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Ionization at relativistic energies and polarization effects.

M. Huybrechts and M. Schönberg
Centre de Physique Nucléaire de l'Université Libre - Bruxelles

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Summary. — A theory of the polarization effects in the energy loss of a charged particle is given, in which the mechanism of the loss is different from that of the Fermi theory. This theory leads to a distribution of the loss between ionization-excitation and emission of Cerenkov radiation which differs considerably from that of the theories on Fermi's lines, although the stopping power is not essentially changed, being somewhat less than in those theories. This treatment leads to an increase of the direct ionization at impact parameters larger than atomic dimensions after the relativistic minimum, the increase being large in the case of gases. The Fermi saturation effect of the loss at distances larger than atomic dimensions does also exist in the present theory, although due to a different mechanism. The relativistic increase of the loss in distant interactions is due largely to an increase of the radii of action for ionization and excitation (Bohr-Williams mechanism), but there is also a contribution of the Cerenkov radiation (Fermi mechanism). The saturation arises from a limitation of the increase of the radii of action due to the polarization of the medium and to the saturation of the emission of Cerenkov radiation. This theory leads to a modification of the formula of Frank and Tamm for the rate of emission of Cerenkov radiation, the emission of high energy Cerenkov radiation being considerably reduced.

Introduction.

1. – The first analysis of the polarization effects in the theory of ionization is due to Fermi (1). Fermi's work was extended and improved by Wick (2),

(1) E. Fermi: *Phys. Rev.*, **57**, 485 (1940).

(2) G. C. Wick: *Ric. Scient.*, **11**, 273 (1940); **12**, 858 (1941), *Nuovo Cimento*, **1**, 302 (1943).

HALPERN and HALL (3), A. BOHR (4) and SCHÖNBERG (5,6). With the exception of A. BOHR's paper, all the above papers follow the line of FERMI's analysis. The work of FERMI, WICK and HALPERN and HALL had mainly in view the saturation of the rate of loss due to distant interactions, although in FERMI's paper it is indicated that the Cerenkov radiation is included in the loss. The importance of the Cerenkov radiation was first emphasized by A. BOHR (4), who showed by an intuitive argument, and also by using FERMI's results, that the whole increase of the loss at relativistic energies was accounted for by the emission of Cerenkov radiation. At that time this conclusion did not attract much attention, but; later on, OCCHIALINI's measurements (7) of the tracks of relativistic particles in photographic emulsions showed that there was no increase of the grain density with energy within the experimental errors (10%). These measurements have stimulated theoretical work on ionization and led MESSEL and RITSON (8) and SCHÖNBERG (5) to the same conclusions as A. BOHR, using FERMI's methods. SCHÖNBERG (5) has shown that the conclusion regarding the increase of the loss in the relativistic region was largely independent of the actual expression of the dielectric constant used in the Maxwell equations for the dispersive medium, but he remarked that such a conclusion might not hold in a rigorous quantum treatment and also that the spectrum of the Cerenkov radiation obtained from the theory of FRANK and TAMM (9) and FERMI (1) required important quantum corrections.

The conclusions of the analysis of the polarization effects on Fermi's lines seemed sufficient to explain the experimental data of OCCHIALINI and the Bristol group (10), as well as those of BOWEN and ROSER (11). However, recently, experimental data were obtained by VOYVODIC (12) and GOSH, JONES and WILSON (13) which cannot be explained by the theory on Fermi's lines, because it is necessary to assume an exceedingly strong absorption of the high

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(3) O. HALPERN and H. HALL: *Phys. Rev.*, **57**, 459 (1940); **73**, 477 (1948).

(4) A. BOHR: *Det. Kgl. Dans. Vid. Sels.*, **24**, n. 19 (1948).

(5) M. SCHÖNBERG: *Bull. Cent. Phys. Nucl. Brux.*, n. 20 (1950).

(6) M. SCHÖNBERG: *Nuovo Cimento*, **8**, 159 (1951).

(7) G. P. S. OCCHIALINI: *Como Congress* (*Nuovo Cimento, Suppl.*, **6**, 377 (1949)).

(8) H. MESSEL and D. M. RITSON: *Phil. Mag.*, **41**, 1129 (1950).

(9) I. FRANK and I. TAMM: *Compt. Rend. Ac. Sci. USSR*, **14**, 109 (1937); I. TAMM: *Journ. of Phys. USSR*, **1**, 439 (1939).

(10) P. H. FOWLER: *Phil. Mag.*, **41**, 169 (1950); U. CAMERINI, P. H. FOWLER, W. O. LOCK and H. MUIRHEAD: *Phil. Mag.*, **41**, 413 (1950).

(11) T. BOWEN and F. X. ROSER: *Phys. Rev.*, **83**, 689 (1951).

(12) L. VOYVODIC: *Bristol Conference* (1951); E. PICKUP and L. VOYVODIC: *Phys. Rev.*, **80**, 89 (1950).

(13) S. G. GOSH, G. M. D. B. JONES and J. G. WILSON: *Proc. Phys. Soc.*, **65**, 68 (1952).

energy Cerenkov radiation to get the experimental values of the increase of the ionization after the minimum. SCHÖNBERG pointed out that the emission of the high energy Cerenkov radiation would be improbable, according to quantum theory ([14]), and proposed a modification of Fermi's methods ([15]), which reduces considerably the intensity of the high energy Cerenkov bands and gives a larger amount of ionization. In this paper we shall work out in detail this modification of the Fermi theory and improve the formulas of reference ([15]).

In the existing theory, the polarization effects are taken into account by assuming that the ionizing particle moves in a classical homogeneous dispersive medium and using the solution of the Maxwell equations for such a medium given by FRANK and TAMM ([9]) and FERMI ([1]). Actually the solution of the Maxwell equations is used to compute the energy loss due to interactions at distances larger than 10^{-8} cm, the loss at smaller distances being taken from the quantum theory of BETHE and BLOCH ([16]). This procedure, originally due to FERMI ([1]), was never satisfactorily justified. It seems that the validity of the theorem of WILLIAMS ([17]) and BLOCH ([16]), concerning the equivalence of the classical and quantum methods of computing the average loss due to interactions at distances larger than atomic dimensions, has been extrapolated to the theory including polarization effects. Such an extrapolation is actually not justified, specially in the determination of the relative contributions of the ionization-excitation and the Cerenkov emission to the total loss. The relative contributions depend critically on the way in which the interactions between different atoms influence the resonance and non resonance transfers. It is shown by the formulas given in references (5) and (6) that the resonances are shifted from the oscillator frequencies ω_j to the zeros $\bar{\omega}_j$ of the dielectric constant $\varepsilon(\omega)$; thus a range of frequencies preceding immediately an oscillator frequency ω_j becomes a non resonance region in which coherent emission of radiation is important. On the other hand, the shift of the resonance from ω_j to $\bar{\omega}_j$ and the fact that $\sqrt{1-\beta^2}$ is replaced by $\sqrt{1-\beta^2\varepsilon(\omega)}$, in the Fermi theory, cancel completely the relativistic increase of the ionization-excitation loss, since $\sqrt{1-\beta^2\varepsilon(\bar{\omega}_j)}=1$. The shift of the resonance frequency ω_j is approximately $(4\pi ne^2/3m)(f_j/\omega_j)$, n being the number of electrons per unit volume, e and m the charge and the rest mass of the electron and f_j the oscillator percentual ($\sum f_j=1$). In the neighbourhood of the path, the polarization

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([14]) M. SCHÖNBERG: *Nuovo Cimento*, **9**, 210 (1952).

([15]) M. SCHÖNBERG: *Nuovo Cimento*, **9**, 372 (1952).

([16]) H. BETHE: *Zeits. f. Phys.*, **76**, 293 (1932); *Ann. d. Phys.*, **5**, 325 (1930); F. BLOCH: *Ann. d. Phys.*, **16**, 285 (1933); *Zeits. f. Phys.*, **81**, 363 (1933); C. MÖLLER: *Ann. d. Phys.*, **14**, 531 (1932); E. J. WILLIAMS: *Proc. Roy. Soc.*, A **135**, 108 (1932).

([17]) E. J. WILLIAMS: *Proc. Roy. Soc.*, A **139**, 163 (1933).

effects are small and the total field is nearly the Lorentz transformed Coulomb field, hence the times of collision are essentially the same as in the Williams theory. The short times of collision do not allow such a sharpness of definition of the resonances as the classical theory requires, in order to separate the shifted resonance $\bar{\omega}_j$ from the peak of the corresponding Cerenkov band, because of the uncertainty principle. At distances of the order of $R=\sqrt{mc^2/4\pi ne^2}$ the polarization effects are strong and the times of collision become much larger than in the Williams theory, so that at such distances the classical theory may be applied. It will be shown that the amount of high energy Cerenkov radiation which remains, when polarization effects are neglected up to distances of the order of R, is rather small.

It results from the preceding considerations that, at distances from the path much smaller than R, the interaction between different atoms may be completely neglected, because the polarization is not large and the clean cut separation of the frequency ranges relevant for ionization-excitation and Cerenkov emission on both sides of an oscillator frequency ω_j is not allowed by the uncertainty principle. The adequate treatment in this interval of distances is therefore the Bethe-Bloch-Möller-Williams theory, which is actually equivalent to a classical theory with the oscillator model, in what regards the value of the average loss at distances larger than 10^{-8} cm.

Following strictly the lines of reference ([15]), we should apply the Bethe-Bloch-Möller-Williams theory up to the distance R, and a modified form of the Fermi theory at distances larger than R. This procedure is not very accurate, because the polarization effects are already considerable at distances somewhat less than R. It is, however, possible to introduce a correction to the results of the Bethe-Bloch-Möller-William theory for distances less than R, in order to take into account the non negligible polarization effects. Instead of solving the Maxwell equations for a dispersive medium by assuming the dispersive medium to occupy the entire space, as was done by FERMI, we shall first assume that there are no electrons up to the distance R and determine the solution of the Maxwell equations for a dispersive non homogeneous medium in which the dielectric constant has the value 1, at distances less than R, and the Lorentz value $\varepsilon(\omega)$ at distances larger than R from the path. Thus we neglect the effects of the polarization at distances less than R in the computation of the field at distances larger than R. Our solution of the Maxwell equations allows us to determine with sufficient accuracy the effect of the polarization at distances larger than R on the field at distances less than R, and thus to correct the rate of loss obtained with the Bethe-Bloch-Möller-Williams methods.

