}m z_l . \tag{44}$$

The above definition of the inner product can be easily extended to functions of a countable infinity of complex variables; it leads to the definition (6) of the inner product of two functionals and to a more general definition of the $d\psi^*$ integration than (18). Indeed we get from (4):

$$\chi[\psi^*] = A_0 + \sum_{n=1}^{\infty} \sum_\lambda A_{\lambda_1,...,\lambda_n} a^*_{\lambda_1} ... a^*_{\lambda_n} , \tag{45}$$

$$A_0 = \chi[0] , \quad A_{\lambda_1,...,\lambda_n} = \frac{1}{\sqrt{n!}} \int_\Omega \Psi_n(x_1,..., x_n) \varphi^*_{\lambda_1}(x_1) ... \varphi^*_{\lambda_n}(x_n) dx_1 ... dx_n . \tag{46}$$

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(3) V. FOCK: *Zeits. f. Phys.*, **49**, 339 (1928); *Phys. Zeits. Sow. Un.*, **6**, 428 (1934).
(4) P. A. M. DIRAC: *Comm. Dublin Inst. f. Adv. Stud.*, **A**, **1** (1943).

By applying the definition (43) of the inner product extended to a countable infinity of complex variables we get:

$$(47)\qquad (\chi, \chi') = A_0^* A_0' + \sum_{n=1}^{\infty} n! \sum_{\lambda_1,\dots,\lambda_n} A^*_{\lambda_1,\dots,\lambda_n} A'_{\lambda_1,\dots,\lambda_n} = \\ = \chi^*[0]\chi'[0] + \sum_{n=1}^{\infty} \int_{\Omega} \Psi_n^*(x_1,\dots,x_n)\Psi_n'(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

In the ordinary treatment of the second quantization, the N_λ are diagonalized and the wave functional becomes a function of the eigenvalues N'_λ of the N_λ. In this representation the inner product of two functionals is taken as follows:

$$(48)\qquad (\chi(N'), \chi'(N')) = \sum_{N'} \chi^*(N')\chi'(N')\,.$$

We shall see in section 3 that this definition is also equivalent to (6).

The replacement of the wave function Ψ by a functional $\chi[\psi^*]$ corresponds to a generalization of the field model and can also be applied to other field theories, to the classical electromagnetic theory for instance. The χ-formalism is a technique for the treatment of non coherent fields, as well as of random fields, and it is not restricted to the quantum mechanics. The standard method of dealing with non coherent fields corresponds to the introduction of some kind of density matrix; the χ-formalism has the advantage of giving a direct generalization of the field quantities, instead of the bilinear combinations of field quantities of the density matrix method. A general discussion of the generalization of the field theories by means of functionals will be given elsewhere.

2. – The χ-formalism.

Equation (41) shows that the functional Δ is the analogue of the Dirac δ-function in the theory of the functionals χ:

$$(49)\qquad \Delta[\psi^*, \psi'] = \langle \psi^* | 1 | \psi' \rangle, \qquad \Delta[\psi^{*\prime}, \psi] = \{ \Delta[\psi^*, \psi'] \}^*.$$

It follows from (41) that:

$$(50)\qquad \int \Delta[\psi^*, \Psi] \Delta[\Psi^*, \psi'] \mathrm{d}\Psi^* = \Delta[\psi^*, \psi'].$$

Hence

$$(51)\qquad \int \Delta[\psi^*, \Psi] \Delta[\Psi^*, \psi] \mathrm{d}\Psi^* = \Delta[\psi^*, \psi] = \exp\left[\int_{\Omega} |\psi|^2 \mathrm{d}x\right],$$

so that $\Delta[\psi^* \Psi]$ is normalizable, although non normalized.

It results from (41) that

(52) $$\int \psi^*(x)\,\Delta[\psi^*, \Psi]\,\chi[\Psi^*]\,d\Psi^* = \psi^*(x)\chi[\psi^*] = \overset{*}{\psi}_{op}(x)\,\chi[\psi^*]\,,$$

and by taking into account (40) we get:

(53) $$\int \Delta[\psi^*, \Psi]\Psi(x)\chi[\Psi^*]\,d\Psi^* = \int\left\{\frac{\delta}{\delta\psi^*(x)}\,\Delta[\psi^*, \Psi]\right\}\chi[\Psi^*]\,d\Psi^* =$$
$$= \frac{\delta}{\delta\psi^*(x)}\int \Delta[\psi^*, \Psi]\,\chi[\Psi^*]\,d\Psi^* = \frac{\delta}{\delta\psi^*(x)}\,\chi[\psi^*] = \psi_{op}(x)\,\chi[\psi^*]\,.$$

It follows from (52) and (53) that:

(54) $$\langle\psi'^*\,|\,\overset{*}{\psi}_{op}(x)\,|\,\psi''(x)\rangle = \psi'^*(x)\,\Delta[\psi'^*(x), \psi''(x)]\,,$$

(55) $$\langle\psi'^*\,|\,\psi_{op}(x)\,|\,\psi''(x)\rangle = \psi''(x)\,\Delta[\psi'^*(x), \psi''(x)]\,.$$

Since

(56) $$\int\left\{\Delta[\psi^*, \Psi]\int_\Omega \psi^*(x)A\Psi(x)\,dx\right\}\chi[\Psi^*]\,d\Psi^* =$$

$$= \int_\Omega \overset{*}{\psi}_{op}(x)A\psi_{op}(x)\,dx\int \Delta[\psi^*, \Psi]\,\chi[\Psi^*]\,d\Psi^* = \int_\Omega \overset{*}{\psi}_{op}(x)A\psi_{op}(x)\,dx\,\chi[\psi^*] = \mathcal{A}\,\chi[\psi^*]\,,$$

we have:

(57) $$\langle\psi'^*\,|\,\mathcal{A}\,|\,\psi''\rangle = \Delta[\psi'^*, \psi'']\int_\Omega \psi'^*(x)\,A\psi''(x)\,dx\,.$$

Since

(58) $$\int \Delta[\psi'^*, \psi]\{\mathcal{A}_{\psi^*}\,\Delta[\psi^*, \psi'']\}\,d\psi^* = \mathcal{A}_{\psi'^*}\Delta[\psi'^*, \psi''] =$$

$$= \int_\Omega \psi'^*A\,\frac{\delta}{\delta\psi'^*(x)}\,\Delta[\psi'^*, \psi'']\,dx = \int_\Omega \psi'^*(x)A\psi''(x)\,dx\,\Delta[\psi'^*, \psi'']\,,$$

we have

(59) $$\langle\psi'^*\,|\,\mathcal{A}\,|\,\psi''\rangle = \int \Delta[\psi'^*, \psi]\{\mathcal{A}_{\psi^*}\,\Delta[\psi^*, \psi'']\}\,d\psi^*,$$

although $\Delta[\psi^*, \Psi]$ is not a normalized functional.

Let us introduce the unitary operator of evolution $\mathcal{U}(t)$ of the χ-formalism, i.e. the operator defined by the equations:

$$i\hbar\frac{d\mathcal{U}}{dt} = \mathcal{H}\mathcal{U}\,, \qquad \mathcal{U}(0) = 1\,. \tag{60}$$

We have:

$$\chi[t;\psi^*] = \mathcal{U}(t)\chi[0;\psi^*]\,. \tag{61}$$

The unitary operator of the motion of Σ is defined by the equations:

$$i\hbar\frac{dU}{dt} = HU\,, \qquad U(0) = 1\,. \tag{62}$$

The wave function of Σ at time t corresponding to the wave function $\psi(x)$ at the time 0 is $\psi(t;x)$:

$$\psi(t;x) = U(t)\psi(x)\,, \qquad \psi(0;x) = \psi(x)\,. \tag{63}$$

We have

$$i\hbar\frac{d}{dt}\Delta[\psi^*(x),\psi'(t;x)] = \Delta[\psi^*(x),\psi'(t;x)]\, i\hbar\frac{d}{dt}\int_\Omega \psi^*(x)\psi'(t;x)dx = \tag{64}$$
$$= \Delta[\psi^*(x),\psi'(t;x)]\int_\Omega \psi^*(x)H\psi'(t;x)\,dx = \mathcal{H}\Delta[\psi^*(x),\psi'(t;x)]\,,$$

hence

$$\Delta[\psi^*(x),\psi'(t;x)] = \mathcal{U}(t)\,\Delta[\psi^*,\psi']\,, \tag{65}$$

so that:

$$\langle\psi'^*|\mathcal{U}(t)|\psi''\rangle = \Delta[\psi'^*(x),\psi''(t;x)] = \exp\left[\int_\Omega \psi'^*(x)\,U(t)\psi''(x)\,dx\right]. \tag{66}$$

In the Heisenberg representation of the χ-formalism, the time independent operators $\mathcal{K}$ are replaced by the dependent ones $\mathcal{K}(t)$:

$$\mathcal{K}(t) = \mathcal{U}^{-1}(t)\,\mathcal{K}\,\mathcal{U}(t)\,, \qquad i\hbar\frac{d}{dt}\mathcal{K}(t) = [\mathcal{K}(t),\mathcal{H}]\,. \tag{67}$$

It is easily seen that $\psi_{op}(t;x)$ has an equation of motion of the same form as the Schrödinger equation (1):

$$i\hbar\frac{\partial}{\partial t}\psi_{op}(t;x) = H\psi_{op}(t;x)\,. \tag{68}$$

Hence we have:

$$\psi_{op}(t;x) = U(t)\psi_{op}(x)\,, \qquad \psi^*_{op}(t;x) = U^{-1}(t)\psi^*_{op}(x)\,. \tag{69}$$

The ψ_{op} taken at different times do commute, and the ψ^*_{op} also:

$$[\psi_{op}(t;x),\, \psi_{op}(t';x')] = 0\,, \qquad [\psi^*_{op}(t;x),\, \psi^*_{op}(t';x')] = 0\,. \tag{70}$$

Since

$$\begin{aligned}\langle\psi'^*|\psi_{op}(t;x)|\psi''\rangle &= U(t)\psi_{op}(x)\,\Delta[\psi'^*,\psi''] = \\ &= U(t)\psi''(x)\,\Delta[\psi'^*,\psi''] = \psi''(t;x)\,\Delta[\psi'^*,\psi'']\,,\end{aligned} \tag{71}$$

the matrix elements of $\psi_{op}(t;x)$ satisfy the Schrödinger equation (1). We get from (71):

$$\langle\psi'^*|\psi_{op}(t;x)|\psi'\rangle = \psi'(t;x)\exp\left[\int_\Omega |\psi'|^2\,dx\right]. \tag{72}$$

This equation shows that the wave function Ψ of Σ is the expectation value of $\psi_{op}(t;x)$ taken in the condition of the χ-formalism described by a functional $\Delta[\psi^*,\Psi]$:

$$\Psi(t;x) = (\Delta[\psi^*,\Psi]\;\psi_{op}(t;x)\Delta[\psi^*,\Psi])\exp\left[-\int_\Omega |\Psi|^2\,dx\right]. \tag{73}$$

This result follows of course immediately from the fact that $\psi_{op}(x)\Delta[\psi^*,\Psi] = \Psi(x)\,\Delta[\psi^*,\Psi]$.

