M. SCHÖNBERG
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A General Theory of the Second Quantization Methods.

M. SCHÖNBERG

Centre de Physique Nucléaire de l'Université Libre - Bruxelles

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Summary. — It is shown that the second quantization methods are a general mathematical technique applicable to formalisms involving linear equations of change, differential with respect to the time variable. The ordinary second quantization formalism for systems of bosons or fermions, and the « second quantization » of the classical theory developed by the author are obtained as particular cases of the general methods. There are several ways of applying the second quantization methods to the same linear problem, which lead to different formalisms. A very simple kind of second quantization is applied to the Schrödinger equation of an arbitrary mechanical system in order to develop a new treatment of the statistical quantum mechanics (the Gibbs second quantization). In this way a straightforward method of introducing the Gibbs ensembles and grand ensembles is obtained. The entropy is discussed with the Gibbs second quantization. Some ergodic theorems of the general second quantization and the quantum mechanics are derived.

1. – Introduction.

The first application of a second quantization method was made by DIRAC [1] in the quantum theory of systems of non interacting bosons. JORDAN and KLEIN [2] extended Dirac's method to the case of interacting bosons and JORDAN and WIGNER [3] showed that a second quantization formalism can also be developed for interacting fermions. The technique of second

[1] P. A. M. DIRAC: *Proc. Roy. Soc.*, A **114**, 243 (1927).
[2] P. JORDAN and O. KLEIN: *Zeits. f. Phys.*, **45**, 751 (1927).
[3] P. JORDAN and E. WIGNER: *Zeits. f. Phys.*, **47**, 631 (1928).

quantization was considerably improved by FOCK (4) and JORDAN (5), in the sense of simplifying the passage from the formalism in configuration space to that of second quantization. We have shown recently (6) that the methods of second quantization can also be applied to the classical mechanics, by using the first order linear partial differential equation for the integrals of the motion (Liouville equation of the classical statistical mechanics) as the wave equation submitted to the « second quantization » procedure. In this case the denomination of second quantization is clearly inadequate, since the Planck constant does not appear in the theory. The results of reference (6) indicate that the second quantization method is a mathematical technique which can be applied to linear differential equations others than the wave equations of the quantum theory. In the present paper we shall prove that the second quantization methods can be applied to any linear equation of the form

$$(1) \qquad i\frac{\partial}{\partial t}\Psi_n(t;\tau_1,\ldots,\tau_n) = K_n(t;\tau_1,\ldots,\tau_n)\Psi_n(t;\tau_1,\ldots,\tau_n),$$

Ψ_n being a function of n similar sets τ_l of r variables $x_1^{(l)}, x_2^{(l)}, \ldots, x_r^{(l)}$, continuous or discrete, and $K_n(t;\tau)$ a linear operator of the form

$$(2) \qquad K_n(t;\tau_1,\ldots,\tau_n) = \sum_{\alpha=1}^{s}\frac{1}{\alpha!}\sum_{l_1,l_2,\ldots,l_\alpha}{}' K^{(\alpha)}(t;\tau_{l_1},\tau_{l_2},\ldots,\tau_{l_\alpha}) \qquad (s\leqslant n),$$

$\sum'$ denoting that the summation is taken over different values of $l_1, l_2, \ldots, l_\alpha$ and the $K^{(\alpha)}(t;\tau_{l_1},\tau_{l_2},\ldots,\tau_{l_\alpha})$ being linear operators symmetrical with respect to the $\tau_{l_1}, \tau_{l_2}, \ldots, \tau_{l_\alpha}$. We shall treat all the x as continuous variables for the sake of simplicity, but it will be obvious that the modifications necessary when some of the x are discrete are of a trivial nature. We shall assume that the $x_1^{(l)}, \ldots, x_r^{(l)}$ are the coordinates of a point τ_l in a r dimensional space Ω and we shall denote by $\mathrm{d}\tau$ the element of volume in Ω.

The basic operators of the « second quantization » are the hermitian conjugated $\psi(\tau)$ and $\psi^*(\tau)$ characterized by the following commutation rules:

$$(3) \qquad [\psi(\tau),\psi^*(\tau')]_{\pm} = \delta(\tau-\tau'), \qquad [\psi(\tau),\psi(\tau')]_{\pm} = [\psi^*(\tau),\psi^*(\tau')]_{\pm} = 0,$$

$$(4) \qquad \delta(\tau-\tau') = \delta(x_1-x_1')\,\delta(x_2-x_2')\ldots\delta(x_r-x_r'),$$

$$(5) \qquad [A,B]_{\pm} = AB \pm BA.$$

(4) V. FOCK: *Zeits. f. Phys.*, **75**, 522 (1932).
(5) P. JORDAN: *Zeits. f. Phys.*, **75**, 648 (1932).
(6) M. SCHÖNBERG: *Nuovo Cimento*, **9**, 1139 (1952); **10**, 419 (1953).

The signs + and — correspond to two different forms of the « second quantization ». In the ordinary second quantization they correspond to the cases of bosons (—) and fermions (+). We shall introduce the operators $\mathcal{K}_\alpha$ and $\mathcal{K}$

$$\mathcal{K}_\alpha = \frac{1}{\alpha!}\int_\Omega \psi^*(\tau_1)\psi^*(\tau_2)\ldots\psi^*(\tau_\alpha)K^{(\alpha)}(\tau_1,\ldots,\tau_\alpha)\{\psi(\tau_\alpha)\psi(\tau_{\alpha-1})\ldots\psi(\tau_1)\}\mathrm{d}\tau_1\ldots\mathrm{d}\tau_\alpha\,, \tag{6}$$

$$\mathcal{K} = \sum_{\alpha=1}^{s}\mathcal{K}_\alpha \tag{7}$$

and the functionals χ defined by the equation:

$$i\frac{\mathrm{d}\chi}{\mathrm{d}t} = \mathcal{K}\chi\,. \tag{8}$$

The general solution of (8) is given by a series of the form (Fock exp ansion)

$$\chi(t) = \Psi_0\chi_0 + \sum_{k=1}^{\infty}\frac{1}{\sqrt{k!}}\int_\Omega \Psi_k(t;\,\tau_1,\ldots,\tau_k)\chi_k(\tau_1,\ldots,\tau_k)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_k\,, \tag{9}$$

χ_0 being a functional to be defined later and

$$\chi_k(\tau_1,\ldots,\tau_k) = \psi^*(\tau_1)\psi^*(\tau_2)\ldots\psi^*(\tau_k)\chi_0\,. \tag{10}$$

The Ψ_k are solutions of equation (1).

The operator N_{op}

$$N_{\mathrm{op}} = \int_\Omega \psi^*(\tau)\psi(\tau)\,\mathrm{d}\tau \tag{11}$$

has the eigenvalues 0, 1, 2, 3 ... (∞). N_{op} is the operator for the number of particles in the usual form of second quantization, as well as in the theory of reference (6). χ_0 is the eigenfunctional of N_{op} corresponding to the eigenvalue 0, with a suitable normalization

$$N_{\mathrm{op}}\chi_0 = 0\,, \tag{12}$$

$$\int_{\mu\text{-space}}\chi_0^*\chi_0\,\mathrm{d}\mu = 1\,, \tag{13}$$

the integration in (13) being performed with respect to the variables involved in χ_0. χ_0 is the wave functional of the vacuum in the usual forms of second quantization. The $\chi_k(\tau_1, ..., \tau_k)$ are also eigenfunctionals of N_{op}

$$N_{op}\chi_k(\tau_1, \tau_2, ..., \tau_k) = k\chi_k(\tau_1, ..., \tau_k)\,, \tag{14}$$

hence each term of the series in the right hand side of (9) is an eigenfunctional of N_{op}

$$N_{op}\int_{\Omega} \Psi_k(t; \tau_1, ..., \tau_k)\chi_k(\tau_1, ..., \tau_k)\,d\tau_1 \ldots d\tau_k = \tag{15}$$

$$= k\int_{\Omega} \Psi_k(t; \tau_1, ..., \tau_k)\chi_k(\tau_1, ..., \tau_k)\,d\tau_1 \ldots d\tau_k\,.$$

When χ satisfies the condition

$$N_{op}\chi = k\chi\,, \qquad (k = \text{positive integer}) \tag{16}$$

there is only one term in the right hand side of (9)

$$\chi(t) = \frac{1}{\sqrt{k!}}\int_{\Omega} \Psi_k(t; \tau_1, ..., \tau_k)\chi_k(\tau_1, ..., \tau_k)\,d\tau_1 \ldots d\tau_k\,, \tag{17}$$

and

$$\Psi_k(t; \tau_1, ..., \tau_k)\chi_0 = \frac{1}{\sqrt{k!}}\,\psi(\tau_k)\psi(\tau_{k-1}) \ldots \psi(\tau_1)\chi(t)\,. \tag{18}$$

Thus we get a solution of (1) from any solution of (8) satisfying the condition (16). This solution of (1) is symmetrical or anti-symmetrical, according to the sign in the commutation rules (3). Conversely, with any symmetrical solution of (1) we can form a solution (17) of (8), with the sign minus in the commutation rules, and with any anti-symmetrical solution of (1) we can form a solution (17) of (8), with the sign plus in the commutation rules, the condition (16) being satisfied in both cases.

The proofs of the above results are given in sections 2 and 3. These results contain as particular cases the fundamental theorems of the usual second quantization as well as those of reference (6). There are however other interesting applications. Let H be the hamiltonian of a quantal sistem Σ of any

kind, by taking

$$K^{(1)} = \hbar^{-1}H\,, \qquad K^{(2)} = K^{(3)} = \ldots = 0, \qquad \left(\hbar = \frac{h}{2\pi}\right) \tag{19}$$

we get a new kind of a second quantization formalism describing statistica[1] ensembles. Let us assume that at the time 0 it is only known that the system Σ is in one of the states described by the time independent and orthogonal wave functions $\varphi_1, \varphi_2, \ldots, \varphi_p$. We can extend the set $(\varphi_1, \varphi_2, \ldots, \varphi_p)$ by including other orthogonal functions, in order to obtain a complete set of orthogonal functions φ_λ $(\lambda = 1, 2, \ldots (\infty))$, and take the occupation numbers N'_λ of those states as variables to describe the « quantized field ». The wave functional $\chi(0)$ is then

$$\chi(0) = \delta_{N'_1,1}\delta_{N'_2,1} \ldots \delta_{N'_p,1} \prod_{\lambda=p+1}^{\infty} \delta_{N'_\lambda,0}\,. \tag{20}$$

By determining the wave functional $\chi(t)$ we can compute the probabilities of the various distributions of the p similar systems between the various states φ_λ at the time t. It will be shown in section **6** that the wave functional $\chi(t)$ whose initial value is

$$\chi(0) = \delta_{N'_1,n_1}\,\delta_{N'_2,n_2} \ldots = \prod_{\lambda=1}^{\infty} \delta_{N'_\lambda,n_\lambda}\,, \tag{21}$$

describes the same incompletely specified state of motion of a system Σ as the von Neuman density matrix $\langle\tau|R(t)|\tau'\rangle$ whose initial value is:

$$\langle\tau|R(0)|\tau'\rangle = \sum_{\lambda=1}^{\infty} n_\lambda \varphi_\lambda(\tau)\varphi^*_\lambda(\tau')\,. \tag{22}$$

This new kind of second quantization gives the most direct quantal method of introducing the ensembles and grand ensembles of the statistical technique of Gibbs. We shall call it the Gibbs second quantization. The grand ensembles are described by wave functionals χ which are not eigenfunctionals of N_{op}.

The formalism of the Gibbs second quantization becomes particularly interesting in the representation in which the « emission » operators $\psi^*(\tau)$ are diagonalized, the ψ^*-representation. Such a representation was already used by Fock (7) in the quantum electrodynamics, it is closely related to the Fock (8)

(7) V. Fock: *Phys. Zeits. Sow. Un.*, **6**, 428 (1934).
(8) V. Fock: *Zeits. f. Phys.*, **49**, 339 (1928).

theory of the harmonic oscillator. The use of the ψ^*-representation requires an extension of the ordinary technique of the transformation theory of the quantum mechanics, because the basic commutable operators of the representation are not hermitian. DIRAC (9) has given an extension of the transformation theory which allows to deal with such generalized representations. The Dirac technique involves contour integration and does not seem to be the most convenient. In section **4** is given a new treatment of the ψ^*-representation, in which contour integration is avoided by the use of methods of the theory of systems of orthogonal functions of complex variables. In the ψ^*-representation the wave functional χ is a functional of a complex function $\psi^*(\tau)$ and the operators $\mathcal{K}_\alpha$ have the form:

$$(23) \qquad \mathcal{K}_\alpha = \frac{1}{\alpha!}\int_\Omega \psi^*(\tau_1)\dots\psi^*(\tau_\alpha)K^{(\alpha)}(\tau_1,\dots,\tau_\alpha)\frac{\delta^\alpha}{\delta\psi^*(\tau_1)\dots\delta\psi^*(\tau_\alpha)}\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_\alpha\,.$$

The equation (8) for the Gibbs second quantization becomes a first order partial differential equation:

$$(24a) \qquad i\frac{\mathrm{d}}{\mathrm{d}t}\chi[t;\psi^*] = \int_\Omega \psi^*(\tau)K(\tau)\frac{\delta\chi[t;\psi^*]}{\delta\psi^*(\tau)}\,\mathrm{d}\tau.$$

The complex conjugated of $\chi[t;\psi^*]$ is a functional of a function $\psi = (\psi^*)^*$ and satisfies the equation:

$$(24b) \qquad -i\frac{\mathrm{d}}{\mathrm{d}t}\chi^*[t;\psi] = \int_\Omega \frac{\delta\chi^*[t;\psi]}{\delta\psi(\tau)}K(\tau)\psi(\tau)\,\mathrm{d}\tau\,.$$

The product $\chi[t;\psi^*]\chi^*[t;\psi']$ is a functional of the two complex functions ψ^* and ψ' which satisfies a kind of Liouville equation:

$$(24c) \qquad i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[t;\psi^*,\psi'] = \int_\Omega\left\{\psi^*(\tau)K(\tau)\frac{\delta\mathcal{P}}{\delta\psi^*(\tau)} - \frac{\delta\mathcal{P}}{\delta\psi'(\tau)}K(\tau)\psi'(\tau)\right\}\mathrm{d}\tau\,.$$

The equations (24) play a central role in a generalization of the quantum mechanics recently discussed by us (10). Both (24*a*) and (24*b*) are particular cases of (24*c*), so that $\chi[t;\psi^*]$ and $|\chi[t;\psi^*]|^2$ satisfy the same Liouville equation.

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(9) P. A. M. DIRAC: *Comm. Dub. Inst. f. Adv. Stud.*, A, **1** (1943).
(10) M. SCHÖNBERG: *Nuovo Cimento*, **10**, 350 (1953).

The relations between the Gibbs second quantization and the von Neumann statistical formalism are discussed in sections **6** and **7**. It is shown that a non normalized von Neumann operator $R(t)$ can be defined in terms of any normalized solution $\chi(t)$ of the equation (8) of the Gibbs second quantization:

$$\langle \tau' | R(t) | \tau'' \rangle = \int_{\mu\text{-space}} \chi^*(t)\psi^*(\tau'')\psi(\tau')\chi(t)\,\mathrm{d}\mu\,. \tag{25}$$

Thus it becomes possible to apply the von Neumann definition of the entropy to the Gibbs second quantization by means of (25). The entropy thus defined is time independent. An interesting feature of the Gibbs second quantization consists in the possibility of defining a kind of entropy operator $S_{\mathrm{op}}^{(\varphi)}$:

$$S_{\mathrm{op}}^{(\varphi)} = -k \sum_{\lambda} f_\lambda \log f_\lambda\,. \qquad (k = \text{Boltzmann constant}). \tag{26}$$

The f_λ are the $N_\lambda/N_{\mathrm{op}}$, conveniently defined in order to avoid the difficulties arising from the eigenvalue 0 of N_{op}. The expectation value $\int_{\mu\text{-space}} \chi^*(t) S_{\mathrm{op}}^{(\varphi)} \chi(t)\,\mathrm{d}\mu$ has a behaviour similar to that of an entropy. Thus in the case of a wave functional of the type (21) the expectation value of S_{op} coincides with the von Neumann entropy of the assembly described by the density operator (22). The entropy is discussed in section **8**.