In the theory developed in this paper, most of the relativistic increase of the loss comes from distances less than R, in which the polarization effects play a minor part, and it is mainly due to an increase of ionization and excitation. The increase of loss due to interactions at distances larger than 10^{-8} cm is essentially due to the increase of the radii of action, i.e. to the Bohr-Williams mechanism limited to distances of the order of R. The upper limitation of the radii of action leads to saturation, whereas the indefinite increase of the radii, that appears when polarization effects are neglected, leads to the indefinite logarithmic increase of the Bethe-Bloch-Möller-Williams theory. The contribution of the distances larger than R includes the loss due to Cerenkov radiation and is small. This part of the loss also gets saturated, by the same mechanism as in the theory on Fermi's lines. The total rate of loss is often smaller in the present theory than in the corresponding one on Fermi's lines, there is also a very different distribution of the loss between ionization-excitation and Cerenkov radiation. The results of the present theory agree satisfactorily with the existing experimental data.

The solution of the Maxwell equations for the field created by the ionizing particle and the electrons lying at distances larger than R from the path is derived in section 2. The value of the loss due to transfers at distances larger than R is computed in section 3. The modification of the formula of Frank and Tamm is given in section 4. The contribution of the different oscillator frequencies to the ionization is discussed in sections 5 and 6.

The modifications of the Bethe-Bloch-Möller-Williams formulas due to the limitation of the impact parameters to R are given in section 7. The corrections due to the action of the field of the atoms lying at distances larger than R on the atoms at distances less than R are given in section 8. The quantum effects connected with the short times of collision are discussed in sections 9 and 10.

The application of the theory to the analysis of the experimental data of GOSH, WILSON and JONES and of VOYVODIC is given in section 12.

Modification of the Fermi solution of the Maxwell equations.

2. – In Fermi's theory, the energy transferred to the medium at distances larger than atomic dimensions is computed with the field determined by the Maxwell equations for a dispersive non magnetic medium of dielectric constant $\varepsilon(\omega)$:

$$\left\{\begin{array}{ll} \dfrac{1}{c}\dfrac{\partial}{\partial t}\,\varepsilon_{\text{op}}\boldsymbol{E} = \operatorname{rot}\boldsymbol{H} - 4\pi\boldsymbol{j}\,, & \dfrac{1}{c}\dfrac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{rot}\boldsymbol{E}\,, \\ \operatorname{div}\varepsilon_{\text{op}}\boldsymbol{E} = 4\pi j_0\,, & \operatorname{div}\boldsymbol{H} = 0\,, \end{array}\right. \tag{1}$$

ω denotes a circular frequency and the operator ε_{op} is:

$$\varepsilon_{\text{op}} = \varepsilon\left(i\frac{\partial}{\partial t}\right). \tag{2}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic field vectors, respectively. j_0 and $c\boldsymbol{j}$ are respectively the charge and current densities corresponding to the ionizing particle of charge z_1e moving with constant velocity $\boldsymbol{v}$ along the x axis:

$$(3) \qquad j_0 = z_1 e\delta(x - vt)\,\delta(y)\,\delta(z)\,, \qquad \boldsymbol{j} = \frac{\boldsymbol{v}}{c}\,j_0\,.$$

We shall use, instead of Fermi's fields $\boldsymbol{E}^{(F)}$ and $\boldsymbol{H}^{(F)}$, fields which satisfy the Maxwell equations for a non dispersive medium at distances from the path less than R

$$(4) \qquad \left.\begin{array}{ll} \dfrac{1}{c}\dfrac{\partial \boldsymbol{E}^{\text{I}}}{\partial t} = \operatorname{rot} \boldsymbol{H}^{\text{I}} - 4\pi \boldsymbol{j}\,, & \dfrac{1}{c}\dfrac{\partial \boldsymbol{H}^{\text{I}}}{\partial t} = -\operatorname{rot} \boldsymbol{E}^{\text{I}} \\ \operatorname{div} \boldsymbol{E}^{\text{I}} = 4\pi j_0\,, & \operatorname{div} \boldsymbol{H}^{\text{I}} = 0 \end{array}\right\} \begin{array}{l} (\varrho = \sqrt{y^2 + z^2} < R)\,, \\ \left(R = \sqrt{\dfrac{mc^2}{4\pi n e^2}}\right), \end{array}$$

and the Maxwell equations for a dispersive medium at distances from the path larger than R:

$$(5) \qquad \left.\begin{array}{ll} \dfrac{1}{c}\dfrac{\partial}{\partial t}\,\varepsilon_{\text{op}}\boldsymbol{E}^{\text{II}} = \operatorname{rot} \boldsymbol{H}^{\text{II}}\,, & \dfrac{1}{c}\dfrac{\partial \boldsymbol{H}^{\text{II}}}{\partial t} = -\operatorname{rot} \boldsymbol{E}^{\text{II}} \\ \operatorname{div} \varepsilon_{\text{op}}\boldsymbol{E}^{\text{II}} = 0\,, & \operatorname{div} \boldsymbol{H}^{\text{II}} = 0\,, \end{array}\right\} \quad (\varrho > R)\,.$$

We shall denote by C_ϱ the cylinder of radius ϱ with axis on the path. We must fit the solutions of (4) and (5) at the cylinder C_R. It is convenient to use cylindrical coordinates (ϱ, φ, x), φ denoting the azimuthal angle around the path. By taking into account the well known formulas

$$(6) \qquad \operatorname{div} \boldsymbol{U} = \frac{1}{\varrho}\frac{\partial}{\partial \varrho}(\varrho U_\varrho) + \frac{1}{\varrho}\frac{\partial U_\varphi}{\partial \varphi} + \frac{\partial U_x}{\partial x}\,;$$

$$(7) \qquad \left\{\begin{array}{l} (\operatorname{rot} \boldsymbol{U})_\varrho = \dfrac{1}{\varrho}\dfrac{\partial U_x}{\partial x} - \dfrac{\partial U_\varphi}{\partial x}\,, \\ (\operatorname{rot} \boldsymbol{U})_\varphi = \dfrac{\partial U_\varrho}{\partial x} - \dfrac{\partial U_x}{\partial \varrho}\,, \qquad (\operatorname{rot} \boldsymbol{U})_x = \dfrac{1}{\varrho}\dfrac{\partial}{\partial \varrho}(\varrho U_\varphi) - \dfrac{1}{\varrho}\dfrac{\partial U_\varrho}{\partial \varphi}\,, \end{array}\right.$$

we get from (4) and (5) the following boundary conditions for the fields:

$$(8) \qquad E^{\text{I}}_\varrho = \varepsilon_{\text{op}} E^{\text{II}}_\varrho\,, \quad E^{\text{I}}_\varphi = E^{\text{II}}_\varphi\,, \quad E^{\text{I}}_x = E^{\text{II}}_x\,, \quad \boldsymbol{H}^{\text{I}} = \boldsymbol{H}^{\text{II}}\,. \qquad (\varrho = R).$$

It is convenient to introduce potentials $(A_0, \boldsymbol{A})$:

$$(9) \qquad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \operatorname{grad} A_0 , \qquad \boldsymbol{H} = \operatorname{rot} \boldsymbol{A} .$$

We shall denote by $(A_0^{\rm I}, \boldsymbol{A}^{\rm I})$ the potentials inside C_R and by $(A_0^{\rm II}\, \boldsymbol{A}^{\rm II})$ the potentials outside C_R. The equations for the potentials are:

$$(10) \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\{A_0^{\rm I}, \boldsymbol{A}^{\rm I}\} = 4\pi\{j_0, \boldsymbol{j}\}, \qquad \frac{1}{c}\frac{\partial A_0^{\rm I}}{\partial t} + \operatorname{div} \boldsymbol{A}^{\rm I} = 0 ,$$

$$(11) \qquad \left(\frac{\varepsilon_{\rm op}}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\{A_0^{\rm II}, \boldsymbol{A}^{\rm II}\} = 0 , \qquad \frac{\varepsilon_{\rm op}}{c}\frac{\partial A_0^{\rm II}}{\partial t} + \operatorname{div} \boldsymbol{A}^{\rm II} = 0 .$$

Because of the geometrical symmetry of the problem, the vector potentials are parallel to the path

$$(12) \qquad A_y = A_z = 0 ,$$

and the Lorentz conditions can be satisfied by taking

$$(13) \qquad \boldsymbol{A}^{\rm I} = \frac{\boldsymbol{v}}{c} A_0^{\rm I} , \qquad \boldsymbol{A}^{\rm II} = \frac{\boldsymbol{v}}{c} \varepsilon_{\rm op} A_0^{\rm II} ,$$

since the potentials depend on x and t only through the combination $x - vt$.

The boundary conditions for the field (8) are satisfied, by taking:

$$(14) \quad (1-\beta^2)\varepsilon_{\rm op} A_x^{\rm I} = (1-\beta^2\varepsilon_{\rm op}) A_x^{\rm II} , \qquad \frac{\partial A_x^{\rm I}}{\partial \varrho} = \frac{\partial A_x^{\rm II}}{\partial \varrho} . \qquad \left(\varrho = R,\ \beta = \frac{v}{c}\right).$$

In the case of an absorbent medium, the condition that the fields vanish at infinity shows that we must take:

$$(15) \qquad A_x^{\rm II} = \frac{z_1 e}{\pi c}\int_{-\infty}^{+\infty} B(\omega) K_0(\varrho k(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] d\omega ,$$

$$(16) \quad k^2(\omega) = \frac{\omega^2}{v^2}(1-\beta^2\varepsilon(\omega)) , \quad k(-\omega) = k^*(\omega) , \quad -\frac{\pi}{2} \leqslant \arg k(\omega) < \frac{\pi}{2}$$

$$\text{for} \quad \omega > 0 .$$

$B(\omega)$ is determined by the boundary conditions (14). $A_x^{\rm I}$ must be of the form:

$$(17)\quad A_x^1 = \frac{z_1 e}{\pi c}\int_{-\infty}^{+\infty} K_0(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v}-t\right)\right] d\omega +$$

$$+\frac{iz_1 e}{2c}\int_{-\infty}^{+\infty} C(\omega) I_0(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v}-t\right)\right] d\omega\,, \quad \left(k_0(\omega)=\frac{|\omega|}{v}\sqrt{1-\beta^2}\right).$$

The K and I are modified cylindrical functions:

$$(18)\quad K_\nu(u)=\frac{i\pi}{2}\exp\,[i\pi\nu/2] H_\nu^{(1)}(iu)\,, \qquad I_\nu(u)=\exp\,[-i\pi\nu/2] J_\nu(iu)\,.$$

$C(\omega)$ is also determined by the boundary conditions (14). The first term in the right hand side of (17) is the x-component of the vector potential of the field created by the ionized particle moving in vacuum.