Let us consider now the operator L defined by (36). We have

$$L\Delta[\psi^*,\Psi] = \sum_{n=1}^{\infty}\frac{n^{-1}}{n!}\left\{\int_\Omega \psi^*(x)\Psi(x)\,dx\right\}^n = F\left(\int_\Omega \psi^*\Psi\,dx\right), \tag{74}$$

$$F(u) = \int_0^u \frac{e^z-1}{z}\,dz = Ei(u) - \log u - C\,. \tag{75}$$

$Ei(u)$ being the exponential integral of u and C the Euler constant. We get from (74) and (37):

$$\langle\psi^*|\mathcal{A}_{nor}|\psi'\rangle = \frac{\int_\Omega \psi^*(x)A\psi'(x)\,dx}{\int_\Omega \psi^*(x)\psi'(x)\,dx}\,\{\Delta[\psi^*,\psi'] - 1\}\,, \tag{76}$$

$$\langle\psi^*|\mathcal{A}_{nor}|\psi\rangle = \int_\Omega \psi^*(x)A\psi(x)\,dx\;\frac{\exp\left[\int_\Omega |\psi|^2\,dx\right] - 1}{\int_\Omega |\psi|^2\,dx}\,. \tag{77}$$

3. – The N-representation of the χ-formalism.

We shall now consider the representation of the χ-formalism in which the operators N_λ for a complete orthonormal set of functions φ_λ are diagonal. Instead of a wave functional $\chi[\psi^*]$, we have now a wave function $\chi(N'_1, N'_2,...)$ depending on the eigenvalues of the N_λ. In this representation the inner product is defined by (48). The basic functions of this representation are the $\delta(N', n)$.

$$\delta(N', n) = \prod_\lambda \delta_{N'_\lambda, n_\lambda}, \tag{78}$$

the n being non negative integers, since they are eigenfunctions of the N_λ and:

$$\sum_{N'} \delta^*(N', n')\, \delta(N', n'') = \prod_\lambda \delta_{n'_\lambda, n''_\lambda}, \tag{79a}$$

$$\sum_{n} \delta^*(N', n)\, \delta(N'', n) = \prod_\lambda \delta_{N'_\lambda, N''_\lambda}. \tag{79b}$$

Let us introduce the functionals $\langle N' | \psi \rangle$ and $\langle \psi^* | N' \rangle$:

$$\langle N' | \psi \rangle = \prod_\lambda \left\{ (N'_\lambda!)^{-1/2} \left(\int_\Omega \varphi^*_\lambda(x) \psi(x)\, \mathrm{d}x \right)^{N'_\lambda} \right\}, \tag{80}$$

$$\langle \psi^* | N' \rangle = \{ \langle N' | \psi \rangle \}^*. \tag{81}$$

It is easily seen that:

$$\sum_{N'} \langle \psi^* | N' \rangle \langle N' | \psi' \rangle = \tag{82}$$

$$= \exp \left[\sum_\lambda \left(\int_\Omega \varphi^*_\lambda(x) \psi(x)\, \mathrm{d}x \right)^* \left(\int_\Omega \varphi^*_\lambda(x) \psi'(x)\, \mathrm{d}x \right) \right] = \exp \left[\int_\Omega \psi^*(x) \psi'(x)\, \mathrm{d}x \right] = \Delta[\psi^*, \psi'].$$

We have obviously:

$$\langle N' | \psi \rangle = \prod_\lambda \{ (N'_\lambda!)^{-1/2}\, a_\lambda^{N'_\lambda} \}, \tag{83}$$

the a_λ being the complex conjugates of the coefficients in the expansion (42). By taking into account (43) we get:

$$\int \langle N' | \psi \rangle \langle \psi^* | n \rangle\, \mathrm{d}\psi^* = \prod_\lambda \delta_{N'_\lambda, n_\lambda} = \delta(N', n). \tag{84}$$

We get from (27)

(85) $$N_\lambda = \int_\Omega \psi^*_{\text{op}}(x)\varphi_\lambda(x)\,dx \int_\Omega \varphi^*_\lambda(x')\psi_{\text{op}}(x')\,dx,$$

therefore the $\langle\psi^*|n\rangle$ are eigenfunctionals of the N_λ corresponding to eigenvalues n_λ:

(86) $$N_\lambda\langle\psi^*|n\rangle = n_\lambda\langle\psi^*|n\rangle .$$

In (86), the n in $\langle\psi^*|n\rangle$ are treated as indices, $\langle\psi^*|n\rangle$ being considered as a wave functional in the ψ^*-representation.

In the application of the χ-formalism to the second quantization, the wave functional $\chi[\psi^*]$ of a state of the quantized ψ-field is related to the wave function $\chi(N')$ of the same state by the equation:

(87) $$\chi[\psi^*] = \sum_{N'}\langle\psi^*|N'\rangle\chi(N') .$$

The geometrical meaning of (87) becomes clear by taking into account that $\langle\psi^*|N'\rangle$ is an eigenfunctional of the N_λ: $\chi[\psi^*]$ is the component of the vector representing the state of the ψ-field in the direction of the eigenvector of the $\psi^*_{\text{op}}(x)$ corresponding to the eigenvalues $\psi^*(x)$, $\langle\psi^*|N'\rangle$ the projection on the direction on that eigenvector of a normalized eigenvector of the N_λ corresponding to the eigenvalues N'_λ and $\chi(N')$ the projection of the vector of the state on the direction of this eigenvector of the N'. It follows from (87) and (84) that:

(88) $$\chi(N') = \int\langle N'|\psi\rangle\chi[\psi^*]\,d\psi^* .$$

This equation shows that $\langle N'|\Psi\rangle$ must be an eigenfunction of the operators $\psi^{(N)}_{\text{op}}(x)$ which correspond to the $\psi_{\text{op}}(x)$ in the N-representation:

(89) $$\psi^{(N)}_{\text{op}}(x)\langle N'|\Psi\rangle = \Psi(x)\langle N'|\Psi\rangle .$$

Equation (89) can be easily checked by using the well known expression of $\psi^{(N)}_{\text{op}}(x)$:

(90) $$\psi^{(N)}_{\text{op}}(x) = \sum_\lambda \varphi_\lambda(x)\sqrt{N_\lambda+1}\,\exp\left[\frac{\partial}{\partial N_\lambda}\right] .$$

It results from (87) and (81) that:

(91) $$\chi^*[\psi] = \sum_{N'}\chi^*(N')\langle N'|\psi\rangle .$$

Hence

$$\int \chi^*[\psi]\chi'[\psi^*]\,d\psi^* = \sum_{N',N''} \chi^*(N')\chi'(N'') \int \langle N'|\psi\rangle\langle\psi^*|N''\rangle\, d\psi^* = \\ = \sum_{N'} \chi^*(N')\chi'(N')\,, \tag{92}$$

and thus the equivalence of the inner product definitions (6) and (48) is proven.

Let us consider the spectral decomposition of the hermitian operator A with a discrete spectrum:

$$A = \sum_{A'}\sum_{\alpha} A' p_{A'}^{(\alpha)}\,, \tag{93}$$

$p_{A'}^{(\alpha)}$ denoting the projection operator on the direction of the normalized eigenfunction $\varphi_{A'}^{(\alpha)}(\alpha)$. It follows from (93) that the corresponding χ-operator $\mathcal{A}$ can be written as follows

$$\mathcal{A} = \sum_{A'}\sum_{\alpha=1}^{s_{A'}} A' N_{A'}^{(\alpha)}\,, \tag{94}$$

$s_{A'}$ being the number of linearly independent eigenfunctions of the eigenvalue A':

$$N_{A'}^{(\alpha)} = \int_{\Omega} \overset{*}{\psi}_{\text{op}}(x) p_{A'}^{(\alpha)} \psi_{\text{op}}(x)\, dx\,. \tag{95}$$

The eigenvalues of the $N_{A'}^{(\alpha)}$ are the non negative integers. Therefore the eigenvalues of $\mathcal{A}$ are all the linear combinations $\sum_{A',\alpha} n_{A'}^{(\alpha)} A'$ with integer non negative coefficients $n_{A'}^{(\alpha)}$. It follows from (94), (37) and (36) that:

$$\mathcal{A}_{\text{nor}} = \sum_{A',\alpha} A' \sum_{n=1}^{\infty} n^{-1} P_n N_{A'}^{(\alpha)}\,. \tag{96}$$

The eigenvalues of $P_n N_{A'}^{(\alpha)}$ are $0, 1, 2, \dots, n$. Therefore the eigenvalues or $\mathcal{A}_{\text{nor}}$ are linear combinations $\sum_{A',\alpha} r_{A'}^{(\alpha)} A'$, the $r_{A'}^{(\alpha)}$ being non negative rational numbers smaller than 1. The $r_{A'}^{(\alpha)}$ cannot be chosen arbitrarily, because $P_n P_{n'} = 0$, for $n \neq n'$. We can take

$$r_{A'}^{(\alpha)} = l_{A'}^{(\alpha)} \Big/ \sum_{A'',\alpha'} l_{A''}^{(\alpha')} \quad \text{when} \quad \sum_{A',\alpha} l_{A'}^{(\alpha)} \neq 0\,; \qquad r_{A'}^{(\alpha)} = 0 \quad \text{when} \quad \sum_{A',\alpha} l_{A'}^{(\alpha)} = 0\,, \tag{97}$$

the $l_{A'}^{(\alpha)}$ being non negative integers. *The eigenvalues of* $\mathcal{A}_{\text{nor}}$ *cover densely the interval* $A'_{\text{min}} - A'_{\text{max}}$ *, the only eigenvalue of* $\mathcal{A}_{\text{nor}}$ *lying outside that interval being possibly* 0, *in case* 0 *does not belong to that interval.*