In section **9** the ordinary second quantization of the quantum mechanics is obtained as a particular case of the general theory. Some new formulas are given. The « second quantization » formalism of the classical mechanics is obtained from the general theory in section **10**. It is shown that there is a classical analogue of the Gibbs second quantization.

It is shown in section **11** that in the case of hermitian operators $K^{(\alpha)}$ the general formalism of « second quantization » may be obtained by the application of the ordinary procedure of field quantization to a kind of field in the space Ω described by non linear equations. This approach is particularly interesting in the case of the Gibbs second quantization, because it allows to get the interpretation rules of the statistical formalism from the usual interpretation rules of the theory of the quantized fields. The quantum theory of a field whose quanta are bosons appears as a statistical theory of systems of bosons. From this facts results the possibility of assigning an entropy to a pure state of a field of bosons (*).

Ergodic theorems of the general « second quantization » and of the quantum mechanics are derived in sections **13** and **14**. These ergodic theorems give the values of generalized limits for $t = \infty$ of operators $\mathscr{A}(t)$, $A(t)$ of the Heisen-

(*) Prof. PRIGOGINE was led to a similar conclusion by different considerations.

berg representations of the general second quantization and of the quantum mechanics. The generalized Cesàro limits

$$\mathscr{A}(\infty) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathscr{A}(t)\mathrm{d}t\,, \tag{27a}$$

are discussed in section **13**, for hamiltonians $\mathscr{H}$ with a pure point spectrum. The case of general spectra is discussed in section **14** by means of the Abel generalized limit:

$$\mathscr{A}_{\mathrm{ab}}(\infty) = \lim_{\sigma\to 0+} \sigma \int_0^\infty e^{-\sigma t}\mathscr{A}(t)\,\mathrm{d}t\,. \tag{27b}$$

The ergodic theorems allow to give a justification of the fundamental postulate of the equal a priori probabilities, without the unsatisfactory use of the perturbation theory for long intervals of time. These ergodic theorems differ considerably from those of the usual ergodic theory, developed in the last twenty years in connection with the classical mechanics and the theory of stochastic processes, although the generalized limits are used in both.

2. – The general formalism of second quantization.

It follows from the commutation rules (3) that

$$[\varrho(\tau),\ \psi^*(\tau_1)\psi^*(\tau_2)\ldots\psi^*(\tau_k)] = \sum_{l=1}^{k} \delta(\tau-\tau_l)\psi^*(\tau_1)\ldots\psi^*(\tau_k)\,, \tag{28}$$

with

$$\varrho(\tau) = \psi^*(\tau)\psi(\tau) \tag{29}$$

and

$$[A, B] = AB - BA\,. \tag{30}$$

By taking the hermitian conjugates of both sides of (28) we get:

$$[\psi(\tau_k)\psi(\tau_{k-1})\ldots\psi(\tau_1),\ \varrho(\tau)] = \sum_{l=1}^{k} \delta(\tau-\tau_l)\psi(\tau_k)\ldots\psi(\tau_1)\,. \tag{31}$$

Since

$$N_{\mathrm{op}} = \int_\Omega \psi^*(\tau)\psi(\tau)\,\mathrm{d}\tau\,, \tag{32}$$

we get from (28) and (31):

$$[N_{\mathrm{op}}, \psi^*(\tau_1) \dots \psi^*(\tau_k)] = k\psi^*(\tau_1) \dots \psi^*(\tau_k)\,, \tag{33a}$$

$$[N_{\mathrm{op}}, \psi(\tau_k) \dots \psi(\tau_1)] = -k\psi(\tau_k) \dots \psi(\tau_1)\,. \tag{33b}$$

We get from (6) and (7), by taking into account (33):

$$[N_{\mathrm{op}}, \mathcal{K}_\alpha] = 0\,, \qquad [N_{\mathrm{op}}, \mathcal{K}] = 0\,. \tag{34}$$

Let χ be a solution of (8). We have

$$i\frac{\mathrm{d}}{\mathrm{d}t}(N_{\mathrm{op}}\chi) = iN_{\mathrm{op}}\frac{\mathrm{d}\chi}{\mathrm{d}t} = N_{\mathrm{op}}\mathcal{K}\chi = \mathcal{K}(N_{\mathrm{op}}\chi)\,, \tag{35}$$

so that $N_{\mathrm{op}}\chi$ is also a solution of (8). We shall now prove that the condition (16) will be satisfied for all t if it is for a particular value t_0, provided the equation (8) determines $\chi(t)$ for all t when $\chi(t_0)$ is given, as will be always assumed. $N_{\mathrm{op}}\chi(t)$ must coincide with $k\chi(t)$, because both are solutions of (8) and have the same value for $t = t_0$.

The proof that the eigenvalues of N_{op} are $0, 1, 2, \dots (\infty)$ is the same as in the Dirac-Jordan second quantization (see for instance reference (6), section 4). The same considerations given in sections 4 and 5 of reference (6) are applicable in the present case and thus we can establish that the $\chi_k(\tau_1, \dots, \tau_k)$ constitute a complete orthonormal set. From this follows the possibility of expanding any functional $\chi(t)$ in a series of the type (9).

In order to show that the coefficients $\Psi_k(\tau_1, \dots, \tau_k)$ in the expansion (9) of the solutions of (8) do satisfy equations of the form (1), we shall need the following equation

$$\begin{aligned}
[\mathcal{K}_\alpha, \psi^*(\tau_1) \dots \psi^*(\tau_k)] = {} & \sum_{l_1,l_2,\dots,l_\alpha=1,2,\dots,k}^{l_1<l_2<\dots<l_\alpha} (K^{(\alpha)*}(\tau_{l_1}, \dots, \tau_{l_\alpha})\psi(\tau_k) \dots \psi(\tau_1))^* + \\
& + \sum_{l_1,l_2,\dots,l_{\alpha-1}=1,2,\dots,k}^{l_1<l_2<\dots<l_{\alpha-1}} \int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \dots, \tau_{l_{\alpha-1}}, \tau_1')\{\psi(\tau_1')\psi(\tau_k) \dots \psi(\tau_1)\})^*\psi(\tau_1')\,\mathrm{d}\tau_1' + \\
& + \frac{1}{2}\sum_{l_1,l_2,\dots,l_{\alpha-2}=1,2,\dots,k}^{l_1<l_2<\dots<l_{\alpha-2}} \int_\Omega (K^{(\alpha)*}(\tau_{l_1},\dots,\tau_{l_{\alpha-2}}, \tau_1', \tau_2')\{\psi(\tau_2')\psi(\tau_1')\psi(\tau_k)\dots\psi(\tau_1)\})^* \cdot \\
& \qquad \cdot\psi(\tau_2')\psi(\tau_1')\,\mathrm{d}\tau_1'\mathrm{d}\tau_2' + \dots + \\
& + \frac{1}{(\alpha-1)!}\sum_{l_1=1}^{k} \int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \tau_1', \dots, \tau_{\alpha-1}')\{\psi(\tau_{\alpha-1}') \dots \psi(\tau_1')\psi(\tau_k) \dots \psi(\tau_1)\})^* \cdot \\
& \qquad \cdot\psi(\tau_{\alpha-1}') \dots \psi(\tau_1')\,\mathrm{d}\tau_1' \dots \mathrm{d}\tau_{\alpha-1}'\,,
\end{aligned} \tag{36}$$

$K^{(\alpha)*}$ denoting the adjoint or hermitian conjugated of the operator $K^{(\alpha)}$. When $\alpha > k$ the first $\alpha - k$ terms in the right hand side of (36) do not exist. This

equation is satisfied for $k = 1$, as a consequence of (6) and (3):

$$(37)\quad [\mathcal{K}_\alpha, \psi^*(\tau_1)] =$$

$$= \frac{1}{(\alpha-1)!}\int_\Omega (K^{(\alpha)*}(\tau_1, \tau_1', \ldots, \tau_{\alpha-1}')\{\psi(\tau_{\alpha-1}')\ldots\psi(\tau_1')\psi(\tau_1)\})^*\psi(\tau_{\alpha-1}')\ldots\psi(\tau_1')\,\mathrm{d}\tau_1'\ldots\mathrm{d}_{\alpha-1}'.$$

Since

$$(38)\quad [\mathcal{K}_\alpha, \psi^*(\tau_1)\ldots\psi^*(\tau_k)] = [\mathcal{K}_\alpha, \psi^*(\tau_1)\ldots\psi^*(\tau_{k-1})]\psi^*(\tau_k) +$$

$$+ \psi^*(\tau_1)\ldots\psi^*(\tau_{k-1})[\mathcal{K}_\alpha, \psi^*(\tau_k)] = [\mathcal{K}_\alpha, \psi^*(\tau_1)\ldots\psi^*(\tau_{k-1})]\psi^*(\tau_k) +$$

$$+ \frac{1}{(\alpha-1)!}\int_\Omega (K^{(\alpha)*}(\tau_k, \tau_1', \ldots, \tau_{\alpha-1}')\{\psi(\tau_{\alpha-1}')\ldots\psi(\tau_1')\psi(\tau_k)\ldots\psi(\tau_1)\})^*\cdot$$

$$\cdot\,\psi(\tau_{\alpha-1}')\ldots\psi(\tau_1')\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_{\alpha-1}',$$

and

$$(39)\quad \frac{1}{p!}\sum_{l_1,\ldots,l_{\alpha-p}=1,\ldots,k-1}^{l_1<\ldots<l_{\alpha-p}}\int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l_{\alpha-p}}, \tau_1', \ldots, \tau_p')\cdot$$

$$\cdot\{\psi(\tau_p')\ldots\psi(\tau_1')\psi(\tau_{k-1})\ldots\psi(\tau_1)\})^*\psi(\tau_p')\ldots\psi(\tau_1')\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_p'\psi^*(\tau_k) =$$

$$= \frac{1}{p!}\sum_{l_1,\ldots,l_{\alpha-p}=1,\ldots,k-1}^{l_1<\ldots<l_{\alpha-p}}\int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l_{\alpha-p}}, \tau_1', \ldots, \tau_p')\cdot$$

$$\cdot\{\psi(\tau_p')\ldots\psi(\tau_1')\psi(\tau_k)\ldots\psi(\tau_1)\})^*\psi(\tau_p')\ldots\psi(\tau_1')\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_p' +$$

$$+ \frac{1}{(p-1)!}\sum_{l_1,\ldots,l_{\alpha-p}=1,\ldots,k-1}^{l_1<\ldots<l_{\alpha-p}}\int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l_{\alpha-p}}, \tau_k, \tau_1', \ldots, \tau_{p-1}')\cdot$$

$$\cdot\{\psi(\tau_{p-1}')\ldots\psi(\tau_1')\psi(\tau_k)\ldots\psi(\tau_1)\})^*\psi(\tau_{p-1}')\ldots\psi(\tau_1')\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_{p-1}',$$

the validity of (36) for k follows from its validity for $k-1$. Therefore we have proven (36) for all the values of k and α. We get from (36) and (10) the following relation:

$$(40)\quad \mathcal{K}\chi_k(\tau_1, \ldots, \tau_k) = \psi^*(\tau_1)\ldots\psi^*(\tau_k)\mathcal{K}\chi_0 +$$

$$+ \sum_{\alpha=1}^{s \text{ or } k}\sum_{l_1,\ldots,l_\alpha=1,\ldots,k}^{l_1<\ldots<l_\alpha}(K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l_\alpha})\chi_k^*(\tau_1, \ldots, \tau_k))^* +$$

$$+ \sum_{\alpha=1}^{s \text{ or } k+1}\sum_{l_1,\ldots,l_{\alpha-1}=1,\ldots,k}^{l_1<\ldots<l_{\alpha-1}}\int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l\alpha-1}, \tau_1)\{\psi(\tau_1')\psi(\tau_k)\ldots\psi(\tau_1)\})^*\psi(\tau_1')\cdot$$

$$\cdot\,\chi_0\,\mathrm{d}\tau_1' + \ldots + \sum_{\alpha=1}^{s}\frac{1}{(\alpha-1)!}\int_\Omega (K^{(\alpha)*}(\tau_{l_1}, \tau_1', \ldots, \tau_{\alpha-1}')\{\psi((\tau_{\alpha-1}')\ldots\psi(\tau_1')\psi(\tau_k)\ldots\psi(\tau_1)\})^*\cdot$$

$$\cdot\,\psi(\tau_{\alpha-1}')\ldots\psi(\tau_1')\chi_0\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_{\alpha-1}'.$$

By taking into account (33*b*) we get

$$N_{\mathrm{op}}\psi(\tau)\chi_0 = -\psi(\tau)\chi_0 \tag{41}$$

and, since N_{op} has no negative eigenvalues, we have

$$\psi(\tau)\chi_0 = 0 \tag{42}$$

and, (40) can be simplified:

$$\mathcal{R}\chi_k(\tau_1, \ldots, \tau_k) = \sum_{\alpha=1}^{s \text{ or } k} \sum_{l_1,\ldots,l_\alpha=1,\ldots,k}^{l_1<\ldots<l_\alpha} (K^{(\alpha)*}(\tau_{l_1}, \ldots, \tau_{l_\alpha})\chi_k^*(\tau_1, \ldots, \tau_k))^* . \tag{43}$$

By introducing the expansion (9) into (8) and taking into account (43) we get the equations for the:

$$i\frac{\partial \Psi_0}{\partial t} = 0 , \tag{44a}$$

$$i\frac{\mathrm{d}\Psi_k}{\partial t} = \sum_{\alpha=1}^{k} \sum_{l_1,\ldots,l_\alpha=1,\ldots,k}^{l_1<\ldots<l_\alpha} K^{(\alpha)}(\tau_{l_1}, \ldots, \tau_{l_\alpha})\Psi_k(\tau_1, \ldots, \tau_k) , \qquad (k < s) \tag{44b}$$

$$i\frac{\partial \Psi_k}{\partial t} = K_k(\tau_1, \ldots, \tau_k)\Psi_k(\tau_1, \ldots, \tau_k) . \qquad (k \geqslant s) \tag{44c}$$

Until now we have not assumed that the $K^{(\alpha)}$ are hermitian. We made only use of the adjoints $K^{(\alpha)*}$ and of the relation

$$(K^{(\alpha)*})^* = K^{(\alpha)} . \tag{45}$$

The case of hermitian $K^{(\alpha)}$ is particularly important because of the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} |\Psi_k(t; \tau_1, \ldots, \tau_k)|^2 \,\mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k = 0 , \tag{46}$$

which allows to define a probability distribution of density $|\Psi_k(\tau_1, \ldots, \tau_k)|^2$. It is easily seen that the hermiticity of the $K^{(\alpha)}$ implies that of the $\mathcal{R}_\alpha$. It is thereby possible to introduce probability distributions attached to the solutions of (8), since:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mu\text{-space}} |\chi(t)|^2 \,\mathrm{d}\mu = 0 . \tag{47}$$

3. – We must now examine the methods of integration in the μ-space. The simplest way of introducing the integral in the μ-space consists in taking a discrete and complete orthonormal set of functions $\varphi_1(\tau)$, $\varphi_2(\tau)$, ... and expanding $\psi(\tau)$ and $\psi^*(\tau)$ as follows:

$$\psi(\tau) = \sum_{\lambda=1}^{\infty} a_\lambda \varphi_\lambda(\tau) , \qquad \psi^*(\tau) = \sum_{\lambda=1}^{\infty} a_\lambda^* \varphi_\lambda^*(\tau) . \tag{48}$$