We get from (14):

$$(19)\quad B(\omega)K_0(Rk(\omega)) = \left[K_0(Rk_0(\omega)) + \frac{i\pi}{2}\,C(\omega)I_0(Rk_0\,(\omega))\right]\frac{(1-\beta^2)\varepsilon(\omega)}{(1-\beta^2\varepsilon(\omega))}\,,$$

$$(20)\quad B(\omega)k(\omega)K_1(Rk(\omega)) = \left[K_1(Rk_0(\omega)) - \frac{i\pi}{2}\,C(\omega)I_1(Rk_0(\omega))\right]k_0(\omega)\,.$$

By taking into account the well known relation

$$(21)\quad K_0(u)I_1(u) + K_1\,(u)I_0(u) = \frac{1}{u}\,,$$

we get from (19) and (20):

$$(22)\quad B(\omega) = R^{-1}\Big[K_1(Rk(\omega))I_0(Rk_0(\omega))k(\omega) +$$

$$+\frac{1-\beta^2\varepsilon(\omega)}{(1-\beta^2)\varepsilon(\omega)}\,K_0(Rk(\omega))I_1(Rk_0(\omega))k_0(\omega)\Big]^{-1},$$

$$(23)\quad C(\omega) = \frac{2}{i\pi}\left[\frac{1-\beta^2\varepsilon(\omega)}{(1-\beta^2)\varepsilon(\omega)}\,K_0(Rk(\omega))B(\omega) - K_0(Rk_0(\omega))\right]\{\,I_0(Rk_0(\omega))\,\}^{-1}\,.$$

By taking into account the expressions

$$(24)\quad I_0(u)\cong 1\,, \quad I_1(u)\cong\frac{u}{2}\,, \quad K_0(u)\cong\log\frac{2}{\gamma u}\,, \quad K_1(u)\cong\frac{1}{u}\,,$$

$$(24a)\quad (|u|\ll 1,\ \log\gamma = 0{,}577...)\,,$$

and treating R as a variable parameter, we get:

(25) $$\lim_{R\to 0} B(\omega) = 1 .$$

We shall denote by $k_c(\omega)$ the limit of $k(\omega)$ as $v \to c$. It is easily seen that:

(26) $$\lim_{v\to c} B(\omega) = \left[K_1(Rk_c(\omega))Rk_c(\omega) + \frac{1}{2} K_0(Rk_c(\omega))R^2k_c^2(\omega)\varepsilon^{-1}(\omega)\right]^{-1} .$$

The fields outside C_R are:

(27) $$E_x^{\text{II}} = -\frac{iz_1 e}{\pi v^2}\int_{-\infty}^{+\infty} B(\omega)\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) K_0(\varrho k(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] \omega\, d\omega = B_{\text{op}}\, E_x^{(F)} ,$$

(28) $$E_\varrho^{\text{II}} = \frac{z_1 e}{\pi v}\int_{-\infty}^{+\infty} B(\omega)\varepsilon^{-1}(\omega)k(\omega)K_1(\varrho k(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] d\omega = B_{\text{op}} E_\varrho^{(F)} ,$$

(29) $$H^{\text{II}} = H_\varphi^{\text{II}} = \frac{z_1 e}{\pi c}\int_{-\infty}^{+\infty} B(\omega)k(\omega)K_1(\varrho k(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] d\omega = B_{\text{op}} H^{(F)} ,$$

(30) $$B_{\text{op}} = B\left(i\frac{\partial}{\partial t}\right).$$

The fields $(\boldsymbol{E}^{\text{II}}, \boldsymbol{H}^{\text{II}})$ may be obtained from the Fermi fields by the application of the operator B_{op}.

The fields inside C_R are:

(31) $$\boldsymbol{E}^{\text{I}} = \boldsymbol{E}^{(c)} + \boldsymbol{E}' , \qquad \boldsymbol{H}^{\text{I}} = \boldsymbol{H}^{(c)} + \boldsymbol{H}' .$$

$\boldsymbol{E}^{(c)}$ and $\boldsymbol{H}^{(c)}$ are the fields created by the ionizing particle in vacuum, they can be obtained from the Coulomb field by Lorentz transformation:

(32) $$E_x^{(c)} = -\frac{iz_1 e}{\pi v^2}(1 - \beta^2) \times$$
$$\times \int_{-\infty}^{+\infty} K_0(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] \omega\, d\omega = \frac{z_1 e(1 - \beta^2)(x - vt)}{[\varrho^2(1 - \beta^2) + (x - vt)^2]^{3/2}} ,$$

(33) $$E_\varrho^{(c)} = \frac{z_1 e}{\pi v^2}\sqrt{1 - \beta^2} \times$$
$$\times \int_{-\infty}^{+\infty} K_1(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] |\omega|\, d\omega = \frac{z_1 e(1 - \beta^2)\varrho}{[\varrho^2(1 - \beta^2) + (x - vt)^2]^{3/2}} ,$$

(34) $$H^{(c)} = H_\varphi^{(c)} = \beta E_\varrho^{(c)} .$$

$(\boldsymbol{E}', \boldsymbol{H}')$ are the corrections to the field of the particle moving in vacuum due to the polarization outside C_R:

(35) $$E'_x = \frac{z_1 e}{2v^2}(1-\beta^2)\int_{-\infty}^{+\infty} C(\omega) I_0(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v}-t\right)\right]\omega\, \mathrm{d}\omega\,,$$

(36) $$E'_\varrho = -\frac{iz_1 e}{2v^2}\sqrt{1-\beta^2}\int_{-\infty}^{+\infty} C(\omega) I_1(\varrho k_0(\omega)) \exp\left[i\omega\left(\frac{x}{v}-t\right)\right]|\omega|\, \mathrm{d}\omega\,,$$

(37) $$H' = H'_\varphi = \beta E'_\varrho\,.$$

The field $\boldsymbol{E}'$ is finite at the position of the particle at any instant and allows us to get immediately the rate of loss per unit length of path due to distant interactions, since it is the difference between the total field and the Lorentz transformed Coulomb field:

(38) $$-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x} = -\frac{z_1 e}{v}(\boldsymbol{v}\cdot\boldsymbol{E}'(\text{part.})) = -\frac{z_1^2 e^2}{2v^2}(1-\beta^2)\int_{-\infty}^{+\infty} C(\omega)\omega\, \mathrm{d}\omega\,.$$

By distant interactions we mean always those at distances larger than R, whereas usually the distant interactions are taken at distances larger than 10^{-8} cm.

3. – We shall now compute explicitly the value of the integral in (38) by taking $v = c$. It results from (23) that:

(39) $$-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x} = \frac{2z_1^2 e^2}{\pi v^2}\mathcal{R}\int_0^\infty \left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)\frac{K_0(Rk(\omega))}{I_0(Rk_0(\omega))} B(\omega) i\omega\, \mathrm{d}\omega\,,$$

$$(\mathcal{R} = \text{real part}).$$

We shall consider only the limit for $v \to c$, because for low velocities the contribution of the interactions at distances larger than R to the loss are small. It results from the second equation (24) and (26) that:

(40) $$\lim_{v\to c}\left(-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x}\right) = \frac{4z_1^2 e^2}{\pi R^2}\mathcal{R}\int_0^\infty \left[1 + 2\,\frac{K_1(Rk_c(\omega))}{K_0(Rk_c(\omega))}(Rk_c(\omega))^{-1}\varepsilon(\omega)\right]^{-1} i\,\frac{\mathrm{d}\omega}{\omega}\,,$$

(41) $$k_c(\omega) = \lim_{v\to c} k(\omega) = \frac{\omega}{c}\sqrt{1-\varepsilon(\omega)}\,, \qquad \left(\omega > 0,\ -\frac{\pi}{2} \leqslant \arg\sqrt{\ } < \frac{\pi}{2}\right).$$

The dielectric constant is given by the Lorentz formula:

$$(42)\qquad \varepsilon(\omega) = 1 + \frac{\sum_j \dfrac{f_j}{\nu_j^2 - \nu^2 - ig_j\nu}}{1 - \dfrac{1}{3}\sum_j \dfrac{f_j}{\nu_j^2 - \nu^2 - ig_j\nu}}, \qquad \left(\nu = \frac{\omega}{2\pi\alpha},\ \alpha = \sqrt{\frac{ne^2}{\pi m}}\right).$$

The ν_j are the oscillator frequencies measured in units α and the f_j the corresponding oscillator percentuals. The damping constants g are positive and very small, when there are no conduction electrons, as we assume. It is shown in Appendix I that in the case of a complex $\varepsilon(\omega)$ the quantity within the square bracket in (40) has zeros in the neighbourhood of the frequencies ω_j and lying in the lower part of the complex ω-plane, but has no zeros or singularities within the quadrant of the positive real and positive imaginary axes. We shall deform the integration path and replace it by the positive imaginary axis and a quarter of circle at infinity, between the positive imaginary and the positive real axes. We shall then neglect the damping constants. The quantity inside the square bracket in (40) is real on the imaginary axis, so that the only contribution comes from the quarter of circle at infinity. The asymptotic expressions of the dielectric constant ε and of $k_c(\omega)$ are:

$$(43)\qquad \varepsilon(\omega) \sim 1 - \frac{4\pi^2\alpha^2}{\omega^2}, \qquad \varepsilon^{-1}(\omega) \sim 1 + \frac{4\pi^2\alpha^2}{\omega^2}, \qquad k_c(\omega) \sim \frac{2\pi\alpha}{c}.$$

Finally we get:

$$(44)\qquad \lim_{v\to c}\left(-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x}\right) = \frac{2z_1^2e^2}{R^2}\left[1 + 2\frac{K_1(1)}{K_0(1)}\right]^{-1} \cong \frac{z_1^2e^2}{2R^2} = \frac{2\pi n z_1^2 e^4}{mc^2}.$$

Modification of the formula of Frank and Tamm.