It is easily seen that:

$$\int \chi^*[\psi]\,\Delta[\psi^*, \psi']\,d\psi^* = \chi^*[\psi'] . \tag{98}$$

It follows from (88) and (98) that $\langle N'|\Psi\rangle$ is the wave function of the N-representation which corresponds to $\Delta[\psi^*, \Psi]$:

$$\langle N'|\Psi\rangle = \int \langle N'|\psi\rangle\,\Delta[\psi^*, \Psi]\,d\psi^* , \tag{99}$$

$\langle\psi^*|n\rangle$ is the wave functional which corresponds to $\delta(N', n)$:

$$\langle\psi^*|n\rangle = \sum_{A'} \langle\psi^*|N'\rangle\,\delta(N', n) . \tag{100}$$

We shall denote by $\chi[N'; \Psi]$ the wave function which corresponds to the functional (2):

$$\chi[N'; \Psi] = \int \langle N'|\psi\rangle \left\{ \int_\Omega \psi^*(x)\Psi(x)\,dx \right\} d\psi^* = \tag{101}$$

$$= \sum_\lambda \int_\Omega \varphi_\lambda^*(x)\Psi(x)\,dx\,\delta_{N'_\lambda, 1} \prod_{\mu\neq\lambda} \delta_{N'_\mu, 0} .$$

By taking into account that

$$\psi_{\text{op}}^{*(N)}(x) = \sum_\lambda \varphi_\lambda^*(x)\sqrt{N_\lambda}\,\exp\left[-\frac{\partial}{\partial N_\lambda}\right] , \tag{102}$$

we get:

$$\chi[N'; \Psi] = \int_\Omega \psi_{\text{op}}^{*(N)}(x)\Psi(x)\,dx\,\delta(N', 0) . \tag{103}$$

We shall denote by $\chi[N'; \Psi_n(x_1, \ldots, x_n)]$ the N' wave function corresponding to the homogeneous functional (16):

$$\chi[N'; \Psi_n(x_1, \ldots, x_n)] = \tag{104}$$

$$= \frac{1}{\sqrt{n!}} \int \langle N'|\psi\rangle \left\{ \int_\Omega \Psi_n(x_1, \ldots, x_n)\psi^*(x_1) \ldots \psi^*(x_n)\,dx_1 \ldots dx_n \right\} d\psi^* .$$

The ψ^* integration can be easily performed by applying (45) to the homogeneous functional and using the expression (83) of $\langle N'|\psi\rangle$:

$$\chi[N'; \Psi_n(x_1, \ldots, x_n)] = \frac{1}{\sqrt{n!}} \sum_{\lambda_1, \ldots, \lambda_n} \int_\Omega \Psi_n(x_1, \ldots, x_n)\varphi_{\lambda_1}^*(x_1) \ldots \varphi_{\lambda_n}^*(x_n)\,dx_1 \ldots dx_n \cdot \tag{105}$$

$$\cdot \sqrt{s_1!\, s_2! \ldots s_r!}\;\delta_{N'_{\mu_1}, s_1}\,\delta_{N'_{\mu_2}, s_2} \ldots \delta_{N'_{\mu_r}, s_r} \prod_{\lambda\neq\mu} \delta_{N'_\lambda, 0} ,$$

$(\mu_1, \mu_2, ..., \mu_n)$ denote the distinct numbers λ in the sequence $\lambda_1, ..., \lambda_n$ and the s the frequency in that sequence of the corresponding μ.

4. – The general occupation number operators N_M.

Until now we considered only operators N_φ for a state φ of Σ and the operator N_{op}. Let M be any closed linear manifold of the Hilbert space of the ψ and p_M the corresponding projection operator. p_M transforms any vector ψ into its projection on M. The occupation number operator N_M of M is defined as the χ-operator corresponding to p_M:

$$N_M = \int_\Omega \overset{*}{\psi}_{\text{op}}(x) p_M \psi_{\text{op}}(x)\, \mathrm{d}x\,. \tag{106}$$

Let the $\psi_M^{(\varrho)}(x)$ be a complete orthonormal set of vectors for M. Since the projection of a vector on M is the sum of its projections on the directions of the $\psi_M^{(\varrho)}$, the projection operator p_M is the sum of the projection operators $p_M^{(\varrho)}$ on the directions of the vectors $\psi_M^{(\varrho)}$:

$$p_M = \sum_\varrho p_M^{(\varrho)}\,. \tag{107}$$

Hence

$$N_M = \sum_\varrho N_M^{(\varrho)}\,, \qquad N_M^{(\varrho)} = N_{\psi_M^{(\varrho)}}\,;\,, \tag{108}$$

so that the occupation number operator N_M of the manifold M is the sum of the occupation number operators of any complete orthonormal set of vectors of M. N_{op} is simply the occupation number operator of the entire ψ Hilbert space.

To each discrete eigenvalue A' of the operator of a physical quantity (A) corresponds a closed linear manifold $M_{A'}$, whose basic vectors are those corresponding to a complete orthonormal set $\psi_{A'}^{(\alpha)}$ for the eigenvalue A'. We shall denote the projection operator of $M_{A'}$ by $p_{A'}$ and the corresponding occupation number operator by $N_{A'}$. Equations (93) and (94) can be written as follows:

$$A = \sum_{A'} A' p_{A'}\,, \tag{109}$$

$$\mathcal{A} = \sum_{A'} A' N_{A'}\,. \tag{110}$$

The general form of the spectral decomposition of a hermitian operator is:

$$A = \int_{-\infty}^{+\infty} \lambda \, dE_A(\lambda) \,. \tag{111}$$

The $E_A(\lambda)$ are projection operators on manifolds $\mathcal{M}_A(\lambda)$. They satisfy the following equations:

$$E_A(-\infty) = 0 \,, \qquad E_A(\infty) = 1 \,, \qquad E_A(\lambda + 0) = E_A(\lambda) \,, \tag{112}$$

$$E_A(\lambda')E_A(\lambda'') = E_A(\lambda'')E_A(\lambda') = E_A(\lambda') \,, \qquad (\lambda' \leqslant \lambda'') \,. \tag{113}$$

In the case of a pure point spectrum we can take:

$$E_A(\lambda) = \sum_{A' \leqslant \lambda} p_{A'} = \sum_{A'} \theta(\lambda - A') p_{A'} = \theta(\lambda - A) \,, \tag{114}$$

$$\theta(\lambda) = \begin{cases} 1 & \text{for} \quad \lambda \geqslant 0 \,, \\ 0 & \text{for} \quad \lambda < 0 \,. \end{cases} \tag{115}$$

The occupation number operator of $\mathcal{M}_A(\lambda)$ will be denoted by $N_A(\lambda)$:

$$N_A(\lambda) = \int_{\Omega} \overset{*}{\psi}_{\text{op}}(x) E_A(\lambda) \psi_{\text{op}}(x) \, dx \,. \tag{116}$$

Equation (110) can be generalized as follows:

$$\mathcal{A} = \int_{-\infty}^{+\infty} \lambda \, dN_A(\lambda) \,. \tag{117}$$

$E_A(\lambda'') - E_A(\lambda')$, $\lambda' < \lambda''$, is the projection operator of a closed linear manifold $\mathcal{M}_A(\lambda', \lambda'')$. In the case of a discrete spectrum $\mathcal{M}_A(\lambda', \lambda'')$ is the linear manifold whose basic vectors are the $\psi_{A'}^{(\alpha)}$ for all the eigenvalues of A in the semiclosed interval $(\lambda', \lambda'']$, i.e. for $\lambda' < A' \leqslant \lambda''$.

A family of projection operators satisfying the conditions (112) and (113) is called a spectral series. To any spectral series corresponds a set of operators $N(\lambda)$ and there is a representation in which all the $N(\lambda)$ are diagonal. In this representation the wave function is actually a wave functional $\chi[N'(\lambda)]$, depending on a non decreasing function $N'(\lambda)$ with non negative integer values.

p_M is a non normalized density operator:

$$\langle x' | p_M | x'' \rangle = \sum_{\varrho} \langle x' | p_M^{(\varrho)} | x'' \rangle = \sum_{\varrho} \psi_M^{(\varrho)}(x') \psi_M^{(\varrho)*}(x'') \,, \tag{118}$$

$$\text{Trace } p_M = \sum_{\varrho} \delta_{\varrho,\varrho} \,. \tag{119}$$

The states $\psi_M^{(\varrho)}$ are equally probable in the mixture described by p_M. By assigning different weights W_ϱ to the basic vectors of M we get a general density operator R_M attached to the manifold M:

$$R_M = \sum_\varrho W_\varrho p_M^{(\varrho)}\,, \qquad W_\varrho \geqslant 0\,. \tag{120}$$

There is no restriction of generality in considering only density operators attached to the entire Hilbert space, since it suffices to give weights 0 to the basic vectors of the manifold orthogonal to M, in case M is not empty. Thus we shall take

$$R = \sum_\lambda W_\lambda p_\lambda\,, \qquad W_\lambda \geqslant 0 \tag{121}$$

the p_λ being the projection operators of the functions of a complete orthonormal set φ_λ.