The a and a^* have the commutation rules

$$[a_\lambda , a_{\lambda'}^*]_\pm = \delta_{\lambda,\lambda'} , \qquad [a_\lambda , a_{\lambda'}]_\pm = [a_\lambda^* , a_{\lambda'}^*]_\pm = 0 \tag{49}$$

and the operators N_λ

$$N_\lambda = a_\lambda^* a_\lambda , \tag{50}$$

have the eigenvalues 0 and 1 or $0, 1, 2, \ldots (\infty)$ when the signs in the commutation rules are $+$ and $-$, respectively, as well known (see the section 4 of reference (6)). The N_λ are a complete set of commutable operators and the functional χ can be taken as a function of their eigenvalues N'_λ. In this case we have simply:

$$\int_{\mu\text{-space}} \chi_1^* \chi_2 \, \mathrm{d}\mu = \sum_{N'} \chi_1^*(N'_1, N'_2, \ldots) \chi_2(N'_1, N'_2, \ldots) . \tag{51}$$

In this representation the wave functional of the vacuum is obviously:

$$\chi_0(N') = \prod_{\lambda=1}^{\infty} \delta_{N'_\lambda, 0} . \tag{52}$$

The $\chi_n(\tau_1, \ldots, \tau_n)$ can be expanded as follows:

$$\chi_n(\tau_1, \ldots, \tau_n) = \sum_{\lambda_1, \ldots, \lambda_n} \chi_{\lambda_1, \ldots, \lambda_n} \varphi_{\lambda_1}^*(\tau_1) \ldots \varphi_{\lambda_n}^*(\tau_n) , \tag{53}$$

$$\chi_{\lambda_1, \ldots, \lambda_n} = a_{\lambda_1}^* \ldots a_{\lambda_n}^* \chi_0 . \tag{54}$$

It is easily seen that

$$\chi^{(+)}_{\lambda_1, \ldots, \lambda_n} = \pm g_+(\lambda_1, \ldots, \lambda_n) \delta_{N'_{\lambda_1, 1}} \ldots \delta_{N'_{\lambda_n, 1}} \prod_{\lambda \neq \lambda_1, \ldots, \lambda_n} \delta_{N'_\lambda, 0} , \tag{55}$$

$g_+(\lambda_1, ..., \lambda_n)$ being equal to 1 when all the $\lambda_1, ..., \lambda_n$ are different and otherwise equal to 0. When the sign in the commutation rules is — there will in general be s_1 of the $\lambda_1, ..., \lambda_n$ equal to λ_1', s_2 equal to λ_2', ..., s_j equal to λ_j' with $s_1 + s_2 + ... + s_j = n$. In this case we have:

$$\chi^{(-)}_{\lambda_1,...,\lambda_n} = \sqrt{g_-(\lambda_1, ..., \lambda_n)}\, \delta_{N'_{\lambda_1'}, s_1} \delta_{N'_{\lambda_2'}, s_2} ... \delta_{N'_{\lambda_j'}, s_j} \prod_{\lambda \neq \lambda'} \delta_{N'_\lambda, 0}\,, \tag{56}$$

$$g_-(\lambda_1, ..., \lambda_n) = s_1!\, s_2! ... s_j!\,. \tag{57}$$

It is easily seen that the $\chi_n(\tau_1, ..., \tau_n)$ are normalized as follows:

$$(\chi_n^{(\pm)}(\tau_1, ..., \tau_n), \chi_{n'}^{(\pm)}(\tau_1', ..., \tau_n')) = \int_{\mu\text{-space}} \chi_n^{(\pm)*}(\tau_1, ..., \tau_n) \chi_{n'}^{(\pm)}(\tau_1', ..., \tau_n')\, \mathrm{d}\mu = \tag{58}$$

$$= n!\, \delta_n^{(\pm)}(\tau, \tau') \delta_{n,n'}\,,$$

$$\delta_n^{(-)}(\tau, \tau') = \frac{1}{n!} \sum_{\text{permut. } k} \delta(\tau_1 - \tau_{k_1}') \delta(\tau_2 - \tau_{k_2}') ... \delta(\tau_n - \tau_{k_n}')\,, \tag{59}$$

$$\delta_n^{(+)}(\tau, \tau') = \frac{1}{n!} \begin{vmatrix} \delta(\tau_1 - \tau_1') & \delta(\tau_1 - \tau_2') & .... & \delta(\tau_1 - \tau_n') \\ \delta(\tau_2 - \tau_1') & \delta(\tau_2 - \tau_2') & .... & \delta(\tau_2 - \tau_n') \\ \vdots & \vdots & & \vdots \\ \delta(\tau_n - \tau_1') & \delta(\tau_n - \tau_2') & .... & \delta(\tau_n - \tau_n') \end{vmatrix}. \tag{60}$$

It results from (9) and (58) that:

$$\int_{\mu\text{-space}} \chi^* \chi\, \mathrm{d}\mu = \Psi_0^* \Psi_0 + \sum_{n=1}^{\infty} \int_{\Omega} |\Psi_n(t; \tau_1, ..., \tau_n)|^2\, \mathrm{d}\tau_1 ... \mathrm{d}\tau_n\,, \tag{61}$$

and also that

$$\Psi_n(t; \tau_1, ..., \tau_n) = \frac{1}{\sqrt{n!}} \int_{\mu\text{-space}} \chi \chi_n^*(\tau_1, ..., \tau_n)\, \mathrm{d}\mu = \frac{1}{\sqrt{n!}} \int_{\mu\text{-s ace}} \chi_0^* \psi(\tau_n) ... \psi(\tau_1) \chi\, \mathrm{d}\mu\,. \tag{62}$$

By taking into account that $\Psi_n(t; \tau_1, ..., \tau_n)$ is either symmetrical or antisimmetrical we get:

$$\psi(\tau_1) \int_{\Omega} \Psi_n(t; \tau_1', ..., \tau_n') \psi^*(\tau_1') ... \psi^*(\tau_n') \chi_0\, \mathrm{d}\tau_1' ... \mathrm{d}\tau_n' = \tag{63}$$

$$= n \int_{\Omega} \Psi_n(t; \tau_1, \tau_1', ..., \tau_{n-1}') \psi^*(\tau_1') ... \psi^*(\tau_{n-1}') \chi_0\, \mathrm{d}\tau_1' ... \mathrm{d}\tau_{n-1}'\,.$$

Hence

$$\psi(\tau_2)\psi(\tau_1)\int_{\Omega} \Psi_n(t;\tau_1',\ldots,\tau_n')\psi^*(\tau_1')\ldots\psi^*(\tau_n')\chi_0\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_n' = \tag{64}$$

$$= n(n-1)\int_{\Omega} \Psi_n(t;\tau_1,\tau_2,\tau_1',\ldots,\tau_{n-2}')\psi^*(\tau_1')\ldots\psi^*(\tau_{n-2}')\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_{n-2}',$$

and after n steps we get:

$$\psi(\tau_n)\psi(\tau_{n-1})\ldots\psi(\tau_1)\int_{\Omega} \Psi_n(t;\tau_1',\ldots,\tau_n')\chi_n(\tau_1',\ldots,\tau_n')\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_n' = \tag{65}$$

$$= n!\,\Psi_n(t;\tau_1,\ldots,\tau_n)\chi_0\,.$$

By combining (65) and (17) we obtain equation (18).

The general solution of equation (1) can be expressed in terms of an operator $U_n(t)$ such that:

$$\Psi_n(t) = U_n(t-t_0)\Psi_n(t_0)\,, \tag{66}$$

$U_n(t)$ is determined by the equations:

$$i\frac{\mathrm{d}}{\mathrm{d}t}U_n(t) = K_n U_n(t) \qquad U_n(0) = 1\,. \tag{67}$$

In a similar way the general solution of (8) can be expressed in terms of an operator $\mathcal{U}(t)$ such that:

$$\chi(t) = \mathcal{U}(t-t_0)\chi(t_0)\,, \tag{68}$$

$\mathcal{U}(t)$ is determined by the equations:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t) = \mathcal{K}\mathcal{U}(t) \qquad \mathcal{U}(0) = 1\,. \tag{69}$$

When $n < s$ we shall take:

$$K_n(\tau_1,\ldots,\tau_n) = \sum_{\alpha=1}^{n}\frac{1}{\alpha!}\sum_{l_1,\ldots,l_\alpha}{}' K^{(\alpha)}(\tau_{l_1},\ldots,\tau_{l_\alpha})\,. \tag{70}$$

We shall now prove that the relation between $\mathcal{U}(t)$ and the $U_n(t)$ given in

reference (6) is also valid in the general case we are considering:

$$(71) \qquad \mathcal{U}(t) = P_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Omega} \psi^*(\tau_n) \dots \psi^*(\tau_1) U_n(t; \tau_1, \dots, \tau_n) \cdot$$

$$\cdot \{\psi(\tau_1) \dots \psi(\tau_n)\} \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n P_n \, .$$

The P_n are the projection operators on the linear manifolds of the χ-space corresponding to the eigenvalues n of N_{op}:

$$(72) \qquad N_{\mathrm{op}} P_n = n P_n \, , \qquad P_n P_{n'} = \delta_{nn'} P_n \, , \qquad \sum_{n=0}^{\infty} P_n = 1 \, .$$

We must prove that the right hand side of (71) satisfies the two equations (69). The proof regarding the second equation (69) is the same given in section 3 of reference (6). In order to show that the first equation (69) is satisfied, we shall use equation (36) and take into account that $N_{\mathrm{op}} P_n \chi$ is equal to $n P_n \chi$ for any χ, so that the product of more than n operators ψ applied to $P_n \chi$ is always 0, as a consequence of (33*b*). Thus we get:

$$(73) \qquad \frac{1}{n!} \mathcal{R} \int_{\Omega} \psi^*(\tau_n) \dots \psi^*(\tau_1) U_n(t; \tau_1, \dots, \tau_n) \{\psi(\tau_1) \dots \psi(\tau_n)\} \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n P_n \chi =$$

$$= \frac{1}{n!} \int_{\Omega} \psi^*(\tau_n) \dots \psi^*(\tau_1) K_n(\tau_1, \dots, \tau_n) U_n(t; \tau_1, \dots, \tau_n) \{\psi(\tau_1) \dots \psi(\tau_n)\} \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n P_n \chi =$$

$$= \frac{i}{n!} \frac{\partial}{\partial t} \int_{\Omega} \psi^*(\tau_n) \dots \psi^*(\tau_1) U_n(t; \tau_1, \dots, \tau_n) \{\psi(\tau_1) \dots \psi(\tau_n)\} \, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_n \, P_n \chi \, .$$

This equation shows that the series in the right hand side of (71) is a solution of the first equation (69). From (71) we get again the expansion (9) with:

$$(74) \qquad \Psi_n(t; \tau_1, \dots, \tau_n) = U_n(t; \tau_1, \dots, \tau_n) \Psi_n(0; \tau_1, \dots, \tau_n) \, ,$$

$$(75) \qquad \Psi_n(0; \tau_1, \dots, \tau_n) \chi_0 = \frac{1}{\sqrt{n!}} \psi(\tau_n) \dots \psi(\tau_1) P_n \chi(0) \, .$$

Equation (74) is equivalent to the set (44). Therefore equation (71) describes in the most complete way the relations between the « second quantization » formalism and the formalism in the configuration space.

We shall now complete the spectral decomposition of N_{op} by introducing

the projection operators $P(\tau_1, ..., \tau_n)$:

$$P(\tau_1, ..., \tau_n)\chi = \frac{1}{n!}\chi_n(\tau_1, ..., \tau_n)\int_{\mu\text{-space}} \chi_n^*(\tau_1, ..., \tau_n)\chi \, d\mu \,, \tag{76}$$

$$P(\tau_1, ..., \tau_n) = \frac{1}{n!}\psi^*(\tau_1) ... \psi^*(\tau_n)P_0\psi(\tau_n) ... \psi(\tau_1) \,, \tag{77}$$

$$P_n = \int_\Omega P(\tau_1, ..., \tau_n)\, d\tau_1 ... d\tau_n \,. \tag{78}$$

Equation (78) is easily proven by showing that the operator in its right hand side leaves invariant any eigenfunctional $\chi^{(n)}$ of N_{op} corresponding to the eigenvalue n and annihilates all the $\chi^{(n')}$ for $n' \neq n$. We get from (58) the orthogonality relations:

$$P(\tau_1, ..., \tau_n)P(\tau_1', ..., \tau_{n'}') = \delta_{n,n'}\delta_n(\tau, \tau')P(\tau_1, ..., \tau_n) \,. \tag{79}$$

It follows from (78) and the third equation (72) that:

$$P_0 + \sum_{n=1}^{\infty}\int_\Omega P(\tau_1, ..., \tau_n)\, d\tau_1 ... d\tau_n = 1 \,. \tag{80}$$

4. – The ψ^*-representation.

We shall now develop a special formalism for the case of commutation rules with sign minus. This corresponds to the usual second quantization for bosons in the quantum mechanics. The formalism we shall develop is related to the Fock theory of the harmonic oscillator ([8]), which was extended by DIRAC ([9]). Our treatment differs from that of the Dirac formalism by the definition of inner products of functions of a complex variable by means of two-dimensional real integrals, instead of the complex contour integrals used by Dirac. Our point of view corresponds to that more usually taken in the theory of orthogonal systems of functions of complex variables (see S. BERGMANN: *The kernel function and conformal mapping* (New York, 1950)).

The commutability of the $\psi^*(\tau)$ at all the points of Ω allows us to take the operator ψ^* as a multiplicative numerical factor. The commutation rules (3) are satisfied by taking $\psi(\tau)$ as the functional derivative with respect

to $\psi^*(\tau)$:

$$\{\psi^*(\tau)\}_{\mathrm{op}}\chi = \psi^*(\tau)\chi\,, \qquad \{\psi(\tau)\}_{\mathrm{op}}\chi = \frac{\delta\chi}{\delta\psi^*(\tau)}\,, \tag{81}$$

$$\frac{\delta}{\delta\psi^*(\tau)}\{\psi^*(\tau')\chi[\psi^*]\} - \psi^*(\tau')\frac{\delta\chi[\psi^*]}{\delta\psi^*(\tau)} = \delta(\tau-\tau')\chi[\psi^*]\,. \tag{82}$$

In this ψ^*-representation the χ are functionals of a complex function of real variables $\psi^*(\tau)$. This representation was already used by FOCK (7) in his work in quantum electrodynamics. We shall assume that the functionals χ can be expanded in Volterra series

$$\chi[\psi^*] = \chi[0] + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Omega}\left\{\frac{\delta^n\chi[\psi^*(\tau)]}{\delta\psi^*(\tau_1)\ldots\delta\psi^*(\tau_n)}\right\}_{(\psi^*=0)}\psi^*(\tau_1)\ldots\psi^*(\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\,, \tag{83}$$

which are the analogue of the Taylor series in functional analysis. The wave functional of the vacuum is simply a constant and, more generally, the eigenfunctionals of N_{op} corresponding to any integer eigenvalue n are polynomial functionals $\chi^{(n)}$ of order n:

$$\chi^{(n)}[\psi^*] = \frac{1}{n!}\int_{\Omega}\Phi_n(\tau_1,\ldots,\tau_n)\psi^*(\tau_1)\ldots\psi^*(\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n, \tag{84}$$

$$\psi(\tau)\chi^{(n)}[\psi^*] = \frac{\delta\chi^{(n)}}{\delta\psi^*(\tau)} = \frac{1}{(n-1)!}\int_{\Omega}\Phi_n(\tau,\tau_1,\ldots,\tau_{n-1})\psi^*(\tau_1)\ldots\psi^*(\tau_{n-1})\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_{n-1}\,, \tag{85}$$

$$N_{\mathrm{op}}\chi^{(n)}[\psi^*] = \int_{\Omega}\psi^*_{\mathrm{op}}(\tau)\psi_{\mathrm{op}}(\tau)\,\mathrm{d}\tau\chi^{(n)}[\psi^*] = n\chi^{(n)}[\psi^*]. \tag{86}$$

The Volterra series (83) can be written as follows

$$\chi[\psi^*] = \chi[0] + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Omega}\Phi_n(\tau_1,\ldots\tau_n)\psi^*(\tau_1)\ldots\psi^*(\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\,, \tag{87}$$

with

$$\Phi_n(\tau_1,\ldots,\tau_n) = \left\{\frac{\delta^n\chi[\psi^*(\tau)]}{\delta\psi^*(\tau_1)\ldots\delta\psi^*(\tau_n)}\right\}_{(\psi^*=0)}\,, \tag{88}$$

and by taking into account (86) we see that the Volterra series is essentially the same as the Fock expansion (9), for commutation rules (3) with the sign minus.