4. – We shall now examine more in detail the nature of the loss due to distant interactions, in order to separate the ionization-excitation from the loss due to emission of Cerenkov radiation. The energy transmitted to the medium at distances larger than ϱ, ϱ being larger than R, per unit length of path is:

$$(45)\qquad -\frac{\mathrm{d}W_\varrho^{\text{II}}}{\mathrm{d}x} = \frac{c}{4\pi v}\int_{C_\varrho} [\boldsymbol{E}^{\text{II}} \times \boldsymbol{H}^{\text{II}}]_n \,\mathrm{d}S = -\frac{1}{4\pi}\int_{C_\varrho} E_x^{\text{II}}\,\varepsilon_{\text{op}} E_\varrho^{\text{II}}\,\mathrm{d}S.$$

The integral is taken over the surface of the cylinder C_ϱ and n denotes the normal component. It is easily seen that:

$$(46)\qquad -\frac{\mathrm{d}W_\varrho^{\mathrm{II}}}{\mathrm{d}x} = -\frac{\varrho}{2}\int_{-\infty}^{+\infty} E_x^{\mathrm{II}}\,\varepsilon_{\mathrm{op}} E_\varrho^{\mathrm{II}}\,\mathrm{d}x =$$

$$= \frac{iz_1^2e^2\varrho}{\pi v^2}\int_{-\infty}^{+\infty} B(\omega)B(-\omega)\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)k(-\omega)K_1(\varrho k(-\omega))K_0(\varrho k(\omega))\omega\,\mathrm{d}\omega =$$

$$= \frac{2z_1^2e^2\varrho}{\pi v^2}\,\mathcal{R}\int_0^{\infty} |B(\omega)|^2\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)k^*(\omega)K_1(\varrho k^*(\omega))K_0(\varrho k(\omega))i\omega\,\mathrm{d}\omega\,.$$

The corresponding formula in the Fermi theory

$$(47)\qquad -\frac{\mathrm{d}W_\varrho^{(F)}}{\mathrm{d}x} = \frac{2z_1^2e^2\varrho}{\pi v^2}\,\mathcal{R}\int_0^{\infty}\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)k^*(\omega)K_1(\varrho k^*(\omega))K_0(\varrho k(\omega))i\omega\,\mathrm{d}\omega\,,$$

can be obtained from (46) by taking $B=1$.

It was shown in references (5) and (6) that in the case of a non absorbent medium (real $\varepsilon(\omega)$):

$$(48)\qquad -\frac{\mathrm{d}W_\varrho^{(F)}}{\mathrm{d}x} = -\frac{\mathrm{d}W_\infty^{(F)}}{\mathrm{d}x} + \frac{2z_1^2e^2}{v^2}\sum_{\bar\omega>0} a_{\bar\omega}\,\bar\omega\,\frac{|\bar\omega|\varrho}{v}K_1\left(\frac{|\bar\omega|\varrho}{v}\right)K_0\left(\frac{|\bar\omega|\varrho}{v}\right),$$

$$(49)\qquad -\frac{\mathrm{d}W_\infty^{(F)}}{\mathrm{d}x} = -\frac{z_1^2e^2}{v^2}\int_{1-\beta^2\varepsilon<0}\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)\omega\,\mathrm{d}\omega\,,\qquad (\omega>0).$$

where $-\mathrm{d}W_\infty^{(F)}/\mathrm{d}x$ is the rate of emission of Cerenkov radiation per cm. Formula (49) is due to FRANK and TAMM (9). The $a_{\bar\omega}$ are the residues of $\varepsilon^{-1}(\omega)$ at the poles $\bar\omega$. These poles are all real and simple, in the case of a real $\varepsilon(\omega)$:

$$(50)\qquad \frac{1}{\varepsilon(\omega)} = 1+\sum_{\bar\omega}\frac{a_{\bar\omega}}{\omega-\bar\omega} = 1+2\sum_{\bar\omega>0}\frac{\bar\omega a_{\bar\omega}}{\omega^2-\bar\omega^2}\,.$$

The contribution of the residues in (48) gives the rate of loss per cm due to direct ionization and excitation. Equation (48) shows clearly that the resonances occur at the frequencies $\bar\omega$, as it was said in the discussion of section 1.

We may expect to get a formula analogous to (48) in the present theory. There are now some new circumstances due to the presence of the factor $|B(\omega)|^2$ in (46). The $\bar\omega$ are no more poles of the quantity under the integral in (46), but there are poles introduced by the factor $|B(\omega)|^2$, as we shall now prove. It was shown in references (5) and (6) that there is a pole $\overline{\overline{\omega}}_j$ of $\varepsilon(\omega)$ very close to each frequency ω_j ($\overline{\overline{\omega}}_j < \omega_j$). We may write:

$$(51)\quad |B(\omega)|^2 k^*(\omega) K_1(\varrho k^*(\omega)) K_0(\varrho k(\omega)) = B(\omega)B(-\omega)k^*(\omega)K_1(\varrho k^*(\omega))K_0(\varrho k(\omega)) = \\ = \frac{K_1(\varrho k^*(\omega))K_0(\varrho k(\omega))}{K_1(Rk^*(\omega))K_1(Rk(\omega))}\,[R^2 k(\omega)D(\omega)D(-\omega)]^{-1},$$

$$(52)\qquad D(\omega) = I_0(Rk_0(\omega)) + \frac{K_0(Rk(\omega))k(\omega)}{K_1(Rk(\omega))k_0(\omega)}\,\varepsilon^{-1}(\omega) I_1(Rk_0(\omega)) .$$

In the case of a real $\varepsilon(\omega)$, the function D is real and monotonic in the interval $\overline{\overline{\omega}}_j - \overline{\omega}_j$. Since

$$(53)\qquad D(\overline{\overline{\omega}}_j) = I_0(Rk_0(\overline{\overline{\omega}}_j)) > 0 , \qquad D(\overline{\omega}_j - 0) = -\infty ,$$

there is a zero $\tilde{\omega}_j$ of D in that interval. $\tilde{\omega}_j$ is a double pole of the quantity under the integral in (46). When the damping of the electronic vibrations is not neglected, $\varepsilon(\omega)$ is complex and its zeros $\overline{\omega}'_j - i\overline{\eta}_j$ and poles $\overline{\overline{\omega}}'_i - i\overline{\overline{\eta}}_j$ have negative imaginary parts which tend to zero together with the damping constants g in the Lorentz formula (42). In the case of very small damping constants, the phase of $D(\omega)$ varies approximately by $-\pi$ as ω varies from $\overline{\overline{\omega}}'_j$ to $\overline{\omega}'_j$. This shows that the zeros of $D(\omega)$ corresponding to the $\tilde{\omega}_j$ have also negative imaginary parts $-\tilde{\eta}_j$. The corresponding zeros of $D(-\omega)$ have positive imaginary parts. In the limit of zero damping constants, $D(\omega)$ has no zeros in the Cerenkov bands, because the ratio of the K functions in (52) cannot take imaginary values inside a Cerenkov band. (These points are discussed in Appendix I).

The transition to the case of zero damping, in (46), is more complicated than in the case of the theory on Fermi's lines, because the quantity under the integral has poles on both sides of the real ω-axis. In order to obtain a formula analogous to (48), we shall consider equations (27) and (28) as obtained from the inversion of a two sided Laplace transformation, instead of a Fourier transformation, and the integration paths will be taken parallel to the real ω-axis at a distance δ, in the upper part of the ω-plane. The well known formula for the Laplace transform of a product of two functions gives

$$(54)\quad -\frac{\varrho}{2}\int_{-\infty}^{+\infty} E_x^{\text{II}}\,\varepsilon_{\text{op}} E_\varrho^{\text{II}} \exp\left[-2i\delta\frac{x}{v}\right] \mathrm{d}x =$$

$$= \frac{iz_1 e^2 \varrho}{\pi v^2}\int_{-\infty+i\delta}^{+\infty+i\delta} B(\omega)B(2i\delta-\omega)\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right) K_0(\varrho k(\omega))K_1(\varrho k(2i\delta-\omega))k(2i\delta-\omega)\omega\,\mathrm{d}\omega .$$

By taking a sufficiently small δ, we may keep it constant and make the damping constants tend to zero, and then let δ tend to zero. Thus we get:

$$(55)\quad \lim_{g\to 0}\left(-\frac{\mathrm{d}W_\varrho^{\mathrm{II}}}{\mathrm{d}x}\right) = \lim_{\delta\to 0}\left[\lim_{g\to 0}\frac{iz_1e^2\varrho}{\pi v^2}\times\right.$$

$$\left.\times\int_{-\infty+i\delta}^{+\infty+i\delta} B(\omega)B(2i\delta-\omega)\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)K_0(\varrho k(\omega))K_1(\varrho k(2i\delta-\omega))k(2i\delta-\omega)\omega\,\mathrm{d}\omega\right].$$

The important thing is that, in the limit, we do not get the Cauchy principal value definition of the integral of $B(\omega)B(-\omega)[(1/\varepsilon(\omega))-\beta^2]K_0(\varrho k(\omega))\times \times K_1(\varrho k(-\omega))k(-\omega)\omega$ along the real ω-axis, but the integral taken along the real axis indented by infinitesimal semi-circles centred at the $\tilde{\omega}_j$ and lying in the upper part of the ω-plane. We shall denote this integration path by $\tilde{C}$ and drop the lim in the left hand side of (55):

$$(56)\quad -\frac{\mathrm{d}W_\varrho^{\mathrm{II}}}{\mathrm{d}x} = \frac{iz_1e^2\varrho}{\pi v^2}\int_{\tilde{C}} B(\omega)B(-\omega)\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)k(-\omega)K_1(\varrho k(-\omega))K_0(\varrho k(\omega))\omega\,\mathrm{d}\omega.$$