The equation of evolution of the density operator $R(t)$ describing a mixed state of motion of Σ is the von Neumann equation (12). Let us introduce the χ-operator $N_{R(t)}$:

$$N_{R(t)} = \int_\Omega \psi_{\text{op}}^*(x) R(t) \psi_{\text{op}}(x)\, \mathrm{d}x\,. \tag{122}$$

We have

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} N_{R(t)} = \int_\Omega \psi_{\text{op}}^*(x) \left[H, R(t)\right] \psi_{\text{op}}(x)\, \mathrm{d}x = [\mathcal{H}, N_{R(t)}]\,, \tag{123}$$

as a consequence of the equation:

$$\int_\Omega \psi_{\text{op}}^*(x)[A, B]\psi_{\text{op}}(x)\, \mathrm{d}x = [\mathcal{A}, \mathcal{B}]\,. \tag{124}$$

Equation (124) follows immediately from the following equation

$$\begin{aligned} &\int_\Omega \psi_{\text{op}}^*(x) A_x \psi_{\text{op}}(x)\, \mathrm{d}x \int_\Omega \psi_{\text{op}}^*(x') B_{x'} \psi_{\text{op}}(x')\, \mathrm{d}x' = \\ &\quad = \int_\Omega \psi_{\text{op}}^*(x) \psi_{\text{op}}^*(x') A_x B_{x'} \psi_{\text{op}}(x') \psi_{\text{op}}(x)\, \mathrm{d}x\, \mathrm{d}x' + \int_\Omega \psi_{\text{op}}^*(x) A_x B_x \psi_{\text{op}}(x)\, \mathrm{d}x\,, \end{aligned} \tag{125}$$

the index x denoting the variable on which the operator acts, by taking into account that:

$$[A_x, B_{x'}] = 0\,, \tag{126}$$

$N_{R(t)}$ is a density operator of the χ-formalism, because it is hermitian, with positive eigenvalues, and its equation of evolution (123) has the form of a von Neumann equation of the χ-formalism:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{R}(t) = [\mathcal{H}, \mathcal{R}(t)] = \mathcal{H}\mathcal{R}(t) - \{\mathcal{H}\mathcal{R}(t)\}^* . \tag{127}$$

We get from (127):

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle = \langle \psi'^* | \mathcal{H}\mathcal{R}(t) | \psi'' \rangle - \langle \psi''^* | \mathcal{H}\mathcal{R}(t) | \psi' \rangle^* . \tag{128}$$

Since

$$\langle \psi'^* | \mathcal{H}\mathcal{R}(t) | \psi'' \rangle = \{\mathcal{H}\mathcal{R}(t)\}_{\psi'^*} \Delta[\psi'^*, \psi''] = \mathcal{H}_{\psi'^*} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle = \tag{129}$$
$$= \int_\Omega \psi'^*(x) H \frac{\delta}{\delta \psi'^*(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle \, \mathrm{d}x ,$$

we have:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle = \tag{130}$$
$$= \int_\Omega \left\{ \psi'^*(x) H \frac{\delta}{\delta \psi'^*(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle - \left\{ \frac{\delta}{\delta \psi''(x)} \langle \psi'^* | \mathcal{R}(t) | \psi'' \rangle \right\} H \psi''(x) \right\} \mathrm{d}x .$$

The density operators $\mathcal{R}$ describe mixtures of states of the quantized ψ-field. They can also be applied to the one system χ-formalism as will be shown in a following paper.

5. – The R density operators of the χ-formalism.

It is easily seen that the density operator $R(t)$ defined by (11) can be expressed as follows:

$$\langle x' | R(t) | x'' \rangle = (\chi[t; \psi^*], \psi^*_{\mathrm{op}}(x'') \psi_{\mathrm{op}}(x') L \chi[t; \psi^*]) . \tag{131}$$

It results from (61) that

$$\langle x' | R(t) | x'' \rangle = (\chi[0; \psi^*] , \mathcal{U}^{-1}(t) \psi^*_{\mathrm{op}}(x'') \psi_{\mathrm{op}}(x') \mathcal{U}(t) L \chi[0; \psi^*]) , \tag{132}$$

since

$$[L, \mathcal{U}(t)] = 0 , \tag{133}$$

as a consequence of the commutability of $\mathscr{H}$ and the projection operators P_n of N_{op}. By taking into account (67) and (69) we get from (132):

$$(134)\qquad \langle x'|R(t)|x''\rangle = (\chi[0;\psi^*],\, \psi^*_{op}(t;x'')\psi_{op}(t;x')L\chi[0;\psi^*]) =$$
$$= (\psi_{op}(t;x'')\chi[0;\psi^*],\, \psi_{op}(t;x')L\chi[0;\psi^*]) =$$
$$= (U(t)\psi_{op}(x'')\chi[0;\psi^*],\, U(t)\psi_{op}(x')L\chi[0;\psi^*])\,.$$

It follows from (134) and (62) that

$$(135)\qquad i\hbar\frac{\partial}{\partial t}\langle x'|R(t)|x''\rangle = (-H_{x''}\psi_{op}(t;x'')\chi[0;\psi^*],\, \psi_{op}(t;x')L\chi[0;\psi^*]) +$$
$$+(\psi_{op}(t;x'')\chi[0;\psi^*],\, H_{x'}\psi_{op}(t;x')L\chi[0;\psi^*]) =$$
$$= (\chi[0;\psi^*],\, \psi^*_{op}(t;x'')H_{x'}\psi_{op}(t;x')L\chi[0;\psi^*]) -$$
$$- (L\psi^*_{op}(t;x')H_{x''}\psi_{op}(t;x'')\chi[0;\psi^*],\, \chi[0;\psi^*])\,,$$

and we get finally:

$$(136)\qquad i\hbar\frac{\partial}{\partial t}\langle x'|R(t)|x''\rangle = H_{x'}\langle x'|R(t)|x''\rangle - \{H_{x''}\langle x''|R(t)|x'\rangle\}^*\,.$$

Equation (136) is equivalent to (12). $R(t)$ is a density operator, because it is hermitian and has only positive discrete eigenvalues, since Trace $R(t) < \infty$ for a normalizable χ, and

$$(137)\qquad \int_\Omega \Psi^*(x)R(t)\Psi(x)\,\mathrm{d}x = \int_\Omega \Psi^*(x')\langle x'|R(t)|x''\rangle\,\Psi(x'')\,\mathrm{d}x'\,\mathrm{d}x'' =$$
$$= \sum_{n=1}^\infty \int_\Omega \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}\left|\int_\Omega \Psi^*(x)\,\Psi_n(t;x,x_1,\ldots,x_{n-1})\,\mathrm{d}x\right|^2 \geqslant 0\,,$$

so that $R(t)$ is a satisfactory density operator for a system Σ.

It is interesting to remark that we can get any density operator of the form (121) from a quadratic functional. Indeed, by taking

$$(138)\qquad \Psi_2(x_1,x_2) = \sum_\lambda \sqrt{W_\lambda}\,\varphi_\lambda(x_1)\varphi_\lambda(x_2)\,, \qquad \Psi_n = 0 \quad \text{for} \quad n \neq 2\,,$$

we get:

$$(139)\qquad \langle x'|R|x''\rangle = \sum_\lambda W_\lambda\varphi_\lambda(x')\varphi^*_\lambda(x'')\,.$$

We have

$$\text{Trace } R = (\chi, \chi) - |\Psi_0|^2 , \tag{140}$$

thereby R will in general not be a normalized density operator, for a normalized χ, but it will be normalized in the important case of $\Psi_0 = 0$.

It is well known that the expectation value of a physical quantity (A) in the mixed state of Σ described by the normalized density operator R is the Trace of AR. We have

$$\begin{aligned}(\chi, \mathcal{A}_{\text{nor}}\chi) &= \Big(\chi, \int_\Omega \psi^*_{\text{op}}(x) A \psi_{\text{op}}(x)\, \mathrm{d}x\, L\chi\Big) = \\ &= \Big(\chi, \int_\Omega \psi^*_{\text{op}}(x')\langle x'|A|x''\rangle \psi_{\text{op}}(x'')\, \mathrm{d}x'\, \mathrm{d}x''\, L\chi\Big) = \\ &= \int_\Omega \mathrm{d}x'\, \mathrm{d}x'' \langle x'|A|x''\rangle (\chi, \psi^*_{\text{op}}(x')\psi_{\text{op}}(x'') L\chi) = \\ &= \int_\Omega \mathrm{d}x'\, \mathrm{d}x'' \langle x'|A|x''\rangle\langle x''|R|x'\rangle ,\end{aligned} \tag{141}$$

hence:

$$(\chi, \mathcal{A}_{\text{nor}}\chi) = \text{Trace } (AR) . \tag{142}$$

Thereby the expectation value of (A) in the mixed state described by R is given by the χ expectation value $(\chi, \mathcal{A}_{\text{nor}}\chi)$ divided by $1 - |\Psi_0|^2$, for a normalized χ.

Let us consider now a wave functional $\chi(t)$ in whose Volterra expansion (4) there are terms with $n \geqslant r$, r being a positive integer. With that functional $\chi(t)$ we can build a r-system density operator R_r:

$$\begin{aligned}&\langle x'_1, \dots, x'_r | R_r(t) | x''_1, \dots, x''_r\rangle = \\ &= (\chi[t;\psi^*], \psi^*_{\text{op}}(x''_1) \dots \psi^*_{\text{op}}(x''_r)\psi_{\text{op}}(x'_r) \dots \psi_{\text{op}}(x'_1)\chi[t;\psi^*]) = \\ &= (\chi[0;\psi^*], \psi^*_{\text{op}}(t;x''_1) \dots \psi^*_{\text{op}}(t;x''_r)\psi_{\text{op}}(t;x'_r) \dots \psi_{\text{op}}(t;x'_1)\chi[0;\psi^*]) .\end{aligned} \tag{143}$$

We may write

$$\begin{aligned}&\langle x'_1, \dots, x'_r | R_r(t) | x''_1, \dots, x''_r\rangle = \\ &= (\psi_{\text{op}}(t;x''_1) \dots \psi_{\text{op}}(t;x''_r)\chi[0;\psi^*], \psi_{\text{op}}(t;x'_1) \dots \psi_{\text{op}}(t;x'_r)\chi[0;\psi^*]) = \\ &= (U(t)_{x''_1} \dots U(t)_{x''_r}\psi_{\text{op}}(x''_1) \dots \psi_{\text{op}}(x''_r)\chi[0;\psi^*], U(t)_{x'_1} \dots U(t)_{x'_r}\psi_{\text{op}}(x'_1) \dots \psi_{\text{op}}(x'_r)\chi[0;\psi^*]) ,\end{aligned} \tag{144}$$

therefore:

$$(145)\qquad i\hbar\frac{\partial}{\partial t}\langle x'_1,\dots,x'_r|R_r(t)|x''_1,\dots,x''_r\rangle =$$
$$= (H_{x'_1}+\dots+H_{x'_r})\langle x'_1,\dots,x'_1|R_r(t)|x''_1,\dots,x'_r\rangle -$$
$$-\{(H_{x''_1}+\dots+H_{x''_r})\langle x''_1,\dots,x''_r|R_r(t)|x'_1,\dots,x'_r\rangle\}^* .$$

This equation shows that the hermitian operator $R_r(t)$ is a solution of the von Neumann equation for an assembly of r non interacting systems Σ. By a reasoning similar to that applied to $R(t)$, it is seen that the eigenvalues of $R_r(t)$ are non negative. $R_r(t)$ is therefore a density operator for an assembly of r non interacting systems Σ.