The ψ^*-representation is not of the kind ordinarily considered in quantum mechanics, because the basic operators $\psi^*(\tau)$ are not hermitian. Thereby methods of the theory of the orthogonal functions of complex variables must be employed. By introducing the expansion (48) of $\psi^*(\tau)$, χ becomes a function of the complex variables a_λ^*. We shall use the following definition of the inner product of two functions G_1 and G_2 of n complex variables $z_1, \ldots, z_n$:

$$(89) \qquad (G_1(z_1, \ldots, z_n),\ G_2(z_1, \ldots, z_n)) = \\ = \pi^{-n} \int_{-\infty}^{+\infty} \{G_1(z_1, \ldots, z_n)\}^* G_2(z_1, \ldots, z_n) \exp\left[-\sum_{l=1}^{n} |z_l|^2\right] \mathrm{d}x_1\, \mathrm{d}y_1 \ldots \mathrm{d}x_n\, \mathrm{d}y_n ,$$

$$(89a) \qquad x_l = \mathrm{R}z_l , \qquad y_l = \mathrm{Im}\, z_l .$$

It follows from (89) that

$$(90) \qquad (z_{l_1}^{r_1}, z_{l_2}^{r_2}) = r_1!\, \delta_{l_1, l_2} \delta_{r_1, r_2} ,$$

so that the functions $(r_1! \ldots r_n!)^{-1/2} z_1^{r_1} \ldots z_n^{r_n}$ constitute a complete orthonormal set. It is convenient to write:

$$(91) \qquad (G_1(z_1, \ldots, z_n),\ G_2(z_1, \ldots, z_n)) = \int_{-\infty}^{+\infty} \{G_1(z_1, \ldots, z_n)\}^* G_2(z_1, \ldots, z_n)\, \mathrm{d}z_1 \ldots \mathrm{d}z_n .$$

The above method of definition of the inner product can be immediately extended to the case of functions depending on a countable infinity of independent complex variables, such as the wave functionals $\chi[\psi^*]$. By introducing the expansion (48) into (87) we get:

$$(92) \qquad \chi[\psi^*] = A_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda_1, \ldots, \lambda_n} A_{\lambda_1, \ldots, \lambda_n} a_{\lambda_1}^* \ldots a_{\lambda_n}^* ,$$

$$(92a) \qquad A_0 = \chi[0] , \qquad A_{\lambda_1 \ldots, \lambda_n} = \int_{\Omega} \Phi_n(\tau_1, \ldots, \tau_n) \varphi_{\lambda_1}^*(\tau_1) \ldots \varphi_{\lambda_n}^*(\tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n .$$

The inner product of two functionals χ_1 and χ_2 is:

$$(93) \qquad (\chi_1, \chi_2) = A_0^{(1)*} A_0^{(2)} + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda_1, \ldots, \lambda_n} A_{\lambda_1, \ldots, \lambda_n}^{(1)*} A_{\lambda_1, \ldots, \lambda_n}^{(2)} .$$

The $A_{\lambda_1,\ldots,\lambda_n}$ are symmetrical with respect to the indices, so that there are $n!/g_-(\lambda_1, \ldots, \lambda_n)$ terms in the series in (92), for each set of values of the indices, $g_-(\lambda_1, \ldots, \lambda_n)$ being defined by (57). There is a cancellation of two factors g_- which leads to (93). In particular the normalization condition is:

$$(94) \qquad (\chi, \chi) = |A_0|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\lambda_1,\ldots,\lambda_n} |A_{\lambda_1\ldots,\lambda_n}|^2 = \\ = |\chi(0)|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Omega} |\Phi_n(\tau_1, \ldots, \tau_n)|^2 \, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n .$$

The inner product defined by (93) may be considered as obtained by a special kind of integration over the variable ψ^*:

$$(95) \qquad (\chi_1[\psi^*], \chi_2(\psi^*]) = \int \{\chi_1[\psi^*]\}^* \chi_2[\psi^*] \, \mathrm{d}\psi^*,$$

$\{G(z_1, \ldots, z_n)\}^*$ is an analytic function of $z_1^*, \ldots, z_n^*$ when $G(z_1, \ldots, z_n)$ is an analytic function of $z_1, \ldots, z_n$. In a similar way $\{\chi[\psi^*]\}^*$ will be treated as a functional of $\psi(\tau)$. The matrix elements of the linear operators in the ψ^*-representation will be denoted by $\langle \psi^* | \mathcal{A} | \psi' \rangle$, in order that the « index » at the right hand side behave as the argument of a χ^* and the « index » at the left hand side as the argument of a χ. Whenever an integration of the special kind (95) occurs there are two arguments in the integrated quantity which behave as a ψ^* and a ψ. Thus the trace of an operator $\mathcal{A}$ will be the integral $\int \langle \psi^* | \mathcal{A} | \psi \rangle \mathrm{d}\psi^*$.

The ordinary rules of matrix multiplication must be slightly modified in the ψ^*-representation, in order to take into account that the summation over the « indices » is a $\mathrm{d}\psi^*$ integration, in which in one of the factors appears an argument ψ and in the other factor an argument ψ^*. The situation is analogous to that of the tensor calculus in which the summations affect a lower index and an equal higher index.

5. – The Gibbs second quantization.

We shall now consider a quantum mechanical system Σ whose wave function depends on a variable point τ in a certain space Ω. If the Σ system is formed by n spinless particles, its wave function will depend on a point τ of a $3n$ dimensional euclidean space. In the case of a quantized field, the wave functional of the field may be considered as a function of a discrete infinity of variables, τ will then be a point in a suitable sequence space. We shall

assume, as we did until now, that we know how to define an integral over the space Ω, as happens in all the cases of practical importance. The hamiltonian of the system divided by $\hbar$ will be denoted by K. K is a hermitian operator.

In order to get the special kind of second quantization which will be called the Gibbs second quantization, we shall apply the mathematical procedure developed in the preceding sections taking

$$K_1 = K\,, \qquad K_n = 0\,, \qquad \text{for } n > 1 \tag{96}$$

with either of the signs in the commutation rules (3). The operator $\mathcal{H}$ is now simply

$$\mathcal{H} = \int_{\Omega} \psi^*(\tau) K \psi(\tau)\, \mathrm{d}\tau\,, \tag{97}$$

and the equation of type (8) for the wave functional χ becomes:

$$i\frac{\mathrm{d}\chi}{\mathrm{d}t} = \int_{\Omega} \psi^*(\tau) K \psi(\tau)\, \mathrm{d}\tau\, \chi\,. \tag{98}$$

The Fock expansion (9) is

$$\chi(t) = \Psi_0 \chi_0 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}} \int_{\Omega} \Psi_n(t;\, \tau_1, \ldots, \tau_n) \chi_n(\tau_1, \ldots, \tau_n)\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n\,, \tag{99}$$

the $\Psi_n(t;\, \tau_1 \ldots \tau_n)$ being solutions of the Schrödinger equations for n similar non interacting systems:

$$i\frac{\partial}{\partial t} \Psi_n(t;\, \tau_1, \ldots, \tau_n) = \sum_{l=1}^{n} K(\tau_l) \Psi_n(t;\, \tau_1, \ldots, \tau_n)\,. \tag{100}$$

It results from (58) and (99) that:

$$(\chi(t),\, \chi(t)) = |\Psi_0|^2 + \sum_{n=1}^{\infty} \int_{\Omega} |\Psi_n(t;\, \tau_1, \ldots, \tau_n)|^2\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n\,. \tag{101}$$

The structure of (99) and (101) shows that $\chi(t)$ describes a grand ensemble $\mathcal{E}(t)$ of systems Σ, i.e. a set whose elements are assemblies α of systems Σ. We can imagine a random extraction of elements α of the set $\mathcal{E}(t)$ with a

probability $\mathcal{P}(t; n)$ of extracting a n-system assembly at the time t. We shall introduce the following rule of physical interpretation:

(I) *The probability of extracting a n-system assembly in a random choice at the time t from the grand ensemble $\mathfrak{E}$ described by the normalized wave functional $\chi(t)$ is given by* $(\chi(t), P_n\chi(t)) = \int_{\Omega} |\Psi_n(t; \tau_1, ..., \tau_n)|^2 \mathrm{d}\tau_1 ... \mathrm{d}\tau_n$, *$P_n$ being the projection operator of N_{op} corresponding to the eigenvalue n. The wave function of the n-system assembly is $\Psi_n(t; \tau_1, ..., \tau_n)$.*

Having extracted an assembly of n-sytems from $\mathfrak{E}(t)$, we can make measurements of the physical quantities of the systems of the assembly, which are not interacting, as shown by equations (100). The probabilities of the different possible values of a quantity A are given by the ordinary interpretation rules of the quantum mechanics. For instance, when it is possible to define an element of volume in Ω, the probability of finding the n systems of the assembly in the elements of volume $\mathrm{d}\tau_1 ..., \mathrm{d}\tau_n$ is $|\Psi_n(t; \tau_1, ..., \tau_n)|^2 \mathrm{d}\tau_1 ... \mathrm{d}\tau_n / \mathcal{P}(t; n)$.

The wave function $\Psi_n(t; \tau_1, ..., \tau_n)$ of the assembly is either symmetrical in the $\tau_1, ..., \tau_n$, when the sign minus is taken in the commutation rules (3), or anti-symmetrical, when the sign plus is taken. In both cases $|\Psi_n(t; \tau_1, ..., \tau_n)|^2$ is symmetrical and the n systems are not distinguished.

Let us consider an hermitian operator A describing some physical quantity of a system Σ. We shall introduce now the spectral decomposition of A:

$$A = \sum_{A'} A' p_{A'} \qquad \left(\sum_{A'} p_{A'} = 1\,, \qquad p_{A'} p_{A''} = p_{A'} \delta_{A'A''}\right). \tag{102}$$

The A' are the eigenvalues of A and the $p_{A'}$ the corresponding projection operators. The probability of finding the values $A'_1, ..., A'_n$ in measurements made on the n systems of the assembly at the time t is

$$\int_{\Omega} \Psi_n^*(t; \tau_1, ..., \tau_n) p_1(\tau_1) ... p_n(\tau_n) \Psi(t; \tau_1, ..., \tau_n) \mathrm{d}\tau_1 ... \mathrm{d}\tau_n / \mathcal{P}(t; n)$$

according to the rules of the quantum mechanics. For the purposes of the statistical quantum mechanics it is convenient to consider the expectation value of the number of systems in which a measurement of a quantity A at the time t gives the values A'. In this expectation value there is of course an averaging with respect to the possible number of systems in the assemblies α of the grand ensemble $\mathfrak{E}(t)$. We shall prove that this expectation value is $(\chi(t), \mathcal{P}_{A'}\chi(t))$, with a normalized $\chi(t)$, as a consequence of the interpretation rule (I), with

$$\mathcal{P}_{A'} = \int_{\Omega} \psi^*(\tau) p_{A'} \psi(\tau) \mathrm{d}\tau\,. \tag{103}$$

Indeed, we have

$$(104)\qquad (\chi(t), \mathcal{P}_{A'}\chi(t)) = \sum_{n=0}^{\infty}(\chi(t), \mathcal{P}_{A'}P_n\chi(t)) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Omega} \Psi_n^*(t;\tau_1,\ldots,\tau_n)\Psi_n(t;\tau_1',\ldots,\tau_n')\cdot$$

$$\cdot\Big(\chi_n(\tau_1,\ldots,\tau_n), \int_{\Omega}\psi^*(\tau)p_{A'}\psi(\tau)\,\mathrm{d}\tau\,\chi_n(\tau_1',\ldots,\tau_n')\Big)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_n' =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Omega}\Big\{\Psi_n^*(t;\tau_1,\ldots,\tau_n)\sum_{l=1}^{n}p_{A'}(\tau_l')\Psi_n(t;\tau_1',\ldots,\tau_n')\Big\}\cdot$$

$$\cdot(\chi_n(\tau_1,\ldots,\tau_n), \chi_n(\tau_1',\ldots,\tau_n'))\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n\,\mathrm{d}\tau_1'\ldots\mathrm{d}\tau_n' =$$

$$= \sum_{n=1}^{\infty}\int_{\Omega}\Psi_n^*(t;\tau_1,\ldots,\tau_n)\sum_{l=1}^{n}p_{A'}(\tau_l)\Psi_n(t;\tau_1,\ldots,\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n.$$

In the derivation of (104) we used the following formula: ,

$$(105)\qquad \mathcal{B}\chi_n(\tau_1,\ldots,\tau_n) = \sum_{l=1}^{n}\{B(\tau_l)\chi_n^*(\tau_1,\ldots,\tau_n)\}^*$$

$$(106)\qquad \mathcal{B} = \int_{\Omega}\psi^*(\tau)B(\tau)\psi(\tau)\,\mathrm{d}\tau \qquad (B = B^*).$$

It follows from the commutation rules (3) that:

$$(107)\qquad [\mathcal{B}, \psi^*(\tau_1)\ldots\psi^*(\tau_n)] = \{B(\tau_1)\psi(\tau_1)\}^*\psi^*(\tau_2)\ldots\psi^*(\tau_n) +$$

$$+ \psi^*(\tau_1)\{B(\tau_2)\psi(\tau_2)\}^*\psi^*(\tau_3)\ldots\psi^*(\tau_n) + \psi^*(\tau_1)\ldots\psi^*(\tau_{n-1})\{B(\tau_n)\psi(\tau_n)\}^* =$$

$$= \Big\{\sum_{l=1}^{n}B(\tau_l)\psi(\tau_n)\ldots\psi(\tau_1)\Big\}^*.$$

Equation (105) follows immediately from (107) and (10). We get from (107)

$$(108)\qquad \mathcal{B}\chi(t) = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}\int_{\Omega}\Big\{\sum_{l=1}^{n}B(\tau_l)\Psi_n(t;\tau_1,\ldots,\tau_n)\Big\}\chi_n(\tau_1,\ldots,\tau_n)\,\mathrm{d}\tau_1\ldots,\mathrm{d}\tau_n.$$

and thus we obtain a formula more general than (104)

$$(109)\qquad (\chi(t), \mathcal{B}\chi(t)) = \sum_{n=1}^{\infty}\int_{\Omega}\Psi_n^*(t;\tau_1,\ldots,\tau_n)\sum_{l=1}^{n}B(\tau_l)\Psi_n(t;\tau_1,\ldots,\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n.$$

It results from (102) and the above derived expectation value of the number of systems in which the measurement gives a value A' of A, that the expectation value of a quantity A in the state of the grand ensemble described by the normalized $\chi(t)$ is $(\chi(t), \mathcal{A}\chi(t))$, $\mathcal{A}$ being the χ-operator formed with A by the rule (106). Thus we get the second rule of physical interpretation:

(II) *The expectation value of a physical quantity A of the system Σ in the state of the grand ensemble $\mathfrak{E}(t)$ described by the normalized wave functional $\chi(t)$ is* $(\chi(t), \mathcal{A}\chi(t))$.