It was shown in references (5) and (6) that:

$$(57)\quad \mathcal{R}[i\varrho k(-\omega)K_1(\varrho k(-\omega))K_0(\varrho k(\omega))] = \arg k(\omega) \qquad (\omega>0).$$

Therefore the only contributions to the right hand side of (56) are those of the Cerenkov frequencies and the residues at the $\tilde{\omega}_j$:

$$(58)\quad -\frac{\mathrm{d}W_\varrho^{\mathrm{II}}}{\mathrm{d}x} = -\frac{z_1^2e^2}{v^2}\int_{\left(\substack{1-\beta^2\varepsilon<0\\ \omega>0}\right)}\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)|B(\omega)|^2\omega\,\mathrm{d}\omega + $$

$$+\frac{2z_1^2e^2}{v^2}\sum_j \tilde{a}_{\tilde{\omega}_j}(\varrho)\tilde{\omega}_j\varrho k(\tilde{\omega}_j)K_1(\varrho k(\tilde{\omega}_j))K_0(\varrho k(\omega_j)),$$

$$(58a)\quad \tilde{a}_{\tilde{\omega}_j}(\varrho) = \frac{\underset{\tilde{\omega}_j}{\mathrm{Residue}}\left[B^2(\omega)\left(\frac{1}{\varepsilon(\omega)}-\beta^2\; k(\omega)K_1(\varrho k(\omega))K_0(\varrho k(\omega))\omega\right]\right.}{\tilde{\omega}_j k(\tilde{\omega}_j)K_1(\varrho k(\tilde{\omega}_j))K_0(\varrho k(\tilde{\omega}_j))}.$$

The flux of the Poynting vector through the cylinder at infinity gives the rate of emission of Cerenkov radiation:

$$(59)\quad -\frac{\mathrm{d}W_\infty^{\mathrm{II}}}{\mathrm{d}x} = -\frac{z_1^2e^2}{v^2}\int_{\left(\substack{1-\beta^2\varepsilon<0\\ \omega>0}\right)}\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)|B(\omega)|^2\omega\,\mathrm{d}\omega = -\frac{\mathrm{d}W_{\mathrm{Cer}}^{\mathrm{II}}}{\mathrm{d}x}.$$

This formula differs from that of FRANK and TAMM (49) by the presence of the factor $|B(\omega)|^2$ which reduces considerably the intensity of the high energy radiation. We may write:

$$(60)\qquad -\frac{\mathrm{d}W_\varrho^{\mathrm{II}}}{\mathrm{d}x} = -\frac{\mathrm{d}W_{\mathrm{Cer}}^{\mathrm{II}}}{\mathrm{d}x} + \frac{2z_1^2e^2}{v^2}\sum_j \tilde{a}_{\tilde{\omega}_j}(\varrho)\tilde{\omega}_j\varrho k(\tilde{\omega}_j)K_1(\varrho k(\tilde{\omega}_j))K_0(\varrho k(\tilde{\omega}_j)).$$

The second term in the right hand side of (60) gives the rate of loss per cm due to direct ionization and excitation at distances larger than $\varrho > R$.

5. – We can compute the rate of loss of energy due to distant interactions by using the Poynting vector of the field inside C_R:

$$(61)\qquad -\frac{\mathrm{d}W_{>R}}{\mathrm{d}x} = \frac{c}{4\pi v}\int_{C_\varrho}[\boldsymbol{E}^{\mathrm{I}}\times\boldsymbol{H}^{\mathrm{I}}]_n\,\mathrm{d}S = -\frac{1}{4\pi}\int_{C_\varrho}E_x^{\mathrm{I}}E_\varrho^{\mathrm{I}}\,\mathrm{d}S =$$

$$= -\frac{\varrho}{2}\int_{-\infty}^{+\infty}[E_x^{(C)}E_\varrho^{(C)} + E_x^{(C)}E'_\varrho + E'_xE_\varrho^{(C)} + E'_xE'_\varrho]\mathrm{d}x\,.$$

It is easily seen that the terms $E_x^{(C)}E_\varrho^{(C)}$ and $E'_xE'_\varrho$ do not give any contribution to (61). By working out the contributions of the other two terms, we find again the formula (38). We shall now separate the contributions of the direct ionization-excitation and the Cerenkov radiation in (39), by indenting the positive real axis at the $\tilde{\omega}_j$ corresponding to zero damping and making the damping constants tend to zero. Thus we get:

$$(62)\qquad -\frac{\mathrm{d}W_{>R}}{\mathrm{d}x} = \frac{2z_1^2e^2}{\pi v^2}\mathcal{R}\int_{\text{Cer bands}}\frac{K_0(Rk(\omega))}{I_0(Rk_0(\omega))}\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)B(\omega)i\omega\,\mathrm{d}\omega\,-$$

$$-\frac{2z_1^2e^2}{v^2}\sum_j\left[I_1(Rk_0(\tilde{\omega}_j))\frac{\tilde{\omega}_jR}{v\sqrt{1-\beta^2}}\right]^{-1}\tilde{\omega}_j\operatorname*{Residue}_{\tilde{\omega}_j}[D(\omega)]^{-1}.$$

It seems likely that the real part of the integral over the Cerenkov bands is the rate of emission of Cerenkov radiation per cm and the contributions of the residues the rate of loss due to direct ionization and excitation at distances larger than R. It is shown in Appendix II that the first term in the right hand side of (62) coincides with $-\mathrm{d}W_\infty^{\mathrm{II}}/\mathrm{d}x$. Thus we get a more convenient form of the loss by ionization and excitation.

The residue in (62) can be computed by the well known formula:

$$(63)\qquad \operatorname*{Residue}_{\tilde{\omega}_j}[D(\omega)]^{-1} = \left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)}.$$

In many cases of practical interest $\omega_j R/v \gg 1$ and we can determine with sufficient accuracy $\tilde{\omega}_j$ by the equation

$$(64) \qquad [\varepsilon(\tilde{\omega}_j)]^{-1}\sqrt{1-\beta^2\varepsilon(\tilde{\omega}_j)} = -\sqrt{1-\beta^2}\,\frac{I_0(Rk_0(\omega_j))}{I_1(Rk_0(\omega_j))},$$

and take

$$(65) \qquad \left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{\tilde{\omega}_j} \cong \left[\left(1+\frac{\beta^2\varepsilon}{2(1-\beta^2\varepsilon)}\right)\frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\,I_0(Rk_0(\omega))\right]^{-1}_{(\omega=\tilde{\omega}_j)}.$$

For extremely relativistic particles and $\omega_j R/v \gg 1$ we get

$$(66) \qquad \varepsilon(\tilde{\omega}_j) \cong -\left(\frac{\omega_j R}{2v}\right)^2, \qquad \left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{\tilde{\omega}_j} \cong -\frac{4\pi n e^2}{m}\,\frac{f_j}{\omega_j}\left(\frac{\omega_j R}{2v}\right)^{-2},$$

by using the approximate expression

$$(67) \qquad \varepsilon(\omega) \cong \frac{4\pi^2\alpha^2 f_j}{\omega_j^2-\omega^2}, \qquad (\omega \cong \omega_j),$$

for values of ω very close to $\tilde{\omega}_j$. The contribution of the residue at ω_j to $-\mathrm{d}W_{>R}/\mathrm{d}x$ is approximately

$$(68) \qquad \frac{4\pi n z_1^2 e^4}{mv^2}\,f_j\left(\frac{\omega_j R}{2v}\right)^{-4}, \qquad \left(\frac{\omega_j R}{v} \gg 1,\ \frac{\omega_j R}{v}\sqrt{1-\beta^2} \ll 1\right).$$

A general approximate expression of the contributions of the residues can be obtained from the formula (9) of Appendix III. It results from (68) that the contribution of the large ω_j is negligible, at distances larger than R. It is interesting to remark that their contribution is nevertheless larger than in the theory on Fermi's lines. Indeed, since

$$(69) \qquad \bar{\omega}_j a_{\bar{\omega}_j} = -\bar{\omega}_j a_{-\bar{\omega}_j} \cong 2\pi^2\alpha^2 f_j,$$

it follows from (48) that the contribution of a large frequency ω_j, in the theory on Fermi's lines, is:

$$(70) \qquad \frac{4\pi n z_1^2 e^4}{mv^2}\,\frac{\pi f_j}{2}\exp\left[-\frac{2\omega_j R}{v}\right], \qquad \left(\frac{\omega_j R}{v} \gg 1\right).$$

When $\omega_l R/v \ll 1$, we may use the approximate expressions (24), so that:

$$(71) \qquad \varepsilon(\tilde{\omega}_l) \cong -\frac{1}{2}\left(\frac{\tilde{\omega}_l R}{v}\right)^2 K_0\left(\frac{\tilde{\omega}_l R}{v}\right), \qquad \left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)_{(\omega=\tilde{\omega}_l)} \cong \left(\frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right)_{(\omega=\tilde{\omega}_l)}.$$

It follows from (62) and (63) that the contribution of ω_l to $-\mathrm{d}W_{>R}/\mathrm{d}x$ is approximately

$$\frac{2z_1^2e^2}{v^2}\,\bar{\omega}_l K_0\left(\frac{\bar{\omega}_l R}{v}\right)\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_l)}, \qquad \left(\frac{\omega_l R}{v}\ll 1\right), \tag{72}$$

and does differ little from the corresponding value in the theory on Fermi's lines:

$$\frac{2z_1^2e^2}{v^2}\,\bar{\omega}_l K_0\left(\frac{\bar{\omega}_l R}{v}\right)\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\bar{\omega}_l)}, \qquad (\tilde{\omega}_l\cong\bar{\omega}_l). \tag{72a}$$

Discussion of the energy loss due to distant interactions.