It is interesting to remark that with a functional $\chi[t, \psi^*]$ it is sometimes possible to build a solution of the Schrödinger equation (1):

$$(146)\qquad \Psi(t;x) = (\chi[t;\psi^*], \psi_{\text{op}}(x)\chi[t;\psi^*]) = (\chi[0;\psi^*], \psi_{\text{op}}(t;x)\chi[0;\psi^*]) .$$

Any density operator R of the system Σ can be obtained in an infinite number of different ways from functionals χ. In order to obtain the density operator of (139) by means of a homogeneous functional of the type (16), it suffices to take

$$(147)\qquad \Psi_n(x_1,\dots,x_n) = \sum_\lambda \sqrt{W_\lambda}\,\varphi_\lambda(x_1)\dots\varphi_\lambda(x_n) ,$$

since the operator $R^{(n)}$ associated to a χ_n has the matrix elements:

$$(148)\qquad \langle x'|R^{(n)}|x''\rangle = \int_\Omega \Psi_n(x', x_1,\dots,x_{n-1})\Psi_n^*(x'', x_1,\dots,x_{n-1})\,dx\dots dx_{n-1} .$$

In particular the homogeneous functional χ_n whose Ψ_n is

$$(149)\qquad \Psi_n(x_1,\dots,x_n) = \varphi(x_1)\dots\varphi(x_n) ,$$

corresponds to a density operator describing the pure state of wave function φ.

The density operator corresponding to the functional whose Ψ_n are

$$(150)\qquad \Psi_0 = 0 , \quad \Psi_n(x_1,\dots,x_n) = \sqrt{W_n}\,\varphi_n(x_1)\dots\varphi_n(x_n) \quad \text{for} \quad n \geqslant 1 ,$$

is the operator (139). It is interesting to remark that each term in the expansion of the type (4) of the functional defined by (150) has a density matrix describing a pure state.

In the particular case of normalized functionals χ with $\Psi_0 = 0$, the corresponding operators R have trace 1. Thereby Trace (AR) is the expectation value of the quantity (A) in the mixed state described by R. By taking into account (142), we see that the field expectation value $(\chi, \mathcal{A}_{\text{nor}}\chi)$ of $\mathcal{A}_{\text{nor}}$ coincides with that expectation value of (A). Since the nor χ-operator which corresponds to the projection operator $p_{A'}$, is $LN_{A'}$, as shown by (106), the probability of obtaining the value A' in a measurement of (A) in the mixed state described by the R corresponding to χ is $(\chi, LN_{A'}\chi)$:

$$\text{Trace}\,(p_{A'}R) = (\chi, LN_{A'}\chi)\,, \qquad ((\chi, \chi) = 1,\; \Psi_0 = 0)\,. \tag{151}$$

6. – Description of mixtures of states of Σ by functionals χ.

The results of section 5 show that it is possible to describe the mixtures of states of Σ, as well as the pure states, by means of functionals χ. It suffices to take the R operators attached to χ as describing the mixtures and pure states of Σ. In this description of the states of Σ, it is convenient to consider only normalized functionals with $\Psi_0 = 0$. With this choice of the χ the physical quantity will be described by the operator $\mathcal{A}_{\text{nor}}$, provided the spectral decomposition of $\mathcal{A}_{\text{nor}}$ be replaced by the following decomposition:

$$\mathcal{A}_{\text{nor}} = \sum_{A'} A' LN_{A'}\,. \tag{152}$$

Equation (152) is a consequence of (110). Thus the probability of obtaining the value A' of the quantity (A) in the mixed state described by χ is given by the expectation value $(\chi, LN_{A'}\chi)$, as was shown in section 5. This corresponds to the ordinary rule for the computation of the probability of obtaining the value A' of (A) in the pure state of Σ described by the normalized wave function Ψ, that probability being the expectation value $\int_\Omega \Psi^* p_{A'} \Psi \, dx$ of $p_{A'}$. Now both the pure and mixed states of Σ are described by functionals χ i.e. correspond to pure states of the χ-formalism, and the projection operator $p_{A'}$ is replaced by $LN_{A'}$.

The above interpretation of the χ-formalism by means of the functionals R corresponds to a probability distribution in the ψ-space, in which probabilities are assigned to linear manifolds. Indeed, let us assign to the linear manifolds M the probabilities $\mathcal{P}[M]$:

$$\mathcal{P}[M] = (\chi, LN_M\chi)\,, \qquad ((\chi, \chi) = 1,\; \Psi_0 = 0)\,. \tag{153}$$

Let M_1 and M_2 be two orthogonal linear manifolds. Since

$$p_{M_1+M_2} = p_{M_1} + p_{M_2}\,, \tag{154}$$

we have:

$$N_{M_1+M_2} = N_{M_1} + N_{M_2} . \tag{155}$$

This equation shows that two orthogonal manifolds correspond to exclusive events:

$$\mathcal{P}[M_1 + M_2] = \mathcal{P}[M_1] + \mathcal{P}[M_2] . \tag{156}$$

The probability of the total Hilbert space S is 1

$$\mathcal{P}[S] = (\chi, LN_{\text{op}}\chi) = 1 , \tag{157}$$

since $LN_{\text{op}}\chi = \chi$, because $\Psi_0 = 0$. We get the above rule of interpretation by assuming that the probability of obtaining a value A' of the quantity (A) in a measurement performed in the mixed state described by χ coincides with the probability of the manifold $M_{A'}$ in the probability distribution associated to χ.

In the case of the general spectrum we must introduce the linear manifolds $\mathcal{M}_A(\lambda', \lambda'')$ of section 4. The probability of $\mathcal{M}_A(\lambda', \lambda'')$

$$\mathcal{P}[\mathcal{M}_A(\lambda', \lambda'')] = (\chi, L\{N_A(\lambda'') - N_A(\lambda')\}\chi) , \tag{158}$$

coincides with that of obtaining a value λ of (A) in the semi-closed interval $(\lambda', \lambda'']$, $\lambda' < \lambda \leqslant \lambda''$.

Let us consider the particular case of a linear functional (2). The density operator R describes now a pure state, as shown by (13), precisely the pure state Ψ. The probability of the ray defined by the normalized vector φ is $|\chi[\varphi^*]|^2$:

$$(\chi, LN_\varphi\chi) = (\chi, N_\varphi\chi) = \left|\int_\Omega \Psi(x)\varphi^*(x)\,\mathrm{d}x\right|^2 = |\chi[\varphi^*]|^2 . \tag{159}$$

Therefore in the case of a linear functional the probability of the ray φ coincides with the probability of the normalized function φ according to the probability distribution of the wave functions corresponding to $\chi[\psi^*]$. In this case $|\chi[\psi^*]|^2$ does not depend on the choice of the constant phase factor in the normalization of φ.

We shall now introduce a functional $\bar{\Delta}[\psi^*, \Psi]$ which differs slightly from $\Delta[\psi^*, \Psi]$

$$\bar{\Delta}[\psi^*, \Psi] = (1 - P_0)\, \Delta[\psi^*, \Psi] = \Delta[\psi^*, \Psi] - 1 , \tag{160}$$

$$\bar{\Delta}[\psi^*, \psi'] = \langle \psi^* | 1 - P_0 | \psi' \rangle = \left\langle \psi^* \middle| \sum_{n=1}^{\infty} P_n \middle| \psi' \right\rangle . \tag{161}$$

We have obviously

$$\int \bar{\Delta}[\psi^*, \psi']\chi[\psi'^*]\,\mathrm{d}\psi'^* = \chi[\psi^*] - \Psi_0\,, \tag{162}$$

in particular

$$(\bar{\Delta}[\psi^*, \Psi], \bar{\Delta}[\psi^*, \Psi]) = \bar{\Delta}[\Psi, \Psi] = \exp\left[\int_\Omega |\Psi|^2\,\mathrm{d}x\right] - 1\,. \tag{163}$$

It is easily seen that:

$$\psi_{\text{op}}(x)\,\bar{\Delta}[\psi^*, \Psi] = \psi_{\text{op}}(x)\,\Delta[\psi^*, \Psi] = \Psi(x)\{\bar{\Delta}[\psi^*, \Psi] + 1\}\,, \tag{164}$$

$$\mathcal{A}\bar{\Delta}[\psi^*, \Psi] = \mathcal{A}\Delta[\psi^*, \Psi] = \Delta[\psi^*, \Psi]\int_\Omega \psi^* A \Psi\,\mathrm{d}x\,, \tag{165}$$

$$L\bar{\Delta}[\psi^*, \Psi] = L\Delta[\psi^*, \Psi]\,, \tag{166}$$

$$\mathcal{A}_{\text{nor}}\bar{\Delta}[\psi^*, \Psi] = \mathcal{A}L\bar{\Delta}[\psi^*, \Psi] = \mathcal{A}_{\text{nor}}\Delta[\psi^*, \Psi]\,. \tag{167}$$

It follows from (167) and (76) that:

$$\mathcal{A}_{\text{nor}}\bar{\Delta}[\psi^*, \psi'] = \langle \psi^* | \mathcal{A}_{\text{nor}} | \psi' \rangle = \bar{\Delta}[\psi^*, \psi'] \frac{\int_\Omega \psi^* A \psi'\,\mathrm{d}x}{\int_\Omega \psi^* \psi'\,\mathrm{d}x}\,, \tag{168}$$

In particular we have:

$$\mathcal{A}_{\text{nor}}\bar{\Delta}[\psi^*, \psi_{A'}] = A'\,\Delta[\psi^*, \psi_{A'}]\,. \tag{169}$$

Therefore $\bar{\Delta}[\psi^*, \Psi_{A'}]$ is an eigenfunctional of $\mathcal{A}_{\text{nor}}$ corresponding to the eigen-value A'.