The $\mathcal{P}_{A'}$ are closely related to the operators N_λ defined by (50). Indeed, let the p_λ be the projection operators associated to the functions $\varphi_\lambda(\tau)$ of a complete orthonormal set

$$p_\lambda \Psi(\tau) = \varphi_\lambda(\tau) \int_\Omega \varphi_\lambda^*(\tau')\Psi(\tau')\,d\tau', \tag{110}$$

we have

$$\mathcal{P}_\lambda = \int_\Omega \psi^*(\tau) p_\lambda \psi(\tau)\,d\tau = \sum_{\lambda',\lambda''} a_{\lambda'}^* a_{\lambda''} \int_\Omega \varphi_{\lambda'}^*(\tau) p_\lambda \varphi_{\lambda''}(\tau)\,d\tau = a_\lambda^* a_\lambda, \tag{111}$$

hence:

$$\mathcal{P}_\lambda = N_\lambda\,. \tag{112}$$

When the eigenvalues of A are degenerated, the corresponding projection operators $p_{A'}$ are sums of one dimensional projection operators $p_{A'}^{(\alpha)}$ corresponding to a complete set of orthonormal functions for the eigenvalue A':

$$p_{A'} = \sum_\alpha p_{A'}^{(\alpha)} \qquad p_{A'}^{(\alpha)} p_{A'}^{(\beta)} = p_{A'}^{(\alpha)} \delta_{\alpha,\beta}\,. \tag{113}$$

$$\mathcal{P}_{A'} = \sum_\alpha \int_\Omega \psi^*(\tau) p_{A'}^{(\alpha)} \psi(\tau) d\tau = \sum_\alpha N_{A'}^{(\alpha)}\,. \tag{114}$$

It is convenient to use the following notation:

$$N_{A'} = \mathcal{P}_{A'}\,. \tag{115}$$

Thus we have:

$$\mathcal{A} = \sum_{A'} A' N_{A'}\,. \tag{116}$$

The $N_{A'}^{(\alpha)}$ are of the same kind as the N_λ defined by (50). They have the eigenvalues 0 and 1 or $0, 1, 2, \ldots (\infty)$, for commutation rules with the signs plus and minus, respectively. Thereby the eigenvalues of the $N_{A'}$ are always positive integers or 0. The $N_{A'}^{(\alpha)}$ are operators for the occupation numbers of the pure states $\varphi_{A'}^{(\alpha)}$. $N_{A'}$ is the operator for the occupation number of the mixed state corresponding to the projection operator (or von Neumann density operator) $p_{A'}$. We shall call $\mathcal{A}$ the χ-operator for the physical quantity A. Equation (116) shows in a suggestive way the relation between the Gibbs second quantization and the ordinary quantum mechanics.

In the ordinary form of the statistical quantum mechanics, the condition of a system Σ is described by a von Neumann density operator R

$$R = \sum_\lambda w_\lambda p_\lambda , \qquad w_\lambda \geqslant 0 \tag{117}$$

the p_λ being the one-dimensional projection operators corresponding to the orthonormal states φ_λ and the w_λ their weights in the mixture R. It is often convenient to normalize these weights:

$$\sum_\lambda w_\lambda = 1 . \tag{118}$$

We can generalize the definition of the $N_{A'}$ in order to get an operator for the occupation number of a mixed state R:

$$N_R = \int_\Omega \psi^*(\tau) R \psi(\tau)\, \mathrm{d}\tau . \tag{119}$$

The Schrödinger equation for the pure states of Σ is replaced by the von Neumann equation in the case of mixed states:

$$i\hbar \frac{\mathrm{d}R}{\mathrm{d}t} = [H, R] \qquad (H = \hbar K) . \tag{120}$$

We get from (120):

$$i \frac{\mathrm{d}}{\mathrm{d}t} N_{R(t)} = \int_\Omega \psi^*(\tau)[K, R(t)]\psi(\tau)\, \mathrm{d}\tau = [\mathcal{K}, N_{R(t)}] . \tag{121}$$

In the derivation of (121) we used the formula:

$$[\mathcal{A}, \mathcal{B}] = \int_\Omega \psi^*(\tau)[A, B]\psi(\tau)\, \mathrm{d}\tau . \tag{122}$$

It follows from the comutation rules (3) that:

$$\mathcal{AB} = \int_{\Omega} \psi^*(\tau)A(\tau)\psi(\tau)\psi^*(\tau')B(\tau')\psi(\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau' = \tag{123}$$

$$= \int_{\Omega} \psi^*(\tau)A(\tau)B(\tau)\psi(\tau)\,\mathrm{d}\tau + \int_{\Omega} \psi^*(\tau)\psi^*(\tau')A(\tau)B(\tau')\psi(\tau')\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau'.$$

Equation (122) follows immediately from (123). Equation (122) shows that the χ-operators corresponding to commutable quantities are also commutable. In particular we have:

$$[\mathcal{P}_{A'}, \mathcal{P}_{A''}] = 0\,. \qquad (A', A'' = \text{eigenvalues of } A). \tag{124}$$

The unitary operator of the motion of the Gibbs second quantization for a time independent K is:

$$\mathcal{U}(t) = \exp[-it\mathcal{K}] = \prod_{K'} \exp[-itK'\mathcal{P}_{K'}]\,. \tag{125}$$

The operators which commute with $\mathcal{K}$ are the integrals of the motion. Their expectation values are time independent:

$$(\chi(t), \mathcal{A}\chi(t)) = (\chi(0), \exp[it\mathcal{K}]\mathcal{A}\exp[-it\mathcal{K}]\chi(0)) = (\chi(0), \mathcal{A}\chi(0))\,. \tag{126}$$

In particular the χ-operators $\mathcal{A}$ of the quantities A which commute with K are integrals of the motion since:

$$[\mathcal{A}, \mathcal{K}] = \int_{\Omega} \psi^*(\tau)[A, K]\psi(\tau)\,\mathrm{d}\tau = 0\,. \tag{127}$$

The occupation number operators corresponding to the eigenvalues of an operator which commutes with K are integrals of the motion of the Gibbs second quantization, because they correspond to the $p_{A'}$ and the $p_{A'}$ are commutable with K:

$$[N_{A'}, \mathcal{K}] = 0 \quad \text{when} \quad [A, K] = 0\,. \tag{128}$$

It is important to notice that the $N^{(\alpha)}_{A'}$ in general are not integrals of the motion' since in general K and $p^{(\alpha)}_{A'}$ are not commutable.

We shall consider now the states of the form:

$$\chi(t) = \int_{\Omega} \Psi(t;\tau)\chi_1(\tau)\,\mathrm{d}\tau\,, \quad (\chi(t), \chi(t)) = \int_{\Omega} |\Psi(t;\tau)|^2\,\mathrm{d}\tau = 1\,. \tag{129}$$

These χ are eigenfunctionals of N_{op} corresponding to the eigenvalue 1. In this particular case we have:

$$(130) \quad (\chi(t), N_\lambda \chi(t)) = \int_\Omega \Psi^*(t;\tau) p_\lambda \Psi(t;\tau)\,\mathrm{d}\tau =$$

$$= \int_\Omega \Psi^*(0;\tau)\exp[itK] p_\lambda \exp[-itK]\Psi(0;\tau)\,\mathrm{d}\tau =$$

$$= \int_\Omega \Psi^*(0;\tau)\exp[itK(\tau)]\varphi_\lambda(\tau)\,\mathrm{d}\tau \int_\Omega \varphi_\lambda^*(\tau')\exp[-itK(\tau')]\Psi(0;\tau')\,\mathrm{d}\tau' =$$

$$= \left| \int_\Omega \varphi_\lambda^*(\tau)\exp[-itK]\,\Psi(0;\tau)\,\mathrm{d}\tau \right|^2 .$$

The expectation value of N_λ at the time t is now simply the transition probability of Σ from the pure state $\Psi(0;\tau)$ to the state $\varphi_\lambda(\tau)$. In particular when $\Psi(0;\tau) = \varphi_{\lambda_0}(\tau)$

$$(131a) \qquad (\chi(0), N_\lambda \chi(0)) = \delta_{\lambda,\lambda_0} .$$

We see that $\Psi(0;\tau)$ is simply the wave function of Σ at the time 0 and $\Psi(t,\tau)$ its wave function at the time t.

The above results show that the Gibbs second quantization may be considered as a statistical generalization of the quantum mechanics. In the next section it will be seen that the results usually obtained with the von Neumann density operators can also be derived with the Gibbs second quantization.

6. – The Gibbs second quantization and the von Neumann statistical formalism.

Let us consider a complete orthonormal set of functions φ_λ. In the representation in which the N_λ are diagonal, the wave functional χ is a function of the eigenvalues N'_λ of the N_λ. We shall now examine especially the wave functionals of the form:

$$(132) \qquad \chi(0, N') = \prod_\lambda \delta_{N'_\lambda, n_\lambda} ,$$

$\chi(t; N')$ describes the motion of an assembly of non interacting systems Σ, such that at the time 0 there are n_λ systems in each of the states φ_λ. In the von Neumann statistical formalism the same assembly of systems is described by the operator $R(t)$ satisfying the equation of motion (120) and having the

initial value:

$$R(0) = \sum_{\lambda} n_{\lambda} p_{\lambda} . \tag{133}$$

The wave functionals $\chi(t)$ which take the form (132) at one instant of time are not the only solutions of (98). Thus we see that the Gibbs second quantization is a more general statistical formalism than that of the von Neumann assemblies.

In order to see more clearly the equivalence of the above class of solutions of (98) and the assembly of systems defined by $R(t)$, we shall compute the expectation value $(\chi(t), N_{\lambda}\chi(t))$. We have:

$$(\chi(t), \mathscr{A}\chi(t)) = (\chi(0), \mathscr{U}^{-1}(t)\mathscr{A}\mathscr{U}(t)\chi(0)) = (\chi(0), \mathscr{A}(t)\chi(0)) , \tag{134}$$

$$\mathscr{A}(t) = \mathscr{U}^{-1}(t)\mathscr{A}\mathscr{U}(t) . \tag{135}$$

$\mathscr{A}(t)$ is the time dependent operator which corresponds to $\mathscr{A}$ in the Heisenberg representation of the Gibbs second quantization. In particular when $\mathscr{A}$ corresponds to the quantity A we have

$$\mathscr{A}(t) = \int_{\Omega} \psi^*(\tau) A(t) \psi(\tau) \, \mathrm{d}\tau , \tag{136}$$

$A(t)$ being the operator for the quantity A in the Heisenberg representation of Σ:

$$A(t) = U^{-1}(t) A U(t) . \tag{137}$$

$U(t)$ denotes the unitary operator of the motion of Σ, which was denoted by $U_1(t)$ in the notation used in equation (67). Equation (136) is obviously satisfied at the time 0. Since

$$i \frac{\mathrm{d}}{\mathrm{d}t} \mathscr{A}(t) = [\mathscr{A}(t), \mathscr{K}] , \tag{138}$$

and

$$i \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \psi^*(\tau) A(t) \psi(\tau) \, \mathrm{d}\tau = \tag{139}$$

$$= \int_{\Omega} \psi^*(\tau) [A(t), K] \psi(\tau) \, \mathrm{d}\tau = \left[\int_{\Omega} \psi^*(\tau) A(t) \psi(\tau) \, \mathrm{d}\tau, \mathscr{K} \right] ,$$

equation (136) is proven for all the values of t. It results from (111), (112) and (136) that

$$N_\lambda(t) = \int_\Omega \psi^*(\tau) p_\lambda(t) \psi(\tau)\, d\tau = \sum_{\lambda',\lambda''} a^*_{\lambda'} a_{\lambda''} \int_\Omega \varphi^*_{\lambda'}(\tau) p_\lambda(t) \varphi_{\lambda''}(\tau)\, d\tau \tag{140}$$

and since

$$\int_\Omega \varphi^*_{\lambda'}(\tau) p_\lambda(t) \varphi_{\lambda''}(\tau)\, d\tau = \int_\Omega \varphi^*_{\lambda'}(\tau) U^{-1}(t) \varphi_\lambda(\tau)\, d\tau \cdot \int_\Omega \varphi^*_\lambda(\tau) U(t) \varphi_{\lambda''}(\tau) d\tau, \tag{141}$$

we have:

$$N_\lambda(t) = \sum_{\lambda',\lambda''} a^*_{\lambda'} a_{\lambda''} \langle \lambda' | U^{-1}(t) | \lambda \rangle \langle \lambda | U(t) | \lambda'' \rangle . \tag{142}$$

$\langle \lambda' | B | \lambda'' \rangle$ denotes a matrix element of a Σ-operator:

$$\langle \lambda' | B | \lambda'' \rangle = \int_\Omega \varphi^*_{\lambda'}(\tau) B \varphi_{\lambda''}(\tau)\, d\tau . \tag{143}$$

The operator $U(t)$ being unitary, $U^{-1} = U^*$ and we have:

$$N_\lambda(t) = \sum_{\lambda',\lambda''} a^*_{\lambda'} a_{\lambda''} \langle \lambda | U(t) | \lambda' \rangle^* \langle \lambda | U(t) | \lambda'' \rangle . \tag{144}$$

It follows from (134) and (144) that:

$$(\chi(t; N'), N_\lambda \chi(t; N')) = \sum_{\lambda'} n_{\lambda'} |\langle \lambda | U(t) | \lambda' \rangle|^2 . \tag{145}$$

The expectation value of the number of systems in the assembly described by $R(t)$ in which a measurement of a quantity $F = \sum_\lambda F'_\lambda p_\lambda$, with non degenerated eigenvalues, gives the value F'_λ is:

$$\text{Trace}\{p_\lambda R(t)\} = \langle \lambda | R(t) | \lambda \rangle . \tag{146}$$

It follows from (120) that:

$$R(t) = U(t) R(0) U^{-1}(t) = \sum_{\lambda'} n_{\lambda'} U(t) p_{\lambda'} U^{-1}(t) . \tag{147}$$

Thereby we have:

$$\langle\lambda|R(t)|\lambda\rangle = \sum_{\lambda'} n_{\lambda'} \int_{\Omega} \varphi_{\lambda}^{*}(\tau) U(t) p_{\lambda'} U^{*}(t) \varphi_{\lambda}(\tau)\, \mathrm{d}\tau = \tag{148}$$

$$= \sum_{\lambda'} n_{\lambda'} |\langle\lambda|U(t)|\lambda'\rangle|^2 = (\chi(t; N'), N_{\lambda}\chi(t; N'))\,.$$

This equation shows that the description of the motion of the assembly given by the von Neumann operator $R(t)$ is equivalent to that given by $\chi(t; N')$.

The $U(t)p_{\lambda}U^{-1}(t)$ are the projection operators corresponding to the wave functions $U(t)\varphi_{\lambda}$. Therefore (147) gives the spectral decomposition of $R(t)$. In the assembly described by $R(t)$ there are n_{λ} systems Σ in each of the states $U(t)\varphi_{\lambda}$. Since

$$\int_{\Omega} \psi^{*}(\tau) p_{\lambda}(-t)\psi(\tau)\, \mathrm{d}\tau = N_{\lambda}(-t) = \mathcal{U}(t) N_{\lambda} \mathcal{U}^{-1}(t)\,, \tag{149}$$

and

$$\int_{\Omega} \psi^{*}(\tau) U(t) p_{\lambda} U^{-1}(t) \psi(\tau)\, \mathrm{d}\tau\, \chi(t; N') = \mathcal{U}(t) N_{\lambda} \chi(0; N') = n_{\lambda}\chi(t; N')\,, \tag{150}$$

there are n_{λ} systems Σ in each of the states $U(t)\varphi_{\lambda}$ in the assembly described by $\chi(t; N')$. This shows clearly the reason of the equivalence of the two descriptions of the assembly.