6. – We shall now compare the results of the present theory and that on Fermi's lines. We shall first assume that

$$\frac{\omega_j R}{v}\sqrt{1-\beta^2}\gg 1, \tag{73}$$

for all the frequencies and we shall replace the I and K functions by their asymptotic expressions

$$I(u)\cong\frac{e^u}{\sqrt{2\pi u}}, \qquad K(u)\cong\sqrt{\frac{\pi}{2u}}\exp[-u] \qquad (|u|\gg 1), \tag{74}$$

in equations (22) and (52). Thus we get $\varepsilon(\tilde{\omega}_j)\cong -1$ for $\beta\ll 1$ and using formula (9) of Appendix III

$$-\frac{2z_1^2e^2}{v^2}\,[I_1(Rk_0(\tilde{\omega}_j))Rk_0(\tilde{\omega}_j)]^{-1}\tilde{\omega}_j\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)}\cong\frac{4\pi n z_1^2 e^4}{mv^2}\,f_j\,\frac{\omega_j R}{v}\,K_1^2\left(\frac{\omega_j R}{v}\right). \tag{75}$$

Hence the results of the present theory do not differ practically from that on Fermi's lines for non relativistic velocities and distances larger than R.

The total loss at distances larger than R is given approximately by (44), for extremely relativistic particles. It is easily seen that the formulas given in references (5) and (6) lead to the following result

$$\lim_{v\to c}\left(-\frac{\mathrm{d}W_\infty^{(F)}}{\mathrm{d}x}\right)=\frac{4\pi n z_1^2 e^4}{mc^2}\log\frac{\prod_j\bar{\omega}_j f'_j}{2\pi\alpha}\cong\frac{4\pi n z_1^2 e^4}{mc^2}\log\frac{I_z}{2\pi\alpha\hbar}, \tag{76}$$

$$f'_j=\frac{a_{\bar{\omega}_j}\bar{\omega}_j}{2\pi^2\alpha^2}\cong f_j\,. \tag{77}$$

I_z being the mean ionization potential of the atoms of the medium (atomic number Z). The value (76) is, in most usual cases, considerably larger than (44), particularly for gases. The total amount of Cerenkov radiation is, therefore, much smaller in the present theory than in that on Fermi's lines, in most cases. We shall now prove that this is due to the strong reduction of the high energy Cerenkov bands, i.e. those in which $\omega R/v \gg 1$. $|Rk(\omega)|$ is equal to zero at the lower limits of those bands, but has large values on a large part of the bands and becomes infinite at the upper limits. The term with the K_0 function in (22) is equal to zero at both the limits of a band and is not very important, so that:

$$B(\omega) \cong [I_0(Rk_0(\omega))K_1(Rk(\omega))Rk(\omega)]^{-1}\,. \tag{78}$$

In the part of a band that gives the strong intensity in the theory of Frank and Tamm, we may use the asymptotic expression (74) of K_1, and since $k(\omega)$ is imaginary:

$$|B(\omega)|^2 \cong \frac{2}{\pi}\left[I_0(Rk_0(\omega))\sqrt{R|k(\omega)|}\right]^{-2} \to \frac{2}{\pi}\left[\frac{|\omega|R}{c}(\varepsilon(\omega)-1)\right]^{-1}_{(v=c)}. \tag{79}$$

The factor $|B(\omega)|^2$ in formula (59) cuts down strongly the intensity when $|\omega|R/v \gg 1$.

It is interesting to remark that the factor $|B(\omega)|^2$ in (59) does not change appreciably the intensity of the low energy Cerenkov radiation. When $|\omega|R/v \ll 1$ we get from (22):

$$B(\omega) \cong 1\,, \qquad \left(\frac{|\omega|R}{v} \ll 1, \qquad |\omega| \neq \bar{\bar{\omega}}\right). \tag{80}$$

The preceding discussion shows that the essential difference between the present theory and that on Fermi's lines comes from the drastic reduction of the high energy Cerenkov radiation. Whenever the theory on Fermi's lines does not lead to a large contribution of the Cerenkov radiation to the total loss, the results of both theories are not very different. It results from (76) that this may only happen when

$$\log \frac{RI_z}{\hbar c} \cong \frac{1}{2}\,, \tag{81}$$

i.e. for condensed media of low atomic number. It will be shown later that the loss at distances less than R is usually less than in the theory on Fermi's lines.

The loss of energy in close collisions.

7. – Quantum corrections become important in the computation of the average loss at distances less than R, as it was shown in the Introduction. It results from those considerations that the uncertainty principle leads to a broadening of the frequency ranges in which resonance occurs and the emission of coherent radiation is not possible in these ranges. Since the main polarization effect at such distances consists in cutting down the direct ionization-excitation and replacing it by the emission of coherent radiation, it is satisfactory to neglect altogether the polarization effects for $\varrho \ll R$ and to apply the methods of Bethe-Bloch-Möller-Williams. At distances close to R the polarization effects become considerable, as it was shown by A. BOHR ([4]).

In reference ([15]), one of us applied the Bethe-Bloch-Möller-Williams methods up to distances R', R' being close to our R, and the modified form of the Fermi theory to distances larger than R'. Such a treatment is satisfactory, when a high accuracy is not required, and it has the advantage of simplicity. It can be improved by computing the energy transfers at distances less than R with the field $(\boldsymbol{E}^{\text{I}}, \boldsymbol{H}^{\text{I}})$, instead of the fields $(\boldsymbol{E}^{(C)}, \boldsymbol{H}^{(C)})$ used in the quantum theory of Bethe-Bloch-Möller-Williams.

BLOCH ([16]) and WILLIAMS ([17]) have shown, by neglecting polarization effects, that, for distances less than $R_1 \simeq 10^{-8}$ cm, it is necessary to use quantum mechanics. But this is no more necessary for distances larger than R_1, if only the average loss is required, provided the atom is assimilated to a system of classical harmonic oscillators with frequencies ω_j corresponding to the quantum jumps, there being Zf_j oscillators of circular frequency ω_j. We shall separate the contributions of distances less than R_1 and distances between R_1 and R:

$$-\frac{\mathrm{d}W_{<R}}{\mathrm{d}x} = -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x} - \frac{\mathrm{d}W^{\text{I}}_{R_1}}{\mathrm{d}x}, \tag{82}$$

$-\mathrm{d}W^{\text{I}}_{R_1}/\mathrm{d}x$ being the contribution of the atoms between the cylinders C_{R_1} and C_R. We shall derive $-\mathrm{d}W_{<R_1}/\mathrm{d}x$ from the Bethe-Bloch quantum treatment and compute $-\mathrm{d}W^{\text{I}}_{R_1}/\mathrm{d}x$ classically as the rate of work per cm done by the field $\boldsymbol{E}^{\text{I}}$ on the electrons between C_{R_1} and C_R. If we neglect completely the polarization field $\boldsymbol{E}'$, $-\mathrm{d}W^{\text{I}}_{R_1}/\mathrm{d}x$ is replaced by the rate of work done by the field $\boldsymbol{E}^{(C)}$. It is shown in references (5) and (6) that this rate of work $-\mathrm{d}W_{R_1}/\mathrm{d}x$ is:

$$-\frac{\mathrm{d}W_{R_1}}{\mathrm{d}x} = \frac{2\pi n z_1^2 e^4}{mv^2} \sum_j f_j k_0^2(\omega_j)[U(\omega_j, R_1) - U(\omega_j, R)], \tag{83}$$

$$U(\omega, \varrho) = \varrho^2 \left[\beta^2 K_0^2(\varrho k_0(\omega)) - \beta^2 K_1^2(\varrho k_0(\omega)) + \frac{2}{\varrho k_0(\omega)} K_0(\varrho k_0(\omega)) K_1(\varrho k_0(\omega))\right]. \tag{84}$$

For $(z_1e^2/\hbar v)\ll 1$ the total loss is given by the Bethe formula

$$(85)\qquad \left(-\frac{\mathrm{d}W}{\mathrm{d}x}\right)_{\text{Bet.}} = \frac{2\pi n z_1^2 e^4}{mv^2}\left[\log\frac{2mv^2E_{\max}}{I_z^2(1-\beta)^2}-2\beta^2\right], \qquad \left(\frac{z_1e^2}{\hbar v}\ll 1\right),$$

when polarization effects are neglected, $E_{\max}$ being the maximum energy transferable to an electron by the ionizing particle. In order to obtain $-\mathrm{d}W_{<R_1}/\mathrm{d}x$, we must subtract from (85) the contribution of the impact parameters larger than R_1:

$$(86)\qquad -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x} = \left(-\frac{\mathrm{d}W}{\mathrm{d}x}\right)_{\text{Bet.}} - \left(-\frac{\mathrm{d}W_{R_1}}{\mathrm{d}x}\right)_{(R=\infty)}, \qquad \left(\frac{z_1e^2}{\hbar v}\ll 1\right),$$

$$(87)\qquad -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x} = \frac{2\pi n z_1^2 e^4}{mv^2}\left[\log\frac{2mv^2E_{\max}}{I_z^2(1-\beta^2)}-2\beta^2-\sum_j f_j k_0^2(\omega_j)U(\omega_j, R_1)\right], \qquad \left(\frac{z_1e^2}{\hbar v}\ll 1\right).$$

A more general expression can be obtained by replacing the Bethe formula by that of Bloch, which is valid for any value of $z_1e^2/\hbar v$

$$(88)\qquad -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x} = \frac{2\pi n z_1^2 e^4}{mv^2}\left[\log\frac{2mv^2E_{\max}}{I_z^2(1-\beta^2)} - 2\beta^2 + 2\psi(1) - 2\mathcal{R}\left\{\psi\left(1+\frac{iz_1e^2}{\hbar v}\right)\right\} - \sum_j f_j k_0^2(\omega_j)U(\omega_j, R_1)\right],$$

ψ denoting the logarithmic derivative of the gamma function: $\psi(u) = \mathrm{d}/\mathrm{d}u \log\Gamma(u)$. It is important to remark that (87) and (88) do depend on the f_j and ω_j individually and not only through the mean ionization potential I_z. This is a very unpleasant circumstance, because these quantities are in general unknown. However, if we limit the energy transfers to a value E' and consider the limit of very high energies, we get (*):

$$(89)\qquad \lim_{v\to c}\left(-\frac{\mathrm{d}W^{(E')}_{<R_1}}{\mathrm{d}x}\right) = \frac{2\pi n z_1^2 e^4}{mc^2}\left[\log\frac{m\gamma^2R_1^2E'}{2\hbar^2} - 1 + 2\psi(1) - 2\mathcal{R}\left\{\psi\left(1+\frac{iz_1e^2}{\hbar c}\right)\right\}\right].$$

This limit value does not depend any more on the f_j and ω_j.