It results from (168), with a normalized Ψ, that

$$\frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], \mathcal{A}_{\text{nor}}\bar{\Delta}[\psi^*, \Psi]) = \frac{1}{e-1}\{\mathcal{A}_{\text{nor}}\Delta[\psi^*, \Psi]\}_{(\psi=\Psi)} = \int_\Omega \Psi^* A \Psi\,\mathrm{d}x, \tag{170}$$

and by taking $A = p_M$ we get:

$$\frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], LN_M\bar{\Delta}[\psi^*, \Psi]) = \int_\Omega \Psi^* p_M \Psi\,\mathrm{d}x\,. \tag{171}$$

Therefore the probability distributions corresponding to $(1/\sqrt{e-1})\bar{\Delta}[\psi^*, \Psi]$

and to $\int\limits_{\Omega} \psi^* \Psi \, dx$ are the same. It is interesting to notice that $\int\limits_{\Omega} \psi^* \psi_{A'} \, dx$ is also an eigenfunctional of $\mathcal{A}_{\text{nor}}$ corresponding to the eigenvalue A':

$$(172) \qquad \mathcal{A}_{\text{nor}} \int\limits_{\Omega} \psi^* \psi'_{A'} \, dx = \mathcal{A} \int\limits_{\Omega} \psi^* \psi_{A'} \, dx = \int\limits_{\Omega} \psi^* A \psi_{A'} \, dx = A' \int\limits_{\Omega} \psi^* \psi_{A'} \, dx \, .$$

The correspondence between the normalized wave functions Ψ of Σ and the functionals $\bar{\Delta}[\psi^*, \Psi]$ has the remarkable property of conserving the orthogonality. Indeed we have:

$$(173) \qquad (\bar{\Delta}[\psi^*, \Psi_1], \Delta[\psi^*, \Psi_2]) = \bar{\Delta}[\Psi_1^*, \Psi_2] = \exp\left[\int\limits_{\Omega} \Psi_1^* \Psi_2 \, dx\right] - 1 \, .$$

In the correspondence between wave functions $\Psi(x)$ and linear functionals $\int\limits_{\Omega} \psi^* \Psi \, dx$ not only the orthogonality but the inner products themselves are conserved, as shown by (6*a*). $\int\limits_{\Omega} \psi^* \Psi \, dx$ is an eigenfunctional of N_{op} and of N_Ψ corresponding to the eigenvalue 1, as shown by (33) and the following equation:

$$(174) \qquad N_\Psi \int\limits_{\Omega} \psi^* \Psi \, dx =$$

$$= \int\limits_{\Omega} \psi_{\text{op}}^*(x) \Psi(x) \, dx \int\limits_{\Omega} \Psi^*(x') \psi_{\text{op}}(x') \, dx' \int\limits_{\Omega} \psi^*(x'') \Psi(x'') \, dx'' = \int\limits_{\Omega} \psi^* \Psi \, dx \, .$$

On the other hand, in the case of $\bar{\Delta}$, N_{op} and N_Ψ do not have definite values, but their expectation values are $e/(e-1)$, with a normalized Ψ:

$$(175) \qquad \frac{1}{e-1} (\bar{\Delta}[\psi^*, \Psi], N_{\text{op}} \bar{\Delta}[\psi^*, \Psi]) = \frac{1}{e-1} \left(\bar{\Delta}[\psi^*, \Psi], \Delta[\psi^*, \Psi] \int\limits_{\Omega} \psi^* \Psi \, dx\right) =$$

$$= \frac{1}{e-1} \Delta[\Psi^*, \Psi] \int\limits_{\Omega} \Psi^* \Psi \, dx = \frac{e}{e-1}$$

$$(176) \qquad \frac{1}{e-1} (\bar{\Delta}[\psi^*, \Psi], N_\Psi \bar{\Delta}[\psi^*, \Psi]) =$$

$$= \frac{1}{e-1} \left(\bar{\Delta}[\psi^*, \Psi], \int\limits_{\Omega} \psi^*(x) \Psi(x) \, dx \int\limits_{\Omega} \Psi^*(x') \Psi(x') \, dx' \, \Delta[\psi^*, \Psi] \right) = \frac{e}{e-1} \, .$$

We have however,

$$(177)\qquad \frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], LN_\Psi \bar{\Delta}[\psi^*, \Psi]) = \frac{1}{e-1}(\bar{\Delta}[\psi^*, \Psi], LN_{\text{op}} \bar{\Delta}[\psi^*, \Psi]) = 1\,,$$

Ψ being normalized.

The quantal superposition of states arises from an interesting property of the probability distribution of the linear manifolds in the Hilbert space. Let us consider the probability of the ray Ψ in the distribution corresponding to $\int_\Omega \psi^*\Psi\,\mathrm{d}x$:

$$(178)\qquad \left(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\ N_\Psi \int_\Omega \psi^*\Psi\,\mathrm{d}x\right) = \left(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\ \int_\Omega \psi^*\Psi\,\mathrm{d}x\right) = 1\,.$$

The probability of the ray corresponding to a function Ψ', which is not orthogonal to Ψ, in the same distribution is $\left|\int_\Omega \Psi^*\Psi'\,\mathrm{d}x\right|^2$

$$(179)\qquad \left(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\ N_{\Psi'} \int_\Omega \psi^*\Psi\,\mathrm{d}x\right) =$$

$$= \left(\int_\Omega \psi^*\Psi\,\mathrm{d}x,\ \int_\Omega \psi^*\Psi'\,\mathrm{d}x\right)\int_\Omega \Psi'^*\Psi\,\mathrm{d}x = \left|\int_\Omega \Psi^*\Psi'\,\mathrm{d}x\right|^2,$$

since:

$$(180)\qquad N_{\Psi'}\int_\Omega \psi^*\Psi\,\mathrm{d}x = \int_\Omega \psi^*_{\text{op}}(x)\Psi'(x)\,\mathrm{d}x \int_\Omega \Psi'^*(x')\psi_{\text{op}}(x')\,\mathrm{d}x' \int_\Omega \psi^*(x'')\Psi(x'')\,\mathrm{d}x =$$

$$= \int_\Omega \Psi'^*(x')\Psi(x')\,\mathrm{d}x' \int_\Omega \psi^*(x)\Psi'(x)\,\mathrm{d}x\,.$$

7. – Discussion of the linear functionals χ.

We have seen in section 1 that the passage from the wave function $\Psi(x)$ to the corresponding linear functional $\int_\Omega \psi^*\Psi\,\mathrm{d}x$ and the inverse passage can be performed by means of the complex conjugated transformation functionals $\langle\psi^*|x\rangle$ and $\langle x|\psi\rangle$ respectively

$$(17)\qquad \chi[\psi^*] = \int_\Omega \langle\psi^*|x\rangle\Psi(x)\,\mathrm{d}x\,,$$

$$(21)\qquad \Psi(x) = \int \langle x|\psi\rangle\,\chi[\psi^*]\,\mathrm{d}\psi^*\,,$$

with $\langle\psi^*|x\rangle = \psi^*(x)$. The situation is different from that of the ordinary transformation theory, because we do not have operators for the quantities $(\psi(x))$ in the x-representation. The analogy with the ordinary transformation theory suggests that $\langle\psi^*|x'\rangle$ be the eigenfunctional of the operators of the quantities (x), corresponding to the eigenvalues x'. This is indeed true, since by taking $\Psi(x) = \delta(x - x')$ in (17) we get:

$$\langle\psi^*|x'\rangle = \int_\Omega \langle\psi^*|x\rangle\, \delta(x - x')\, dx\,. \tag{181}$$

The above analogy suggests that $\langle x|\psi'\rangle$ be the eigenfunction of the operators for the quantities $(\psi(x'))$ in the x-representation, corresponding to the eigenvalues $\psi'(x')$. This requirement cannot be fulfilled because we do not have operators for the $(\psi(x'))$ in the x-representation.

We have seen in section 1 that the ordinary rule of interpretation of the transformation functions can be applied to the $\langle\psi^*|x\rangle$, $|\langle\psi^*|x\rangle|^2 dx$ being the probability of finding values of the x in the ranges $x - x + dx$, when the quantities $(\psi(x'))$ have the values $\psi(x')$, if it is assumed that the quantities $(\psi(x'))$ have the values $\psi(x')$ when Σ is in the state described by the normalized wave function ψ . The application of the above interpretation rule requires, of course, that the quantities $(\psi(x'))$ take only the values of normalized functions $\psi(x')$.

Let us consider now the wave function $\Phi(A')$ in a representation in which the complete set of commutable variables (A) is diagonal:

$$\Phi(A') = \int_\Omega \langle A'|x\rangle \Psi(x)\, dx\,, \tag{182}$$

$$\Psi(x) = \int \langle x|A'\rangle \Phi(A')\, dA'\,. \tag{183}$$

The integration with respect to the A' becomes a sum in the cases of discrete spectra of the A. Let us write:

$$\varphi(A') = \int_\Omega \langle A'|x\rangle \psi(x)\, dx\,. \tag{184}$$

We have obviously

$$\chi[\psi^*] = \int_\Omega \Psi(x)\psi^*(x)\, dx = \int_\Omega \Phi(A')\varphi^*(A')\, dA'\,, \tag{185}$$

hence:

$$\langle\psi^*|A'\rangle = \varphi^*(A')\,. \tag{186}$$

Let us assume that the A have discrete spectra. The interpretation rule of the transformation theory shows that $|\langle\psi^*|A'\rangle|^2$ is the probability of the values $\psi^*(x)$ of the quantities $(\psi^*(x))$ when the A have the values A'. Since

$$\langle\psi^*|A'\rangle = \int_\Omega \psi^*(x)\langle x|A'\rangle\,\mathrm{d}x \qquad \left(\int|\langle x|A'\rangle|^2\,\mathrm{d}x = 1\right). \tag{187}$$

$\langle\psi^*|A'\rangle$ is the linear functional which corresponds to $\Psi(x) = \langle x|A'\rangle$. The above result shows that the interpretation rule of the transformation theory leads to the following interpretation rule for the linear functionals: $|\chi[\psi^*]|^2$ *is the probability of the state of* Σ *described by the normalized wave function* ψ^*, $\chi[\psi^*]$ *being normalized.* Thus we obtained from the transformation theory the probability distribution of the rays in the ψ Hilbert space discussed in section **6**.

The theory of the linear functionals χ must be physically equivalent to the ordinary form of the quantum mechanics. There is however one thing which appers in a somewhat different form: the transition probabilities. In the ordinary formulation of the quantum mechanics there are no true transition probabilities from an initial state to the possible states at a later time. Thus the transition probability from the state ψ_0 at the time t_0 to the state $\psi_{A'}$ at the time t must be taken as the probability of obtaining the values A' in a measurement performed at the time t. There will be an effective transition only when the measurement is effectively performed, otherwise the system will be in the state $U(t)\psi_0$. In the χ-formalism there is a true transition probability, because probabilities are assigned to all the states at any instant of time, the transition probability from the state ψ_0 to the state $\psi_{A'}$ being the probability of $\psi_{A'}$ at the time t in the spontaneous evolution corresponding to $\chi[0;\psi^*] = \int_\Omega \psi^*\psi_0\,\mathrm{d}x$, i.e. to $|\chi[0;\psi_0^*]|^2 = 1$. Since $\chi[t;\psi_{A'}^*] = \int_\Omega \psi_{A'}^* U(t)\psi_0\,\mathrm{d}x$, the transition probability has the correct value: $\left|\int_\Omega \psi_{A'}^* U(t)\psi_0\,\mathrm{d}x\right|^2$.