It results from (138) that:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t; \tau) = K(\tau)\psi(t; \tau)\,, \tag{151a}$$

$$-i\frac{\mathrm{d}}{\mathrm{d}t}\psi^{*}(t; \tau) = \{K(\tau)\psi(t; \tau)\}^{*}\,. \tag{151b}$$

$\psi(t; \tau)$ has the same equation of motion as the wave function of Σ, hence:

$$\psi(t; \tau) = U(t)\psi(\tau)\,. \tag{152}$$

It is easily seen that as a consequence of equations (151) we have:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\{\psi^{*}(t; \tau')\psi(t; \tau'')\} = K(\tau'')\psi^{*}(t; \tau')\psi(t; \tau'') - \{K(\tau')\psi^{*}(t; \tau'')\psi(t; \tau')\}^{*}\,. \tag{153}$$

The matrix elements of the von Neumann operators $R(t)$ in the representation

in which the τ are diagonal have an equation of motion of the same form as (153):

$$(154) \qquad i\frac{\mathrm{d}}{\mathrm{d}t}\langle\tau''|R(t)|\tau'\rangle = K(\tau'')\langle\tau''|R(t)|\tau'\rangle - \{K(\tau')\langle\tau'|R(t)|\tau''\rangle\}^* .$$

Let us introduce the operator $\varrho(\tau', \tau'')$:

$$(155) \qquad \varrho(\tau', \tau'') = \psi^*(\tau')\psi(\tau''), \qquad \varrho(t; \tau', \tau'') = \psi^*(t; \tau')\psi(t; \tau'') .$$

We shall now prove that:

$$(156) \quad \langle\tau''|R(t)|\tau'\rangle = (\chi(t; N'), \varrho(\tau', \tau'')\chi(t; N')) = (\chi(0; N'), \varrho(t; \tau', \tau'')\chi(0; N')) .$$

The above results show that the expectation values in (156) satisfy (154), so that it is sufficient to prove (156) for $t = 0$. We have

$$(157) \quad (\chi(0, N'), \varrho(\tau', \tau'')\chi(0, N')) = \sum_{\lambda', \lambda''}(\chi(0, N'), a^*_{\lambda'}a_{\lambda''}\chi(0, N'))\cdot\varphi^*_{\lambda'}(\tau')\varphi_{\lambda''}(\tau'') =$$

$$= \sum_{\lambda'} n_{\lambda'}\varphi^*_{\lambda'}(\tau')\varphi_{\lambda'}(\tau'') = \langle\tau''|R(0)|\tau'\rangle ,$$

thereby (156) is proven. We may consider $\varrho(\tau', \tau'')$ as a matrix element of a hermitian operator $\mathcal{R}$

$$(158) \qquad \varrho(\tau', \tau'') = \langle\tau''|\mathcal{R}|\tau'\rangle$$

and write quite generally for a normalized $\chi(t)$:

$$(159) \qquad R(t) = (\chi(t), \mathcal{R}\chi(t)) .$$

Equation (159) allows us to associate a hermitian operator $R(t)$, which satisfies the von Neumann equation of motion (120), to any solution $\chi(t)$ of (98), not necessarily of the form defined by the initial value (132). It is obvious that:

$$(160) \qquad \langle\lambda'|\mathcal{R}|\lambda''\rangle = a^*_{\lambda''}a_{\lambda'} .$$

Hence

$$(161) \qquad \langle\lambda'|R(t)|\lambda''\rangle = (\chi(t), a^*_{\lambda''}a_{\lambda'}\chi(t)) ,$$

so that (148) is a particular case of (159).

It follows from (161) that

$$\text{Trace } R(t) = (\chi(t), N_{\text{op}}\chi(t)) = (\chi(0), N_{\text{op}}\chi(0)) . \tag{162}$$

Let us assume that the expectation value $(\chi(0), N_{\text{op}}\chi(0))$ is finite but not an integer, as occurs in general when the expectation number of systems in the assembly described by $\chi(0)$ is finite. The trace of $R(0)$ being a non integral number, the eigenvalues of R are not all integers and $R(t)$ does not describe the motion of an assembly of systems Σ. Such operators R have nevertheless being used and they are necessary to treat in a general way the incompletely specified states of motion of Σ.

The operator $R(t)$ defined by (159) has in general a trace different from 1. In order to normalize the trace of $R(t)$ we may take

$$R_{\text{nor}}(t) = \left(\chi(t), \frac{\mathcal{R}}{N_{\text{op}}}\chi(t)\right) = (\chi(t), \mathcal{R}_{\text{nor}}\chi(t)) , \tag{163}$$

defining $\mathcal{R}/N_{\text{op}}$ by the equations

$$\frac{\mathcal{R}}{N_{\text{op}}} P_0 = 0 , \qquad \frac{\mathcal{R}}{N_{\text{op}}} P_n = \frac{1}{n}\mathcal{R} P_n \quad \text{for} \quad n \neq 0 , \tag{164}$$

the P_n being the projection operators of N_{op}. It follows from (164) that:

$$\frac{\mathcal{R}}{N_{\text{op}}}\chi = \frac{\mathcal{R}}{N_{\text{op}}}\sum_{n=0}^{\infty} P_n\chi = \sum_{n=1}^{\infty}\frac{1}{n}\mathcal{R}P_n\chi . \tag{165}$$

$\mathcal{R}\chi$ is of course not a wave functional but a linear operator acting on the functions of τ, as shown by (160). The nature of $\mathcal{R}$ as an operator on the functions of τ can be conveniently characterized by a generalization of (160):

$$\int_{\dot{\Omega}} \Psi_1^*(\tau)\mathcal{R}\Psi_2(\tau)\,\mathrm{d}\tau = \int_{\dot{\Omega}} \psi^*(\tau)\Psi_2(\tau)\,\mathrm{d}\tau \cdot \int_{\dot{\Omega}} \Psi_1^*(\tau)\psi(\tau)\,\mathrm{d}\tau . \tag{166}$$

We can now prove easily that the operator $R(t)$ defined by (159) has no negative eigenvalues. Indeed, it follows from (159) and (166) that

$$\int_{\dot{\Omega}} \Psi^*(\tau)R(t)\Psi(\tau)\mathrm{d}\tau = (\chi(t), \int_{\dot{\Omega}} \psi^*(\tau)\Psi(\tau)\,\mathrm{d}\tau \left\{ \int_{\dot{\Omega}} \psi^*(\tau)\Psi(\tau)\,\mathrm{d}\tau \right\}^* \chi(t)) , \tag{167}$$

hence:

$$\int_{\Omega} \Psi^*(\tau) R(t) \Psi(\tau)\, \mathrm{d}\tau \geqslant 0 \,. \tag{168}$$

This equation shows that $R(t)$ has no negative eigenvalues.

7. – We shall now examine rapidly the application of the Gibbs second quantization to the analysis of the incompletely specified states of motion of a quantal system Σ. A more detailed discussion will be given in a forthcoming paper.

Let us assume that the initial state of Σ is incompletely known. We may suppose that at the time 0 the value of some quantity A with degenerated eigenvalues was measured and that a value A' was obtained, s being the number of linearly independent eigenfunctions of A corresponding to the eigenvalue A'. With the available information we can describe the incompletely specified state at the time 0 by the projection operator $p_{A'}$. The von Neumann operator of trace 1 corresponding to $p_{A'}$ is:

$$R_{\mathrm{nor}}(0) = \frac{1}{s}\, p_{A'} \,. \tag{169}$$

Let the φ_λ be a complete orthonormal set of eigenfunctions of A, the indices being chosen in such a way that the eigenfunctions of the eigenvalue A' be $\varphi_1, \varphi_2, \ldots, \varphi_s$. The simplest wave functional $\chi(0; N')$ which describes the incompletely specified state is:

$$\chi^{(1)}(0; N') = \delta_{N'_1,1}\delta_{N'_2,1} \ldots \delta_{N'_s,1} \prod_{\lambda > s} \delta_{N'_\lambda,0} \,. \tag{170}$$

It is also possible to take any of the functionals $\chi^{(r)}$:

$$\chi^{(r)}(0; N') = \delta_{N'_1,r}\, \delta_{N'_2,r} \ldots \delta_{N'_s,r} \prod_{\lambda > s} \delta_{N'_\lambda,0}, \quad (r = \text{positive integer}). \tag{171}$$

The normalized von Neumann operator $R_{\mathrm{nor}}(t)$ computed with the formula (163) is the same for all the functionals $\chi^{(r)}$. Thus we see that a normalized von Neumann operator does not determine completely the wave functional χ, although the wave functional determines completely the von Neumann operator.

A slightly more complicated case is that in which the wave function $\Psi(0; \tau)$ at the time 0, before the measurement, was known. The probabilities

of the various eigenfunctions of A' are

$$w_l = \left| \int_\Omega \Psi^*(0;\tau)\varphi_l(\tau)\,d\tau \right|^2 \Big/ \sum_{l'=1}^{s} \left| \int_\Omega \Psi^*(0;\tau)\varphi_{l'}(\tau)\,d\tau \right|^2 \qquad (l = 1, 2, ..., s)\,. \tag{172}$$

When the ratios of the w_l are not all rational, it is not possible to describe the incompletely specified state by a monomial $\chi(0; N')$. In the case of rational ratios we can choose integers $n_1, n_2, ..., n_s$ such that

$$w_l = \frac{n_l}{n_s} w_s\,, \tag{173}$$

and take:

$$\chi(0; N') = \delta_{N'_1,n_1}\delta_{N'_2,n_2}\ldots\delta_{N'_s,n_s}\prod_{\lambda>s}\delta_{N'_\lambda,0}\,. \tag{174}$$

It is interesting to remark that in the case of irrational w_l/w_s, there is no von Neumann assembly of systems which gives the exact ratios of the probabilities, although there is no difficulty in representing the incompletely specified state by an operator $R(0)$:

$$R_{\mathrm{nor}}(0) = \sum_{l=1}^{s} w_l p_l\,. \tag{175}$$

In this general case it is necessary to use a multinomial $\chi(0; N')$, i.e. a grand ensemble.

8. – The entropy in the Gibbs second quantization.

A definition of the entropy of the grand ensemble described by $\chi(t)$ can be immediately obtained by taking into account the relation between the wave functional $\chi(t)$ and the von Neumann density operator given in section **6**. We have

$$S[\chi(t)] = -k\ \mathrm{Trace}\,\{R_{\mathrm{nor}}(t)\log R_{\mathrm{nor}}(t)\}\,, \tag{176}$$

k being the Boltzmann constant and R_{nor} being defined by (163). Since we are applying the von Neumann definition to $R_{\mathrm{nor}}(t)$, the entropy $S[\chi(t)]$ is time independent.

There is a very remarkable circumstance in the possibility of defining the entropy by means of the Gibbs second quantization. Let us suppose that the systems Σ obey the Bose statistics; they may be atomic nuclei with even atomic number, for instance. In this case the Gibbs second quantization is equivalent to the theory of the quantized field of the non interacting bosons Σ. *Thus we can assign an entropy to the pure state of the Σ boson field described by the wave functional $\chi(t)$:*

$$S_f(t) = -k \operatorname{Trace} \{R(t) \log R(t)\}, \tag{177}$$

$$R(t) = (\chi(t), \mathcal{R}\chi(t)). \qquad (\chi(t), \chi(t)) = 1. \tag{178}$$

In the case of fields of bosons, the association between the entropy and the incomplete specification of the quantal state is not necessary.

The Gibbs second quantization offers new approaches to the entropy problem. Thus it is possible to introduce a kind of entropy operator:

$$S_{\mathrm{op}}^{(\varphi)} = -k \sum f_\lambda \log f_\lambda . \tag{179}$$

The frequency operator f_λ is defined by the equations

$$f_\lambda P_0 = 0, \qquad f_\lambda P_n = \frac{1}{n} N_\lambda P_n \qquad (n \geqslant 1). \tag{180}$$

and the condition of linearity. We have:

$$f_\lambda \chi = \sum_{n=1}^{\infty} \frac{1}{n} N_\lambda P_n \chi . \tag{181}$$

f_λ may be considered as $N_\lambda / N_{\mathrm{op}}$. The expectation value $(\chi(0; N'), S_{\mathrm{op}}^{(\varphi)} \chi(0; N'))$ of $S_{\mathrm{op}}^{(\varphi)}$ computed with the wave functional (132) has the following value:

$$(\chi(0; N'), S_{\mathrm{op}}^{(\varphi)} \chi(0); N')) = -k \sum_\lambda f'_\lambda \log f'_\lambda , \qquad f'_\lambda = \frac{n_\lambda}{\sum_{\lambda'} n_{\lambda'}} . \tag{182}$$

It coincides with the value of the von Neumann entropy of the assembly described by the $R(0)$ in (133), but the expectation value at the time t will not in general coincide with the corresponding value of the von Neumann entropy.

The operator $S_{\mathrm{op}}^{(\varphi)}$ depends on the choice of the functions φ_λ. Thereby its expectation value does not give a general satisfactory definition of the entropy.

Nevertheless, when the set of the φ_λ plays a significant role, $S_{\mathrm{op}}^{(\varphi)}$ may be conveniently used. In the discussion of the statistical equilibrium the eigenfunctions of K can be chosen as the set φ_λ. With this choice of the φ_λ, $S_{\mathrm{op}}^{(\varphi)}$ becomes an integral of the motion and its expectation value is time independent. In many important applications Σ is formed by several weakly interacting parts and K can be split into the sum K_0 of the hamiltonians of the parts divided by $\hbar$ and a small interaction term K_{int}:

$$K = K_0 + K_{\mathrm{int}} . \tag{183}$$

By taking the φ_λ as a complete orthonormal set of eigenfunctions of K_0, we get a $S_{\mathrm{op}}^{(\varphi)}$ which is not an integral of the motion and whose expectation value is time dependent.

The quantity

$$S_\varphi[\chi(t)] = -k \sum_\lambda (\chi(t), f_\lambda \chi(t)) \log (\chi(t), f_\lambda \chi(t)) , \tag{184}$$

is also related to the entropy. It corresponds in the Gibbs second quantization to the entropy definition of BORN and GREEN [11]. It has also the disadvantage of depending on the choice of the φ_λ. We have

$$S_\varphi[\chi(0; N')] = -k \sum_\lambda f'_\lambda \log f'_\lambda , \tag{185}$$

$\chi(0; N')$ being the wave functional (132), so that $S_\varphi[\chi(0; N')]$ has the same value as the von Neumann entropy of the assembly described by the $R(0)$ in (133). For $t \neq 0$, $S_\varphi[\chi(t; N)]$ does not coincide with the von Neumann entropy of the assembly. $S_\varphi[\chi(t)]$ is less interesting than the expectation value of $S_{\mathrm{op}}^{(\varphi)}$.

A detailed discussion of the entropy and the quantal H theorem with the Gibbs second quantization will be given elsewhere.

9. – The ordinary second quantization.

We shall now apply the general theory of second quantization to the quantum mechanics of systems of similar particles, in order to show how the ordinary second quantization is contained in our general theory. For the sake of simplicity we shall consider only non relativistic spinless particles of

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[11] M. BORN and H. S. GREEN: *Proc. Roy. Soc.*, A **192**, 166 (1948).

mass m interacting through central forces derived from a potential V depending on the distance between two particles.

In the present case the equations (1) are ordinary Schrödinger equations:

$$(186) \qquad i\hbar \frac{\partial}{\partial t} \Psi_n(t; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \left\{ -\frac{\hbar^2}{2m} \sum_{l=1}^{n} \left(\frac{\partial}{\partial \boldsymbol{x}_l} \right)^2 + \frac{1}{2} \sum_{l=1}^{n} \sum_{l' \neq l}^{n} V(\boldsymbol{x}_l - \boldsymbol{x}_{l'}) \right\} \Psi_n .$$

There are only the operators $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$(187) \qquad \hbar \mathcal{R}_1 = \int_{-\infty}^{+\infty} \psi^*(\boldsymbol{x}) \left\{ -\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial \boldsymbol{x}} \right)^2 \right\} \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} ,$$

$$(188) \qquad \hbar \mathcal{R}_2 = \frac{1}{2} \int_{-\infty}^{+\infty} \psi^*(\boldsymbol{x}_1) \psi^*(\boldsymbol{x}_2) V(\boldsymbol{x}_1 - \boldsymbol{x}_2) \psi(\boldsymbol{x}_2) \psi(\boldsymbol{x}_1) \, \mathrm{d}\boldsymbol{x}_1 \, \mathrm{d}\boldsymbol{x}_2 .$$

Equation (8) is now the well known equation of the ordinary second quantization:

$$(189) \qquad i\hbar \frac{\mathrm{d}\chi}{\mathrm{d}t} = \left\{ -\frac{\hbar^2}{2m} \int_{-\infty}^{+\infty} \psi^*(\boldsymbol{x}) \left(\frac{\partial}{\partial \boldsymbol{x}} \right)^2 \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + \right.$$

$$\left. + \frac{1}{2} \int_{-\infty}^{+\infty} \psi^*(\boldsymbol{x}) \psi^*(\boldsymbol{x}') V(\boldsymbol{x} - \boldsymbol{x}') \psi(\boldsymbol{x}') \psi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}\boldsymbol{x}' \right\} \chi .$$

The commutation rules (3) are now:

$$(190) \qquad [\psi(\boldsymbol{x}), \psi^*(\boldsymbol{x}')]_{\pm} = \delta(\boldsymbol{x} - \boldsymbol{x}') , \qquad [\psi(\boldsymbol{x}), \psi(\boldsymbol{x})']_{\pm} = [\psi^*(\boldsymbol{x}), \psi^*(\boldsymbol{x}')]_{\pm} = 0 .$$

The sign minus corresponds to bosons and the sign plus to fermions. This is immediately seen, by noticing that the functions $\Psi_n(t; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ in the Fock expansion are symmetrical in the case of the sign minus in (190), as a consequence of the symmetry of the functionals $\chi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ with respect to the $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. In the case of the sign plus the $\chi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ are anti-symmetrical and the $\Psi_n(t; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ too. The symmetry or anti-symmetry of the Ψ_n is shown directly by (74) and (75).