(*) This formula is valid when $E' > \sqrt{mc^2\hbar\omega_j}$ For small values of E', $E' < \sqrt{mc^2\hbar\omega_j}$ the term -1 inside the bracket in the right hand side is replaced by 0.

A rough approximate value of $-\mathrm{d}W_{<R}/\mathrm{d}x$ can be obtained by adding to (88) the value (83), with R replaced by $R' \approx 0.37R$, in order to take into account the reduction of the loss by the polarization effects at distances close to R:

$$-\frac{\mathrm{d}W_{<R}}{\mathrm{d}x} \approx \frac{2\pi n z_1^2 e^4}{mv^2}\left[\frac{2mv^2 E_{\max}}{I_z^2(1-\beta^2)} - -2\beta^2 + 2\psi(1) - 2\mathcal{R}\left\{\psi\left(1+\frac{iz_1e^2}{\hbar v}\right)\right\} - \sum_j f_j k_0^2(\omega_j) U(\omega_j, R')\right]. \tag{90}$$

In the next section we shall derive a more accurate expression and justify the choice of R'.

8. – In order to determine the density of polarization $\boldsymbol{P}^{\mathrm{I}}$ induced by $\boldsymbol{E}^{\mathrm{I}}$, we shall write

$$\boldsymbol{E}^{\mathrm{I}} = \int_{-\infty}^{+\infty} \mathcal{E}^{\mathrm{I}}(\omega) \exp\left[i\omega\left(\frac{x}{v} - t\right)\right] \mathrm{d}\omega\,, \tag{91}$$

hence

$$\boldsymbol{P}^{\mathrm{I}} = \frac{1}{4\pi}(\varepsilon_{\mathrm{op}} - 1)\boldsymbol{E}^{\mathrm{I}} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}(\varepsilon(\omega) - 1)\mathcal{E}^{\mathrm{I}}(\omega)\exp\left[i\omega\left(\frac{x}{v} - t\right)\right]\mathrm{d}\omega\,. \tag{92}$$

$-\mathrm{d}W^{\mathrm{I}}_{R_1}/\mathrm{d}x$ is equal to the rate of work per cm done by $\boldsymbol{E}^{\mathrm{I}}$ on the electrons between C_{R_1} and C_R:

$$-\frac{\mathrm{d}W^{\mathrm{I}}_{R_1}}{\mathrm{d}x} = \frac{1}{v}\int_{R_1}^{R} \varrho\,\mathrm{d}\varrho \int_0^{2\pi}\mathrm{d}\varphi \int_{-\infty}^{+\infty}\left(\boldsymbol{E}^{\mathrm{I}}\cdot\frac{\partial \boldsymbol{P}^{\mathrm{I}}}{\partial t}\right)\mathrm{d}x = \tag{93}$$

$$= 2\pi\int_{R_1}^{R}\varrho\,\mathrm{d}\varrho\,\mathcal{R}\int_0^{+\infty}(1-\varepsilon(\omega))(\mathcal{E}^{\mathrm{I}}(\omega)\cdot\mathcal{E}^{\mathrm{I}}(-\omega))\,i\omega\,\mathrm{d}\omega\,.$$

It results from (32)-(33) and (35)-(36) that:

$$-\frac{\mathrm{d}W^{\mathrm{I}}_{R_1}}{\mathrm{d}x} = -\frac{\mathrm{d}W_{R_1}}{\mathrm{d}x} + 2\pi\int_{R_1}^{R}\Omega(\varrho)\varrho\,\mathrm{d}\varrho\,, \tag{94}$$

(95) $$\Omega(\varrho) = \frac{z_1^2 e^2}{2\pi v^4}(1-\beta^2)$$

$$\mathcal{R}\cdot\Bigg[\frac{\pi}{2}\int_0^\infty i\varepsilon(\omega)C(\omega)C(-\omega)\{(1-\beta^2)I_0^2(\varrho k_0(\omega)) + I_1^2(\varrho k_0(\omega))\}\omega^3\, d\omega + \int_0^\infty \varepsilon(\omega)(C(\omega)+$$

$$+ C(-\omega))\{(1-\beta^2)K_0(\varrho k_0(\omega))I_0(\varrho k_0(\omega)) - K_1(\varrho k_0(\omega))I_1(\varrho\, k_0(\omega))\}\omega^3\, d\omega\Bigg].$$

We get from (23) and (52):

(96) $$\frac{\omega^2}{v^2}\varepsilon(\omega)[C(\omega)+C(-\omega)] = \frac{4i\omega^2}{\pi v^2}\varepsilon(\omega)\frac{K_0(Rk_0(\omega))}{I_0(Rk_0(\omega))}+$$

$$+\left[\frac{k(\omega)}{D(\omega)}\frac{K_0(Rk(\omega))}{K_1(Rk(\omega))} + \frac{\varepsilon(\omega)}{\varepsilon(-\omega)}\frac{k(-\omega)}{D(-\omega)}\frac{K_0(Rk(-\omega))}{K_1(Rk(-\omega))}\right]\left\{\frac{i\pi}{2}(1-\beta^2)RI_0(Rk_0(\omega))\right\}^{-1}.$$

(97) $$\frac{\omega^2}{v^2}\varepsilon(\omega)C(\omega)C(-\omega) = -\frac{4\omega^2}{\pi^2 v^2}\varepsilon(\omega)\frac{K_0^2(Rk_0(\omega))}{I_0^2(Rk_0(\omega))}+$$

$$+\left[\frac{k(\omega)}{D(\omega)}\frac{K_0(Rk(\omega))}{K_1(Rk(\omega))} + \frac{\varepsilon(\omega)}{\varepsilon(-\omega)}\frac{k(-\omega)}{D(-\omega)}\frac{K_0(Rk(-\omega))}{K_1(Rk(-\omega))}\right]\frac{K_0(Rk_0(\omega))}{I_0^2(Rk_0(\omega))}\left\{\frac{\pi^2}{4}(1-\beta^2)R\right\}^{-1} -$$

$$-\frac{1}{\varepsilon(-\omega)}\frac{k(\omega)k(-\omega)}{D(\omega)D(-\omega)}\frac{K_0(Rk(\omega))K_0(Rk(-\omega))}{K_1(Rk(\omega))K_1(Rk(-\omega))}\left\{\frac{\pi^2}{4}\frac{\omega^2}{v^2}(1-\beta^2)^2R^2I_0^2(Rk_0(\omega))\right\}^{-1}.$$

We must start with a complex $\varepsilon(\omega)$ and then make the damping constants g tend to 0. In the limit of zero damping constants, the only contributions to $\Omega(\varrho)$ come from the poles of the quantities under the integrals in (95), since these quantities are imaginary in the limit. The first term in the right hand side of (96) and the first term in the right hand side of (97) have the same poles as $\varepsilon(\omega)$. In the corresponding integrals the integration paths will be indented with infinitesimal semi circles centred at the poles of $\varepsilon(\omega)$ and lying in the upper part of the ω-plane, because these poles have negative imaginary parts when there is damping. The second group of terms in (96) and the second group of terms in (97) have simple poles due either to $D(\omega)$ or $D(-\omega)$. In the corresponding integrals, the integration paths must be indented with infinitesimal semi-circles centred at the $\tilde{\omega}_j$ and lying above the real ω-axis in the case of $D(\omega)$, and below that axis in the case of $D(-\omega)$. The last term in the right hand side of (97) has double poles at the $\tilde{\omega}_j$, in the limit of zero damping. The corresponding integral can be computed by the same method applied to (46).

We shall denote by $\Omega_1(\varrho)$ the part of $\Omega(\varrho)$ due to the poles of $\varepsilon(\omega)$ and by $\Omega_2(\varrho)$ the part of $\Omega(\varrho)$ due to the poles at the ω_j. It is easily seen that:

$$(98)\qquad \Omega_1(\varrho) = \frac{2z_1^2e^2}{\pi v^4}(1-\beta^2)\sum_j b_j\bar{\bar{\omega}}_j^3\Big[(1-\beta^2)K_0(\varrho k_0(\bar{\bar{\omega}}_j))I_0(\varrho k_0(\bar{\bar{\omega}}_j)) -$$

$$- K_1(\varrho k_0(\bar{\bar{\omega}}_j))I_1(\varrho k_0(\bar{\bar{\omega}}_j)) - \frac{1}{2}\frac{K_0(Rk_0(\bar{\bar{\omega}}_j))}{I_0(Rk_0(\bar{\bar{\omega}}_j))}\{(1-\beta^2)I_0^2(\varrho k_0(\bar{\bar{\omega}}_j)) + I_1^2(\varrho k_0(\bar{\bar{\omega}}_j))\}\Big],$$

$$(99)\qquad b_j = \frac{K_0(Rk_0(\bar{\bar{\omega}}_j))}{I_0(Rk_0(\bar{\bar{\omega}}_j))}\ \underset{\bar{\bar{\omega}}_j}{\text{Residue}}\ \varepsilon(\omega) \simeq -2\pi^2\alpha^2\frac{f_j}{\omega_j}\frac{K_0(Rk_0(\bar{\bar{\omega}}_j))}{I_0(Rk_0(\bar{\bar{\omega}}_j))}.$$

The contribution of the poles $\bar{\bar{\omega}}_j$ is small at relativistic energies, since:

$$(100)\qquad \lim_{v\to c}\Omega_1(\varrho) = 0\,.$$

The contributions of each pair of terms within the square brackets in (96) and (97) cancel, so that the only contribution to $\Omega_2(\varrho)$ comes from the last term in (97):

$$(101)\qquad \Omega_2(\varrho) = -\frac{z_1^2e^2}{\pi v^2R^2}\times$$

$$\times\sum_j \underset{\tilde{\omega}_j}{\text{Res}}\left[\frac{1-\beta^2\varepsilon(\omega)}{(1-\beta^2)\varepsilon(\omega)}\frac{K_0^2(Rk(\omega))}{K_1^2(Rk(\omega))}\left\{(1-\beta^2)\frac{I_0^2(\varrho\, k_0(\omega))}{I_0^2(Rk_0(\omega))} + \frac{I_1^2(\varrho k_0(\omega))}{I_0^2(Rk_0(\omega))}\right\}\omega D^{-2}(\omega)\right].$$