There is no difficulty in extending the linear χ-formalism to the case of eigenfunctions $\psi_{A'}$ of the continuous spectrum of a hermitian operator A. We shall assume that the $\psi_{A'}(x)$ are normalized as follows:

$$\int_\Omega \psi_{A'}^*(x)\psi_{A''}(x)\,\mathrm{d}x = \delta(A' - A'') . \tag{188}$$

$|\chi[\psi_{A'}]|^2\,\mathrm{d}A'$ gives the probability of the states ψ_α with α between A' and $A' + \mathrm{d}A'$.

We shall now consider the functional $\chi[\psi^*;\alpha]$

$$\chi[\psi^*, \alpha] = \int_\Omega \Psi_\alpha(x)\psi^*(x)\,\mathrm{d}x , \tag{189}$$

the functions $\Psi_\alpha(x)$ being normalized as the $\psi_{A'}$:

$$\int_\Omega \Psi^*_{\alpha'}(x)\Psi_{\alpha''}(x)\,\mathrm{d}x = \delta(\alpha' - \alpha'') \,. \tag{190}$$

Now $|\chi[\psi^*; \alpha]|^2$ allows only to compute relative probabilities of different states ψ^*. An important case is that of $\Psi_\alpha(x) = \delta(x - \alpha)$. In this case $\chi[\psi^*, \alpha] = \psi^*(\alpha)$, so that the probability of a ψ is proportional to $|\psi(\alpha)|^2$.

8. – Second quantization interpretation of the functionals χ.

The second quantization of the Schrödinger equation (1) leads to the introduction of a quantized ψ-field whose quanta are systems Σ. Our χ-formalism is mathematically the same as the second quantization formalism for non interacting systems Σ treated as bosons, because of the sign minus in the commutation rules (8). The pure state of the system Σ described by the wave function $\Psi(x)$ corresponds to the state of the quantized ψ-field in which there is a single quantum Σ, the quantum being in the Σ-state Ψ. This state of the field is described by the linear functional $\int_\Omega \Psi(x)\psi^*(x)\,\mathrm{d}x$:

The linear functional (2) *obtained from the wave function* Ψ, *by going over to the dual of the* ψ^**-space, coincides with the wave functional of the state of the quantized* ψ*-field in which there is a single quantum* Σ, *the quantum being in the* Σ*-state* Ψ.

We shall consider now the state of the quantized ψ-field in which the $\psi_{\text{op}}(x)$ have the eigenvalues $\Psi(x)$, i.e. the field state described by the wave functional $\Delta[\psi^*, \Psi]$. Since

$$\Delta[\psi^*, \Psi] = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_\Omega \Psi(x_1) \dots \Psi(x_n)\psi^*(x_1) \dots \psi^*(x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \,, \tag{191}$$

we may write

$$\Delta[\psi^*, \Psi] = 1 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \left(\int_\Omega |\Psi(x)|^2\,\mathrm{d}x \right)^{n/2} \chi_n[\psi^*, \Psi] \,, \tag{192}$$

the $\chi_n[\psi^*, \Psi]$ being normalized eigenfunctionals of N_{op}:

$$\chi_n[\psi^*, \Psi] = \frac{1}{\sqrt{n!}} \left(\int_\Omega |\Psi|^2\,\mathrm{d}x \right)^{-n/2} \int_\Omega \Psi(x_1) \dots \Psi(x_n)\,\psi^*(x_1) \dots \psi^*(x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n \,. \tag{193}$$

$\chi_n[\psi^*, \Psi]$ describes a n quantum state of the ψ-field, in which all the n quanta are in the Σ-state Ψ. It follows from (192) that the probability of finding n quanta in the field-state $\Delta[\psi^*, \Psi]$ is $(1/n!) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\int_\Omega |\Psi|^2 \mathrm{d}x\right)^n$, i. e. the probabilities of the n are given by a Poisson law of average value $\int_\Omega |\Psi|^2 \mathrm{d}x$. This average value is simply the expectation value of N_{op} in the state $\Delta[\psi^*, \Psi]$:

$$(194)\qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\Delta[\psi^*, \Psi], N_{\mathrm{op}}\Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*(x)\Psi(x)\,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left\{\int_\Omega \psi^*\Psi \,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right\}_{(\psi=\Psi)} = \int_\Omega |\Psi|^2 \mathrm{d}x\,.$$

In the particular case of a normalized Ψ, the expectation value of N_{op} is simply 1. The expectation value of the number of quanta in the Σ-state described by the normalized wave function $\varphi(x)$ is $|\int_\Omega \Psi(x)\varphi^*(x)\,\mathrm{d}x|^2$:

$$(195)\qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*_{\mathrm{op}}(x)\varphi(x)\mathrm{d}x \int_\Omega \varphi^*(x')\psi_{\mathrm{op}}(x')\,\mathrm{d}x'\, \Delta[\psi^*, \Psi]\right) =$$

$$= \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\int_\Omega \varphi^*(x')\Psi(x')\,\mathrm{d}x'\left(\Delta[\psi^*, \Psi], \int_\Omega \psi^*_{\mathrm{op}}(x)\varphi(x)\,\mathrm{d}x\, \Delta[\psi^*, \Psi]\right) =$$

$$= \left|\int_\Omega \Psi(x)\varphi^*(x)\,\mathrm{d}x\right|^2.$$

When Ψ is normalized, the expectation value of N_φ coincides with the probability of the value 1 of p_φ in the Σ-state Ψ. It follows from (195) that the expectation value of $N_{\Psi_{\mathrm{nor}}}$ in the field-state $\Delta[\psi^*, \Psi]$ coincides with that of N_{op}:

$$(196)\qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\left(\Delta[\psi, \Psi], N_{\Psi_{\mathrm{nor}}}\Delta[\psi^*, \Psi]\right) = \int_\Omega |\Psi|^2 \mathrm{d}x\,,$$

$$(196a)\qquad \Psi_{\mathrm{nor}}(x) = \Psi(x) \Big/ \left(\int_\Omega |\Psi|^2 \mathrm{d}x\right)^{1/2}.$$

Equation (99) shows that $\langle N'|\Psi\rangle$ is the wave function of the N' representation which corresponds to $\Delta[\psi^*, \Psi]$. It follows from (80) that the probability of finding N'_λ quanta Σ in the Σ-state φ_λ in the field state $\Delta[\psi^*, \Psi]$ is:

$$\text{(197)} \qquad \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \sum_{N'_\mu}^{(\mu\neq\lambda)} |\langle N'|\Psi\rangle|^2 =$$

$$= (N'_\lambda!)^{-1} \left\{\left|\int_\Omega \Psi\varphi_\lambda^* \mathrm{d}x\right|^2\right\}^{N'_\lambda} \exp\left\{-\left|\int_\Omega \Psi\varphi_\lambda^* \mathrm{d}x\right|^2\right\}.$$

Therefore the probability of finding n quanta Σ in the Σ-state φ is given by a Poisson distribution of average value $(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]$:

$$\text{(198)} \qquad (n!)^{-1}\left\{\left|\int_\Omega \Psi\varphi^* \,\mathrm{d}x\right|^2\right\}^n \exp\left[-\left|\int_\Omega \Psi\varphi^* \,\mathrm{d}x\right|^2\right] =$$

$$= (n!)^{-1}\left\{(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\right\}^n \cdot$$

$$\cdot \exp\left\{-(\Delta[\psi^*, \Psi], N_\varphi \Delta[\psi^*, \Psi]) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right]\right\}.$$

The functional $\bar{\Delta}[\psi^*, \Psi]$ defined by (160) describes a state of the quantized ψ-field formed by the superposition of the states $\chi_n[\psi^*, \Psi]$, without a vacuum part:

$$\text{(199)} \qquad \bar{\Delta}[\psi^*, \Psi] = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}\left(\int_\Omega |\Psi|^2 \,\mathrm{d}x\right)^{n/2} \chi_n[\psi^*, \Psi].$$

In the state $\bar{\Delta}[\psi^*, \Psi]$ the probability of finding n quanta Σ in the field is given by a modified Poisson law $(n!)^{-1}\left(\int |\Psi|^2 \,\mathrm{d}x\right)^n \left\{\exp\left[-\int_\Omega |\Psi|^2 \,\mathrm{d}x\right)\right] - 1\right\}^{-1}$.

Equation (195) shows that the expectation value of the number of quanta Σ in the Σ-state φ, when the ψ-field is in the state $\Delta[\psi^*, \Psi]$ with a normalized Ψ, coincides with the probability of obtaining the value 1 of the operator p_φ in a measurement performed on a system Σ in the Σ-state Ψ. More generally, the expectation value of any operator $\mathcal{A}$ in the field state $\Delta[\psi^*, \Psi]$ coincides with the expectation value of the quantity (A) in Σ-state Ψ, provided Ψ is normalized, as shown by (58) and (51). The above results show that the average behaviour of the ψ-field in the state $\Delta[\psi^*, \Psi]$, with a normalized Ψ, is the same as that of a system Σ in the state Ψ. When Ψ is not normalized, the average behaviour of the field is the same as that of an assembly of $\int |\Psi|^2 \,\mathrm{d}x$ systems Σ all in the state Ψ.