The relation (71) seems to be unknown. The use of the projection operators P_n and $P(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ seems also to be new. The relations (74) and (75) are much simpler than those which have been used in the preceding work in second quantization.

The operator N_R introduced in the Gibbs second quantization can also be defined in the ordinary second quantization and corresponds now to the number of particles in a mixed one-particle state. In the particular case of a pure one-particle state described by the wave function $\varphi(\boldsymbol{x})$, N_R can be denoted by N_φ.

$$N_\varphi = \int_{-\infty}^{+\infty} \psi^*(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \cdot \int_{-\infty}^{+\infty} \varphi^*(\boldsymbol{x})\psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{191}$$

The eigenvalues of N are 0 and the positive integers in the case of bosons and 0 and 1 in the case of fermions.

The operator $\mathcal{H}$ being hermitian, equation (47) holds and the normalized wave functional χ may be considered as a probability amplitude. $|\chi|^2$ gives the probability of the different conditions of an assembly of interacting particles, the number of particles in the assembly being one of the variables. In the Fock treatment ([4]) of the second quantization χ is considered as a vector in a χ-space whose components are the Ψ_n, the square of the absolute value of a component $\Psi_n(t; \boldsymbol{x}_1, ..., \boldsymbol{x}_n)$ giving the probability density of finding just n particles, one in each of the elements of volume $\mathrm{d}\boldsymbol{x}_l$ around the point $\boldsymbol{x}_l$.

10. – The " second quantization " in the classical theory.

In the classical theory of indistinguishable particles developed in reference ([6]), the equations (1) are Liouville equations for n particle systems:

$$i\frac{\partial}{\partial t}\Psi_n(t; \tau_1, ..., \tau_n) = \left\{\sum_{l=1}^{n} L_1(\tau_l) + \frac{1}{2}\sum_{l=1}^{n}\sum_{l'\neq l} L_2(\tau_l, \tau_{l'})\right\}\Psi_n(t; \tau_1, ..., \tau_n)\,, \tag{192}$$

$$L_1(\tau_l) = -\frac{i}{m}\boldsymbol{p}_l \cdot \frac{\partial}{\partial \boldsymbol{x}_l}\,, \tag{193}$$

$$L_2(\tau_l, \tau_{l'}) = i\left\{\frac{\partial}{\partial \boldsymbol{x}_l} V(\boldsymbol{x}_l - \boldsymbol{x}_{l'})\right\}\cdot\frac{\partial}{\partial \boldsymbol{p}_l} + \left\{\frac{\partial}{\partial \boldsymbol{x}_{l'}} V(\boldsymbol{x}_l - \boldsymbol{x}_{l'})\right\}\cdot\frac{\partial}{\partial \boldsymbol{p}_{l'}}\,, \tag{194}$$

τ_l denotes now a point in the phase space Ω_1 of a single particle. There are only the operators $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\mathcal{H}_1 = \int_{\Omega_1} \psi^*(\tau) L_1 \tau(\tau)\,\mathrm{d}\tau\,, \tag{195}$$

$$\mathcal{H}_2 = \frac{1}{2}\int_{\Omega_1} \psi^*(\tau_1)\psi^*(\tau_2) L_2 \psi(\tau_2)\psi(\tau_1)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,. \tag{196}$$

In the present case equation (8) becomes:

$$(197)\quad i\frac{\partial\chi}{\partial t}=\left\{\int_{\Omega_1}\psi^*(\tau)L_1\psi(\tau)\,d\tau+\frac{1}{2}\int_{\Omega_1}\psi^*(\tau)\psi^*(\tau')L_2\psi(\tau')\psi(\tau)\,d\tau\,d\tau'\right\}\chi\,.$$

This is the wave equation for the « quantized » field in the phase space Ω_1 given in reference (6). The commutation rules (3) give the commutation rules of reference (6):

$$(198)\quad [\psi(\tau),\psi^*(\tau')]_{\pm}=\delta(\tau-\tau')\,,\qquad [\psi(\tau),\psi(\tau')]_{\pm}=[\psi^*(\tau),\psi^*(\tau')]_{\pm}=0\,.$$

The sign minus corresponds to the classical bosons and the sign plus to the classical fermions. The structure of the formalism is similar to that of the ordinary second quantization, the only essential differences being that in the classical case the space Ω_1 is the six-dimensional phase-space of a single particle and the different forms of the operators K. $\mathcal{R}_1$ and $\mathcal{R}_2$ are hermitian, equation (47) is valid and $|\chi|^2$ can be interpreted as a probability, for a normalized χ. $|\Psi_n(t;\tau_1,\ldots,\tau_n)|^2\,d\tau_1\ldots d\tau_n$ gives the probability of finding just n particles, one in each of the phase-space elements of volume $d\tau_l$ around the points τ_l The probability density $f_n(t;\tau_1,\ldots,\tau_n)$ ordinarily considered in the classical statistical mechanics appears now as the square of the absolute value of a classical wave function Ψ_n in the n-particle phase space. The Liouville equation being a homogeneous linear partial differential equation of the first order, both Ψ_n and f_n satisfy the same Liouville equation.

It is interesting to notice that the analogue of the Gibbs second quantization does also exist in the classical theory. It can be obtained by applying a « second quantization » treatment to the equation (192), the space Ω being taken as the phase-space Ω_n of the n particle system. Each n particle system appears as a « quantum » in this treatment and there is no interaction between these « quanta ». In this classical Gibbs « second quantization » it is not necessary to assume that the particles are indistinguishable, they may even be taken with different masses, since the classical wave functions can be introduced for any hamiltonian mechanical system.

11. – The field hamiltonian formalism.

In the case of hermitian $K^{(\alpha)}$, the general theory of second quantization can be obtained by the quantum mechanical method of field quantization applied to the « field » described by a function $\Psi(t;\tau)$ satisfying the non linear

equation:

$$(199)\qquad i\frac{\partial}{\partial t}\Psi(t;\tau) = K^{(1)}(\tau)\Psi(t;\tau) + \sum_{\alpha=1}^{s}\frac{1}{(\alpha-1)!}\cdot$$

$$\cdot\int_{\Omega}\Psi^*(t;\tau_1)\dots\Psi(t;\tau_{\alpha-1})K^{(\alpha)}(\tau,\tau_1,\dots,\tau_{\alpha-1})\{\Psi(t;\tau_{\alpha-1})\dots\Psi(t;\tau_1)\Psi(t;\tau)\}\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_{\alpha-1}\,.$$

Equation (199) and its adjoint can be derived from the variational principle:

$$(200)\qquad \delta\int_{t_0}^{t_1}\mathcal{L}\,\mathrm{d}t = 0\,,$$

$$(201)\qquad \mathcal{L} = i\int_{\Omega}\Psi^*(t;\tau)\frac{\partial}{\partial t}\Psi(t;\tau)\,\mathrm{d}\tau -$$

$$-\sum_{\alpha=1}^{s}\frac{1}{\alpha!}\int_{\Omega}\Psi^*(t;\tau_1)\dots\Psi^*(t;\tau_\alpha)K^{(\alpha)}(\tau_1,\dots,\tau_\alpha)\{\Psi(t;\tau_\alpha)\dots\Psi(t;\tau_1)\}\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_\alpha\,.$$

We shall now develop a hamiltonian formalism based on the variational principle (200). The conjugated momentum of $\Psi(t;\tau)$ is $\Pi(t;\tau)$:

$$(202)\qquad \Pi(t;\tau) = \frac{\delta\mathcal{L}}{\delta\left\{\dfrac{\partial}{\partial t}\Psi(t;\tau)\right\}} = i\Psi^*(t;\tau)\,.$$

The « hamiltonian » turns out to have the same form as $\mathcal{H}$:

$$(203)\qquad \mathcal{H}_c = \int_{\Omega}\Pi(t;\tau)\frac{\partial}{\partial t}\Psi(t;\tau)\,\mathrm{d}\tau - \mathcal{L} =$$

$$=\sum_{\alpha=1}^{s}\frac{1}{\alpha!}\int_{\Omega}\Psi^*(t;\tau_1)\dots\Psi^*(t;\tau_\alpha)K^{(\alpha)}(\tau_1,\dots\tau_\alpha)\{\Psi(t;\tau_\alpha)\dots\Psi(t;\tau_1)\}\,\mathrm{d}\tau_1\dots\mathrm{d}\tau_\alpha\,.$$

The Hamiltonian equations

$$(204)\qquad \frac{\partial}{\partial t}\Psi(t;\tau) = \frac{\delta\,\mathcal{H}_c(t)}{\delta\,\Pi(t;\tau)}\,,$$

$$(205)\qquad \frac{\partial}{\partial t}\Pi(t;\tau) = -\frac{\delta\,\mathcal{H}_c(t)}{\delta\Psi(t;\tau)}\,,$$

coincide respectively with (199) and its adjoint.

The method of field quantization consists in replacing $\Psi(t;\tau)$ and $\Pi(t;\tau)$ by operators $\Psi_{op}(t;\tau)$ and $\Pi_{op}(t;\tau)$ satisfying the commutation rules:

$$(206)\qquad \begin{cases} [\Psi_{op}(t;\tau), \Pi_{op}(t;\tau')]_{\pm} = i\delta(\tau-\tau') \\ [\Psi_{op}(t;\tau), \Psi_{op}(t;\tau')]_{\pm} = [\Pi_{op}(t;\tau), \Pi_{op}(t;\tau')]_{\pm} = 0\,. \end{cases}$$

These commutation rules can be satisfied by taking

$$(207)\qquad \Psi_{op}(t;\tau) = \psi(t;\tau)\,, \qquad \Pi_{op}(t;\tau) = i\psi^*(t;\tau)\,,$$

with

$$(208)\qquad \begin{cases} [\psi(t;\tau), \psi^*(t;\tau')]_{\pm} = \delta(\tau-\tau')\,, \\ [\psi(t;\tau), \psi(t;\tau')]_{\pm} = [\psi^*(t;\tau), \psi^*(t;\tau')]_{\pm} = 0\,. \end{cases}$$

This quantized « hamiltonian » is:

$$(209)\qquad \mathcal{H}(t) =$$

$$= \sum_{\alpha=1}^{s} \frac{1}{\alpha!} \int_{\Omega} \psi^*(t;\tau_1) \dots \psi^*(t;\tau_\alpha) K^{(\alpha)}(\tau_1, \dots, \tau_\alpha) \psi(t;\tau_\alpha) \dots \psi(t;\tau_1)\, d\tau_1 \dots d\tau_\alpha\,.$$

The preceding equations are taken in the Heisenberg representation. In the Schrödinger representation the operators ψ and ψ^* are time independent and satisfy the commutation rules (3) and $\mathcal{H}(t)$ goes over into the operator $\mathcal{H}$ defined by (6). « The Schrödinger equation » is simply equation (8).

In the ψ^*-representation we have

$$(210)\qquad \mathcal{H} = \sum_{\alpha=1}^{s} \frac{1}{\alpha!} \int_{\Omega} \psi^*(\tau_1) \dots \psi^*(\tau_\alpha) K^{(\alpha)}(\tau_1, \dots, \tau_\alpha) \frac{\delta}{\delta\psi^*(\tau_\alpha)} \dots \frac{\partial}{\partial\psi^*(\tau_1)}\, d\tau_1 \dots d\tau_\alpha\,,$$

so that the « Schrödinger equation » is a partial functional differential equation of order s:

$$(211)\qquad i\frac{d}{dt}\chi[t;\psi^*] =$$

$$= \sum_{\alpha=1}^{s} \frac{1}{\alpha!} \int_{\Omega} \psi^*(\tau_1) \dots \psi^*(\tau_\alpha) K^{(\alpha)}(\tau_1, \dots, \tau_\alpha) \frac{\delta^\alpha \chi[t;\psi^*]}{\delta\psi^*(\tau_1) \dots \delta\psi^*(\tau_\alpha)}\, d\tau_1 \dots d\tau_\alpha\,.$$

12. – The field hamiltonian formalism of a quantal system.

We shall now apply the theory of the preceding section to the case in which only $K^{(1)}$ is different from zero, i.e. the case in which (199) is linear:

$$i\frac{\partial \Psi}{\partial t} = K^{(1)}\Psi \,. \tag{212}$$

We are particularly interested in the case of a quantal system Σ of wave function Ψ whose hamiltonian is $\hbar K$. We shall see that the theory of the Ψ-field leads to the Gibbs second quantization.

Before quantizing the Ψ-field, we shall develop somewhat its c-number hamiltonian formalism. In the « classical » theory of the Ψ-field the quantities analogous to the functions $f(q, p)$ of the classical mechanics are the functionals $\mathcal{F}$ of the independent functions $\Psi(\tau)$ and $\Pi(\tau)$. The Poisson bracket of $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined in the usual way:

$$(\mathcal{F}_1[\Psi;\Pi], \mathcal{F}_2[\Psi;\Pi]) = \int_{\Omega}\left\{\frac{\delta\mathcal{F}_1}{\delta\Psi(\tau)}\frac{\delta\mathcal{F}_2}{\delta\Pi(\tau)} - \frac{\delta\mathcal{F}_1}{\delta\Pi(\tau)}\frac{\delta\mathcal{F}_2}{\delta\Psi(\tau)}\right\}\mathrm{d}\tau \,. \tag{213}$$

The total time derivative of the functional $\mathcal{F}[t;\Psi;\Pi]$ taken along a « trajectory » of the Ψ-field is:

$$\frac{D}{Dt}\mathcal{F}\,[t;\Psi;\Pi] = \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} + (\mathcal{F}, \mathcal{H}_c) \,. \tag{214}$$

Thereby the constants of the motion are the solutions of the equation:

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} + (\mathcal{F}, \mathcal{H}_c) = 0 \,. \tag{215}$$

This is the Liouville equation of the « classical » Ψ-field. In the discussion of this equation, Ψ and Π must be treated as independent variables and we shall write:

$$\Pi = i\Psi'^* \,. \tag{216}$$

Thus the Liouville equation becomes:

$$i\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \int_{\Omega}\left\{\Psi'^*(\tau)K\frac{\delta\mathcal{F}}{\delta\Psi'^*(\tau)} - \frac{\delta\mathcal{F}}{\delta\Psi(\tau)}K\Psi(\tau)\right\}\mathrm{d}\tau \,. \tag{217}$$

The Hamilton-Jacobi equation of the Ψ-field

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}[t;\Psi]-i\int_{\Omega}\frac{\delta\mathcal{V}}{\delta\Psi(\tau)}K\Psi(\tau)\,\mathrm{d}\tau=0\,, \tag{218}$$

has the same form as the Schrödinger equation (24*b*) for the complex conjugated $\chi^*[t;\psi]$ of the wave functional χ of the Gibbs second quantization.