The wave functionals which satisfy (3) form a ring, more precisely a vector ring, because both the sum and the product of two solutions of (3) are solutions of (3). More generally let $f(u)$ be a holomorphic function of the complex variable u, we have:

$$\mathcal{H} f(\chi[t;\psi^*]) = \int_\Omega \mathrm{d}x\, \psi^*(x) H \frac{\delta}{\delta\psi^*(x)} f(\chi[t;\psi^*]) = \tag{200}$$

$$= f'(\chi[t;\psi^*]) \int_\Omega \mathrm{d}x\, \psi^*(x) H \frac{\delta}{\delta\psi^*(x)} \chi[t;\psi^*] = f'(\chi[t;\psi^*]) \mathcal{H} \chi[t;\psi^*].$$

f' denoting the derivative of f. When $\chi[t;\psi^*]$ is a solution of (3) we have:

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} f(\chi[t;\psi^*]) = \mathcal{H} f(\chi[t;\psi^*]) . \tag{201}$$

It follows from this theorem that $\Delta[\psi^*, \Psi(t;x)]$ is a solution of (3). It is easily seen that any holomorphic function of s solutions $\chi^{(1)}[t;\psi^*],\ldots, \chi^{(s)}[t;\psi^*]$ of (3) is also a solution of (3). Since

$$\Delta[\psi^*, \Psi^{(1)}]\, \Delta[\psi^*, \Psi^{(2)}] = \Delta[\psi^*, \Psi^{(1)} + \Psi^{(2)}] , \tag{202}$$

the superposition of the pure states of Ψ gives rise to the following law of composition of the $\Delta[\psi^*, \Psi]$ field states:

$$\Delta[\psi^*, \sum_l a_l \Psi^{(l)}(t;x)] = \prod_l \{\Delta[\psi^*, \Psi^{(l)}(t;x)]\}^{a_l} . \tag{203}$$

We have:

$$(\Delta[\psi^*, \Psi], \Delta[\psi^*, \Psi]) = \Delta[\Psi^*, \Psi] = \exp\left[\int_\Omega |\Psi|^2 \mathrm{d}x\right] . \tag{204}$$

Let us denote by P_Ψ the projection operator on the direction of the vector of the χ-space which corresponds to the field state $\Delta[\psi^*, \Psi]$:

$$\begin{cases} P_\Psi \chi[\psi^*] = \Delta[\psi^*, \Psi](\Delta[\psi^*, \Psi], \chi[\psi^*]) \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] , \\ P_\Psi \chi[\psi^*] = \chi[\Psi^*] \exp\left[-\int_\Omega |\Psi|^2 \mathrm{d}x\right] \Delta[\psi^*, \Psi] . \end{cases} \tag{205}$$

Since

$$\psi_{\mathrm{op}}(x)\chi[\psi^*] = \int \Psi(x) \Delta[\psi^*, \Psi] \chi[\Psi^*]\, \mathrm{d}\Psi^* , \tag{206}$$

we have:

(207) $$\psi_{op}(x)\chi[\psi^*] = \int \exp\left[\int_\Omega \Psi\Psi^*\,dx\right] \Psi(x) P_\Psi \chi[\psi^*]\,d\Psi^* .$$

We shall assume that the expectation value of $\psi_{op}(x)$ is $(\chi[\psi^*], \psi_{op}(x)\chi[\psi^*])$.

Since

(208) $$(\chi[\psi^*], \psi_{op}(x)\chi[\psi^*]) = \int \Psi(x)(\chi[\psi^*], P_\Psi\chi[\psi^*]) \exp\left[\int_\Omega \Psi\Psi^*\,dx\right] d\Psi^* ,$$

the probability of finding the values $\Psi(x)$ of the $\psi_{op}(x)$ may be taken tentatively as:

(209) $$\mathcal{P}[\Psi] = (\chi[\psi^*], P_\Psi\chi[\psi^*]) \exp\left[\int_\Omega |\Psi|^2\,dx\right] .$$

By taking into account (205), we get:

(210) $$\mathcal{P}[\Psi] = |\chi[\Psi^*]|^2 .$$

This result is not satisfactory because the probabilities $\mathcal{P}[\Psi]$ may be larger than 1, as happens already in the case of linear functionals. In the case of linear functionals we can get satisfactory results from (210) by taking only normalized functions Ψ, as shown in section 7.

The above result seems to indicate that, in the state of a field described by a non linear functional χ, it is not possible to get a satisfactory probability distribution for the values $\Psi(x)$ of the $\psi_{op}(x)$. This is not important because the $\psi_{op}(x)$ do not describe ordinary physical quantities of the systems Σ.

9. – Alternative interpretation of the χ-formalism for a single system Σ.

We have already given in section **6** a rule of physical interpretation for the χ-formalism considered as describing kinetic conditions of a single system Σ, by assigning probabilities to the linear manifolds M of the ψ Hilbert space. We shall now see that it is possible to modify the physical interpretation of the formalism of the quantized ψ-field in such a way that it will become a treatment of the motion of a single system Σ. We shall assume the following rules of physical interpretation:

(*a*) *The solutions of equation* (3), *such that* $\chi[t; 0] = 0$, *describe kinetic conditions of a system* Σ.

(*b*) *The possible values of a quantity* (A) *are the eigenvalues* A' *of the operator* A.

(*c*) *The physical quantity* (A) *is described by the corresponding field operator* $\mathcal{A}_{\text{nor}}$.

(*d*) *The expectation value of the quantity* (A) *of* Σ *in the kinetic condition described by the normalized functional* $\chi[\psi^*]$ *is given by the field expectation value* $(\chi[\psi^*], \mathcal{A}_{\text{nor}}\chi[\psi^*])$ *of* $\mathcal{A}_{\text{nor}}$.

(*e*) *The probability of finding the value* A' *of* (A) *in a measurement in the condition described by* $\chi[\psi^*]$ *coincides with the probability of obtaining the value* A' *of the quantity* $(A'p_{A'})$, $p_{A'}$ *being the projection operator of* A *for the eigenvalue* A'.

The field operator corresponding to $A'p_{A'}$ is $A'LN_{A'}$, in the present interpretation. Since the only eigenvalues of $A'p_{A'}$ are 0 and A', the expectation value $(\chi[\psi^*], A'LN_{A'}\chi[\psi^*])$ coincides with the probability of obtaining the value A' of $(A'p_{A'})$. This probability is the same as that of obtaining the value A' of (A), as a consequence of the rule (*e*). Thus we get the same result as in section 6.

The field operator LN_{op} corresponds to the quantity (1). Since

$$LN_{\text{op}}\chi[\psi^*] = \chi[\psi^*] \quad \text{when} \quad \chi[0] = 0\,, \tag{211}$$

the expectation value of LN_{op} is 1, as it should be. Therefore by considering only functionals such that $\chi[0] = 0$ and replacing the operator N_{op} for the total number of quanta by LN_{op} we get indeed a one quantum formalism.

We have:

$$\begin{aligned}(\bar{\Delta}[\psi^*, \Psi], LN_{A'}\bar{\Delta}[\psi^*, \Psi]) &= (\Delta[\psi^*, \Psi], LN_{A'}\,\Delta[\psi^*, \Psi]) = \\ &= \Big(\Delta[\psi^*, \Psi], L\Big\{\int_\Omega \psi^* p_{A'}\Psi\,\mathrm{d}x\,\Delta[\psi^*, \Psi]\Big\}\Big) = \\ &= \Big(L\Big\{\int_\Omega \psi^* p_{A'}\Psi\,\mathrm{d}x\,\Delta[\psi^*, \Psi]\Big\}\Big)_{(\psi=\Psi)} = \frac{\int_\Omega \Psi^* p_{A'}\Psi\,\mathrm{d}x}{\int_\Omega |\Psi|^2\mathrm{d}x}\Big\{\exp\Big[\int_\Omega |\Psi|^2\,\mathrm{d}x\Big] - 1\Big\}.\end{aligned} \tag{212}$$

Since

$$(\bar{\Delta}[\psi^*, \Psi], \bar{\Delta}[\psi^*, \Psi]) = \exp\Big[\int_\Omega |\Psi|^2\,\mathrm{d}x\Big] - 1\,, \tag{213}$$

the expectation value of $LN_{A'}$ in the condition described by $\bar{\Delta}[\psi^*, \Psi]$ coincides with the expectation value of $p_{A'}$. in the Σ-state Ψ, i.e. with the probability of finding the value A' in a measurement of (A) in the state Ψ. Thus in the χ-formalism we can describe the pure state Ψ of Σ by $\overline{\Delta}[\psi^*, \Psi]$. The correspondence between the pure states Ψ of Σ and the functionals $\overline{\Delta}$ is remarkable because an eigenfunction $\psi_{A'}(x)$ of A corresponds to an eigenfunctional $\overline{\Delta}[\psi^*, \psi_{A'}]$ of $\mathcal{A}_{\text{nor}}$, with the same eigenvalue A':

$$\mathcal{A}_{\text{nor}}\overline{\Delta}[\psi^*, \psi_{A'}] = A'\overline{\Delta}[\psi^*, \psi_{A'}] . \tag{169}$$

The pure state Ψ of Σ can also be represented by the linear functional $\int_\Omega \Psi(x)\psi^*(x)\,\mathrm{d}x$ or by any of the homogeneous functionals

$$\frac{1}{\sqrt{n!}}\int_\Omega \Psi(x_1)\dots\Psi(x_n)\,\psi^*(x_1)\dots\psi^*(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n .$$

These functionals are also eigenfunctionals of $\mathcal{A}_{\text{nor}}$ corresponding to the eigenvalue A', when $\Psi = \psi_{A'}$:

$$\begin{aligned}\mathcal{A}_{\text{nor}}\int_\Omega \psi_{A'}(x_1)\dots\psi_{A'}(x_n)\psi^*(x_1)\dots\psi^*(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n =\\ = A'\int_\Omega \psi_{A'}(x_1)\dots\psi_{A'}(x_n)\psi^*(x_1)\dots\psi^*(x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n .\end{aligned} \tag{214}$$

RIASSUNTO (*)

Si dimostra che l'ordinaria equazione di Schrödinger per un sistema dinamico Σ può essere sostituita da un'equazione più generale avente la forma dell'equazione di Schrödinger per un campo di Bose quantizzato i cui quanti sono i sistemi Σ. I funzionali d'onda lineari del campo quantizzato descrivono stati puri del sistema Σ e i funzionali d'onda non lineari descrivono, in generale, stati misti di Σ. La rappresentazione in cui gli operatori di emissione del campo di Σ quantizzato sono diagonali ha una parte preponderante nel presente formalismo. Si dimostra che gli autofunzionali degli operatori di assorbimento del campo di Σ quantizzato possono essere usati per dare una nuova descrizione degli stati di un sistema Σ, in quanto i valori di aspettazione delle grandezze del campo in autostati opportunamente scelti degli operatori di emissione coincidono coi valori di aspettazione delle corrispondenti grandezze negli stati puri di Σ, ma con maggiori fluttuazioni nel primo caso. Gli autofunzionali degli operatori di assorbimento hanno la notevole proprietà di essere elementi di matrice dell'operatore unitario del formalismo del campo e appaiono di natura più fondamentale che non i funzionali lineari corrispondenti all'ordinaria descrizione degli stati per mezzo delle funzioni d'onda di Σ.

(*) *Traduzione a cura della Redazione.*