We may consider (212) as the equation of motion of a classical system with an infinite number of degrees of freedom. If we try to develop a classical statistical mechanics of this system along Gibbs lines, we must try to introduce a probability density in the phase-space of this system with an infinite number of degrees of freedom, i.e. in the space whose points are pairs of functions $\psi(\tau), \Pi(\tau)$. This is in general not possible, because we cannot give any meaning to the elements of volume of such a space. If it would be possible to give a meaning to the probability density $\mathcal{P}[t;\Psi(\tau);\Pi(\tau)]$ it would have to satisfy the Liouville equation:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}[t;\Psi(\tau);i\Psi'^*(\tau)]=\int_{\Omega}\left\{\Psi'^*K\frac{\delta\mathcal{P}}{\delta\Psi'^*(\tau)}-\frac{\delta\mathcal{P}}{\delta\Psi(\tau)}K\Psi(\tau)\right\}\mathrm{d}\tau\,. \tag{219}$$

It is remarkable that by quantizing the Ψ-field we get the Gibbs second quantization, i.e. a statistical quantum mechanics of the quantal system Σ whose Schrödinger equation is precisely (212). The quantum mechanics of Σ is mathematically equivalent to the « classical » theory of the Ψ-field. The statistical quantum mechanics of Σ based on the Gibbs second quantization turns out to be the « quantized » theory of the Ψ-field. Indeed, it results from (211) that the Schrödinger equation of the quantized Ψ-field is

$$i\frac{\mathrm{d}}{\mathrm{d}t}\chi[t;\psi^*]=\int_{\Omega}\psi^*(\tau)K\frac{\delta\chi}{\delta\psi^*(\tau)}\,\mathrm{d}\tau\,, \tag{220}$$

in the representation in which the operators $\psi^*(\tau)$ are diagonal. This is exactly the equation for the wave functional of the Gibbs second quantization in the ψ^*-representation. It follows from (220) and the corresponding equation for the complex conjugated of χ (24*b*) that the product $\chi[t;\psi^*]\chi^*[t;\psi']$ satisfies (24*c*), i.e. the Liouville equation (219) of the « classical » theory of the Ψ-field.

It is well known from the quantum theory of fields that the quantized fields are assemblies of quanta. In the case of the quantized Ψ-field the quanta are systems Σ. When the systems Σ have the Bose statistics, the Gibbs second

quantization with the sign minus in the commutation rules coincides with the ordinary second quantization for non interacting systems. In the case of systems Σ with the Fermi statistics, the Gibbs second quantization with the sign minus, required for the description of the mixed states of Σ, is no more equivalent to the ordinary second quantization for non interacting systems.

The identification of the Gibbs second quantization with the theory of the quantized Ψ-field gives an intuitive picture of the extraction process of section 5. The grand ensemble $\mathscr{E}(t)$ corresponds to a state of the quantized Ψ-field in which the number of quanta Σ is in general not well defined. The interpretation rules I and II do become now equivalent to well known rules of the quantum theory of fields.

13. – Ergodic theorems.

Let us consider the case of hermitian operators $\mathscr{K}$. We shall assume that the time independent hermitian operator $\mathscr{K}$ admits a spectral decomposition:

$$\mathscr{K} = \sum_{\nu} \nu P_{\nu} , \qquad \sum_{\nu} P_{\nu} = 1 , \qquad P_{\nu} P_{\nu'} = \delta_{\nu,\nu'} P_{\nu} . \tag{221}$$

The spectral decomposition of $\exp[\pm it\mathscr{K}]$ is then:

$$\exp[\pm it\mathscr{K}] = \sum_{\nu} \exp[\pm it\nu] P_{\nu} . \tag{222}$$

Let $\mathscr{A}$ be a time independent linear operator acting on the wave functionals χ. We shall consider now the operator $\mathscr{A}(t)$ in the Heisenberg representation of the second quantization formalism:

$$\mathscr{A}(t) = \exp[it\mathscr{K}]\mathscr{A} \exp[-it\mathscr{K}] . \tag{223}$$

We want to compute the time average $\lim_{T\to\infty} (1/T) \int_0^T \mathscr{A}(t)\,\mathrm{d}t$. This time average is also the Cesàro generalized limit of $\mathscr{A}(t)$ for $t = \infty$:

$$\mathscr{A}(\infty) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathscr{A}(t)\,\mathrm{d}t . \tag{224}$$

It results from (222) that

$$\mathscr{A}(t) = \sum_{\nu,\nu'} \exp\{it(\nu - \nu')\} P \mathscr{A} P_{\nu'} , \tag{225}$$

hence:

$$\frac{1}{T}\int_0^T \mathcal{A}(t)\,\mathrm{d}t = \sum_{\nu,\nu'} \frac{\exp\{iT(\nu-\nu')\}-1}{iT(\nu-\nu')} P_\nu \mathcal{A} P_{\nu'} . \tag{226}$$

We get from (226) the ergodic theorem:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{A}(t)\,\mathrm{d}t = \sum_\nu P_\nu \mathcal{A} P_\nu . \tag{227}$$

Since

$$\begin{aligned}(\chi(t), \mathcal{A}\chi(t)) &= (\exp[-it\mathcal{K}]\chi(0), \mathcal{A}\exp[-it\mathcal{K}]\chi(0)) = \\ &= (\chi(0), \exp[it\mathcal{K}]\mathcal{A}\exp[-it\mathcal{K}]\chi(0)) = (\chi(0), \mathcal{A}(t)\chi(0)),\end{aligned} \tag{228}$$

we have the following ergodic theorem for the expectation value $(\chi(t), \mathcal{A}\chi(t))$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (\chi(t), \mathcal{A}\chi(t))\mathrm{d}t = \sum_\nu (P_\nu\chi(0), \mathcal{A}P_\nu\chi(0)) . \tag{229}$$

The P_ν are the projection operators of $\mathcal{K}$ corresponding to the eigenvalues ν, thereby

$$\mathcal{K}P_\nu = \nu\mathcal{K}P_\nu \tag{230}$$

and $P_\nu\chi(0)$ is an eigenfunctional of $\mathcal{K}$:

$$\mathcal{K}P_\nu\chi(0) = \nu P_\nu\chi(0) . \tag{231}$$

Equation (229) gives the time average of the expectation value $(\chi(t), \mathcal{A}\chi(t))$ in terms of the expectation values of $\mathcal{A}$ for the eigenfunctionals of $\mathcal{K}$.

It is easily seen that we have also:

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \mathcal{A}(t)\,\mathrm{d}t = \sum_\nu P_\nu \mathcal{A} P_\nu . \tag{232}$$

It follows from (227) that $\mathcal{A}(\infty)$ commutes with the P_ν and with $\mathcal{K}$:

$$[\mathcal{A}(\infty), P_\nu] = 0 , \qquad [\mathcal{A}(\infty), \mathcal{K}] = 0 . \tag{233}$$

Let us consider now an operator $\mathcal{A}$ of the Gibbs second quantization which corresponds to a quantity A of Σ:

$$\mathcal{A} = \int_{\Omega} \psi^*(\tau) A \psi(\tau)\, \mathrm{d}\tau \,. \tag{234}$$

We have

$$\mathcal{A}(t) = \int_{\Omega} \psi^*(\tau) A(t) \psi(\tau)\, \mathrm{d}\tau \,, \tag{235}$$

$$A(t) = \exp[itK] A \exp[-itK] \,, \tag{236}$$

because both sides of (235) coincide for $t = 0$ and they have the same rate of time variation, since

$$i \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \psi^*(\tau) A(t) \psi(\tau)\, \mathrm{d}\tau = \tag{237}$$

$$= \int_{\Omega} \psi^*(\tau) [A(t), K] \psi(\tau)\, \mathrm{d}\tau = \left[\int_{\Omega} \psi^*(\tau) A(t) \psi(\tau)\, \mathrm{d}\tau, \mathcal{H} \right] .$$

There is an ergodic theorem for $A(t)$ analogous to (227)

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T A(t)\, \mathrm{d}t = \sum_{K'} p_{K'} A p_{L'} \,, \tag{238}$$

the K' being the eigenvalues of K and the $p_{K'}$ the corresponding projection operators. We get from (235) a new form of ergodic theorem:

$$\mathcal{A}(\infty) = \sum_{K'} \int_{\Omega} \psi^*(\tau) p_{L'} A p_{L'} \psi(\tau)\, \mathrm{d}\tau \,. \tag{239}$$

Let the $\varphi_{A'}^{(\alpha)}$ be a complete set of orthonormal eigenfunctions of A and the $\varphi_{K'}^{(\beta)}$ a simila set for K. It follows from (239) that:

$$\mathcal{A}(\infty) = \sum_{A\,,K'} A' \sum_{\alpha,\beta,\beta'} \int_{\Omega} \varphi_{K'}^{(\beta)*}(\tau) p_{A'}^{(\alpha)} \varphi_{K'}^{(\beta')}(\tau)\, \mathrm{d}\tau\, a_{K'}^{*(\beta)} a_{K'}^{(\beta')} \,, \tag{240}$$

$$N_{A'}^{(\alpha)}(\infty) = \sum_{K',\beta,\beta'} \int_{\Omega} \varphi_{K'}^{(\beta)*}(\tau) p_{A'}^{(\alpha)} \varphi_{K'}^{(\beta')}(\tau)\, \mathrm{d}\tau\, a_{K'}^{*(\beta)} a_{K'}^{(\beta')} \,. \tag{241}$$

In most of the usual cases Σ is formed by several weakly interacting parts Σ_r. K is the sum of the operators K_r for the Σ_r and the small interaction K_{int}:

$$K = K_0 + K_{\text{int}}, \qquad K_0 = \sum_r K_r . \tag{242}$$

In the usual applications the energy levels of K_0 are very dense. Because of the smallness of K_{int}

$$\int_{\Omega} \varphi_{K'}^{(\beta)}(\tau)\varphi_{K_0'}^{(\alpha)}(\tau)\,\mathrm{d}\tau \cong 0 , \qquad \text{unless } K' \cong K_0' , \tag{243}$$

hence we have:

$$N_{K_0'}^{(\alpha)}(\infty) = \sum_{K' \cong K_0'} \sum_{\beta,\beta'} \int_{\Omega} \varphi_{K'}^{(\beta)*}(\tau) p_{K_0'}^{(\alpha)} \varphi_{K'}^{(\beta')}(\tau)\,\mathrm{d}\tau\, a_{K'}^{*(\beta)} a_{K'}^{(\beta')} . \tag{244}$$

14. – The ergodic theorems derived in the preceding section can be suitably modified to remain valid in the case in which the spectrum of $\mathcal{K}$ is not a pure point spectrum. It is convenient to introduce the abelian generalized limit:

$$\mathcal{A}_{\text{ab}}(\infty) = \lim_{\sigma \to 0+} \sigma \int_0^{\infty} \exp[-\sigma t]\mathcal{A}(t)\,\mathrm{d}t . \tag{245}$$

The real part of σ must be made to tend to zero by positive values. It is well known from the theory of the generalized limits that the abelian limit exists whenever the Cesàro limit does exist and has then the same value, although the abelian limit may exist in cases in which the Cesàro limit does not exist. We shall now replace (222) by the more general formula:

$$\exp[-it\mathcal{K}] = -\frac{1}{2\pi i}\left(\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} - \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \right) \exp[-it\lambda](\lambda - \mathcal{K})^{-1}\,\mathrm{d}\lambda \qquad (\varepsilon > 0) . \tag{246}$$

(See the book of HILLE: *Functional analysis and semi-groups* (New York, 1948), where this kind of spectral decomposition is studied in extremely general cases). For $t > 0$ we may close the integration path in the second integral in (246) with a semi-circle of infinite radius centred at the origin and lying below the real axis. Thus it is seen that only the first integral survives for $t > 0$, so that:

$$\exp[-it\mathcal{K}] = -\frac{1}{2\pi i} \int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} \exp[-it\lambda](\lambda - \mathcal{K})^{-1}\,\mathrm{d}\lambda \qquad (t > 0) .$$

By taking $\lambda = i\sigma$ we get:

$$\exp[-it\mathcal{K}] = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \exp[\sigma t](\sigma + i\mathcal{K})^{-1}\,\mathrm{d}\sigma \qquad (t>0)\,. \tag{248}$$

This equation shows that $(\sigma + i\mathcal{K})^{-1}$ is the Laplace transform of $\exp[-it\mathcal{K}]$. It follows from the Parseval formula for the Laplace transform (see DOETSCH: *Handbuch der Laplace-Transformation* (Basel, 1950), pags, 251-252) that for real values of σ:

$$\int_0^\infty \exp[-2\sigma t]\mathcal{A}(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\sigma - iy - i\mathcal{K})^{-1}\mathcal{A}(\sigma + iy + i\mathcal{K})^{-1}\,\mathrm{d}y\,. \tag{249}$$

It results from (249) and (245) that:

$$\mathcal{A}_{\mathrm{ab}}(\infty) = \frac{1}{\pi}\lim_{\sigma\to 0+}\left\{\sigma\int_{-\infty}^{+\infty}(\mathcal{K} + y + i\sigma)^{-1}\mathcal{A}(\mathcal{K} + y - i\sigma)^{-1}\,\mathrm{d}y\right\}. \tag{250}$$

This general ergodic theorem contains as a special case (227), as can be easily seen by using the spectral decomposition of $(\mathcal{K} + y \pm i\sigma)^{-1}$:

$$(\mathcal{K} + y \pm i\sigma)^{-1} = \sum_\nu (\nu + y \pm i\sigma)^{-1}P_\nu\,. \tag{251}$$

There is a formula similar to (250) for the abelian generalized limit of $A(t)$:

$$A_{\mathrm{ab}}(\infty) = \frac{1}{\pi}\lim_{\sigma\to 0+}\left\{\sigma\int_{-\infty}^{+\infty}(K + y + i\sigma)^{-1}A(K + y - i\sigma)^{-1}\,\mathrm{d}y\right\}. \tag{252}$$

It follows from (235) that in the Gibbs second quantization we have:

$$\mathcal{A}_{\mathrm{ab}}(\infty) = \int_\Omega \psi^*(\tau)A_{\mathrm{ab}}(\infty)\psi(\tau)\,\mathrm{d}\tau\,. \tag{253}$$

This formula gives $\mathcal{A}_{\mathrm{ab}}(\infty)$ in terms of $A_{\mathrm{ab}}(\infty)$.

Formula (252) gives the general solution of the ergodic problem in the Heisenberg representation of the quantum mechanics. It is well known that the standard form of the ergodic theory gives the generalized limit of operators of the form $\exp[-it\mathcal{K}]$. The content of the ergodic theorems is essen-

tially that some generalized limit of that operator is its projection operator corresponding to the eigenvalue 0. This theorem is of little utility in the quantum mechanics, because in general it gives a trivial result. In order to get an interesting result we must ask not for the generalized limit of $\exp[-itK]\Psi$ but rather for that of $\exp[-itK]p_\Psi \exp[itK]$, p_Ψ being the projection operator corresponding to Ψ. More generally we may ask for the generalized limit of the von Neumann operator $R(t)$:

$$R(t) = \exp[-itK]R(0)\exp[itK]\,. \tag{254}$$

The same reasoning which led to (250) shows that:

$$R_{\mathrm{ab}}(\infty) = \frac{1}{\pi}\lim_{\sigma\to 0+}\left\{\sigma\int_{-\infty}^{+\infty}(K+y-i\sigma)^{-1}R(0)(K+y+i\sigma)^{-1}\,\mathrm{d}y\right. . \tag{255}$$

In the particular case of a point spectrum we get:

$$R_{\mathrm{ab}}(\infty) = R(\infty) = \sum_{K'} p_{K'}R(0)p_{K'}\,, \tag{256}$$

$$p_\Psi(\infty) = \sum_{K'} p_{K'}p_\Psi p_{K'}\,. \tag{257}$$

RIASSUNTO (*)

Si dimostra che i metodi di seconda quantizzazione sono un procedimento matematico generale applicabile a quei formalismi in cui compaiono equazioni lineari di evoluzione, differenziali rispetto alla variabile temporale. Il comune formalismo di seconda quantizzazione per sistemi di bosoni o fermioni, e la « seconda quantizzazione » della teoria classica sviluppata dall'autore si ottengono come casi particolari dei metodi generali. Esistono più modi per applicare i metodi di seconda quantizzazione allo stesso problema lineare, modi che conducono a formalismi assai differenti. Si applica all'equazione di Schrödinger di un sistema meccanico arbitrario una forma semplicissima di seconda quantizzazione allo scopo di sviluppare un nuovo trattamento della meccanica quantica statistica (la seconda quantizzazione di Gibbs). Così operando, si ottiene un metodo immediato per introdurre gli *ensembles* di Gibbs e i *grand ensembles*. Si discute l'entropia con la seconda quantizzazione di Gibbs. Si derivano alcuni teoremi ergodici della seconda quantizzazione generale e della meccanica quantistica.

(*) *Traduzione a cura della Redazione.*