By using the well know Lommel formulas

$$(102)\qquad \int K_l(\lambda u)I_l(\lambda u)u\,du = \frac{u^2}{2}\left[K_l(\lambda u)I_l(\lambda u) + K_{l+1}(\lambda u)I_{l-1}(\lambda u)\right],$$

$$(103)\qquad \int I_l^2(\lambda u)u\,du = \frac{u^2}{2}\left[I_l^2(\lambda u) - I_{l-1}(\lambda u)I_{l+1}(\lambda u)\right];$$

we get:

$$(104)\qquad 2\pi\int_{R_1}^{R}\Omega_1(\varrho)\varrho\,d\varrho \simeq 2\pi\int_0^R\Omega_1(\varrho)\varrho\,d\varrho =$$

$$= -\frac{z_1^2e^2R^2}{v^2c^2}(1-\beta^2)\sum_j b_j\bar{\bar{\omega}}_j^3\left[K_0(Rk_0(\bar{\bar{\omega}}_j))I_0(Rk_0(\bar{\bar{\omega}}_j)) + K_1(Rk_0(\bar{\bar{\omega}}_j))I_1(Rk_0(\bar{\bar{\omega}}_j)) -\right.$$

$$\left.- 2(\beta^2Rk_0(\bar{\bar{\omega}}_j))^{-1}K_0(Rk_0(\bar{\bar{\omega}}_j))I_1(Rk_0(\bar{\bar{\omega}}_j)) + I_1(Rk_0(\bar{\bar{\omega}}_j))\{Rk_0(\bar{\bar{\omega}}_j)I_0(Rk_0(\bar{\bar{\omega}}_j))\}^{-1}\right].$$

The integration od $\varrho\Omega_2(\varrho)$ can also be done by using (102) amd (103):

$$(105)\qquad 2\pi\int_{R_1}^{R}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho \cong 2\pi\int_{0}^{K}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho =$$

$$= -\frac{z_1^2e^2}{c^2}\sum_j \operatorname*{Res}_{\tilde{\omega}_j}\left[\frac{1-\beta^2\varepsilon(\omega)}{(1-\beta^2)\varepsilon(\omega)}\,\frac{K_0^2(Rk(\omega))}{K_1^2(Rk(\omega))}\left\{\frac{I_1^2(Rk_0(\omega))}{I_0^2(Rk_0(\omega))}+\right.\right.$$

$$\left.\left.+\,2(\beta^2Rk_0(\omega))^{-1}\frac{I_1(Rk_0(\omega))}{I_0(Rk_0(\omega))}-1\right\}\omega(D(\omega))^{-2}\right]\cong$$

$$\cong -\frac{z_1^2e^2R^2}{c^2}\sum_j\frac{K_0(Rk(\tilde{\omega}_j))}{K_1(Rk(\tilde{\omega}_j))}\operatorname*{Res}_{\tilde{\omega}_i}\left[\frac{1-\beta^2\varepsilon(\omega)}{(1-\beta^2)\varepsilon(\omega)}B^2(\omega)K_0(Rk(\omega))K_1(Rk(\omega))\times\right.$$

$$\left.\times k^2(\omega)\omega\left\{\frac{I_1^2(Rk_0(\omega))}{I_0^2(Rk_0(\omega))}+2(\beta^2Rk_0(\omega))^{-1}\frac{I_1(Rk_0(\omega))}{I_0(Rk_0(\omega))}-1\right\}\right].$$

It results from the equation (18) of the Appendix II that:

$$(106)\qquad \operatorname*{Res}_{\tilde{\omega}_j}\left[\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)B^2(\omega)K_0(Rk(\omega))K_1(Rk(\omega))k(\omega)\omega\,\times\right.$$

$$\left.\times\left\{\frac{I_1^2(Rk_0(\omega))}{I_0^2(Rk_0(\omega))}+2(\beta^2Rk_0(\omega))^{-1}\frac{I_1(Rk_0(\omega))}{I_0(Rk_0(\omega))}-1\right\}\right]=$$

$$=\left[\frac{I_1^2(Rk_0(\tilde{\omega}_j))}{I_0^2(Rk_0(\tilde{\omega}_j))}+2(\beta^2Rk_0(\tilde{\omega}_j))^{-1}\frac{I_1(Rk_0(\tilde{\omega}_j))}{I_0(Rk_0(\tilde{\omega}_j))}-1\right]\times$$

$$\times\frac{K_0(Rk(\tilde{\omega}_j))}{K_1(Rk(\tilde{\omega}_j))}\,\frac{k(\tilde{\omega}_j)}{\varepsilon(\tilde{\omega}_j)}\left(\frac{\tilde{\omega}_jR}{v}\right)^{-2}\frac{\omega_j\left(\dfrac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)}}{I_0(Rk_0(\tilde{\omega}_j))}.$$

The residue of the product of a function $F_0(u)$, regular at u, by a function $F_2(u)$ having a second order pole at u is given by the formula:

$$(107)\qquad \operatorname*{Residue}_{\tilde{u}}\left[F_2(u)F_0(u)\right]=F_0(\tilde{u})\operatorname*{Residue}_{\tilde{u}}F_2(u)+4\left[\frac{\mathrm{d}F_0}{\mathrm{d}u}F_2^3\left(\frac{\mathrm{d}F_2}{\mathrm{d}u}\right)^{-2}\right]_{(u=\tilde{u})}.$$

It follows from (105), (106) and (107) that:

$$(108)\qquad 2\pi\int_{R_1}^{R}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho\cong\frac{z_1^2e^2}{v^2}\sum_j\frac{K_0(Rk(\omega_j))}{K_1(Rk(\tilde{\omega}_j))}\,\frac{k(\tilde{\omega}_j)}{k_0(\tilde{\omega}_j)}\,\frac{\tilde{\omega}_j\left(\dfrac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)}}{I_0^2(Rk_0(\tilde{\omega}_j))I_1(Rk_0(\tilde{\omega}_j))}\times$$

$$\times\left[\beta^2\{I_1^2(Rk_0(\tilde{\omega}_j))-I_0^2(Rk_0(\tilde{\omega}_j))\}+2(Rk_0(\tilde{\omega}_j))^{-1}I_0(Rk_0(\tilde{\omega}_j))I_1(Rk_0(\tilde{\omega}_j))\right]\times$$

$$\times\left[1+I_0(Rk_0(\tilde{\omega}_j))\left\{\left(\frac{\mathrm{d}}{\mathrm{d}\omega}\log k(\omega)\right)\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}\right\}_{(\omega=\tilde{\omega}_j)}\right].$$

The expression (66) of $(\mathrm{d}D/\mathrm{d}\omega)_{(\omega=\tilde{\omega}_j)}$ is not sufficient, because it is only applicable when $|\varepsilon(\tilde{\omega}_j)|\gg 1$. A more general formula will be derived in Appendix III:

$$\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)} \cong \frac{4\pi ne^2 f_j}{m\tilde{\omega}_j I_0(Rk_0(\tilde{\omega}_j))}\,\frac{1-\beta^2\varepsilon(\tilde{\omega}_j)}{2-\beta^2\varepsilon(\tilde{\omega}_j)}\,\frac{\varepsilon(\tilde{\omega}_j)}{(1-\varepsilon(\tilde{\omega}_j))^2}\,. \tag{109}$$

The approximate expression of $\varepsilon(\omega)$ given in Appendix III shows that:

$$\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\log k(\omega)\right]_{(\omega=\tilde{\omega}_j)} \cong -\frac{\beta^2 m\tilde{\omega}_j}{4\pi ne^2 f_j}\,\frac{(1-\varepsilon(\tilde{\omega}_j))^2}{1-\beta^2\varepsilon(\omega_j)}\,. \tag{110}$$

Finally we get:

$$\begin{aligned} 2\pi\int_{R_1}^{R}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho \cong{}& -\frac{8\pi n z_1^2 e^4}{mv^2}\sum_j f_j\left(\frac{\varepsilon(\tilde{\omega}_j)}{1-\varepsilon(\tilde{\omega}_j)}\right)^2\left(\frac{1-\beta^2\varepsilon(\tilde{\omega}_j)}{2-\beta^2\varepsilon(\tilde{\omega}_j)}\right)^2\times\\ &\times[\beta^2\{I_1^2(Rk_0(\tilde{\omega}_j))-I_0^2(Rk_0(\tilde{\omega}_j))\}+2(Rk_0(\tilde{\omega}_j))^{-1}I_0(Rk_0(\tilde{\omega}_j))I_1(Rk_0(\tilde{\omega}_j))]\times\\ &\times[I_0(Rk_0(\tilde{\omega}_j))I_1(Rk_0(\tilde{\omega}_j))]^{-2}\,. \end{aligned} \tag{111}$$

This correction remains finite as v tends to c:

$$\lim_{v\to c}\left[2\pi\int_{R_1}^{R}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho\right] \cong -\frac{4\pi n z_1^2 e^4}{mc^2}\sum_j f_j\left[\left(1+2\left(\frac{\tilde{\omega}_j R}{2c}\right)^{-2}\right)\left(\frac{\varepsilon(\tilde{\omega}_j)}{2-\varepsilon(\tilde{\omega}_j)}\right)^2\right]_{(v=c)}\,. \tag{112}$$

For non relativistic energies the $Rk_0(\omega)$ have high values and the corrections due to due to Ω_1 and Ω_2 are negligible, because of the asymptotic behaviour of the I functions:

$$I_\nu(u) \sim \frac{e^u}{\sqrt{2\pi u}}\left[1+\frac{1-4\nu^2}{8u}+\dots\right]. \tag{113}$$

We get from (111) and (113):

$$\begin{aligned} 2\pi\int_{R_1}^{R}\Omega_2(\varrho)\varrho\,\mathrm{d}\varrho \cong{}& \\ \cong -\frac{16\pi^2 n z_1^2 e^4}{mv^2}(2-\beta^2)\sum_j&\left[\frac{\varepsilon(\tilde{\omega}_j)(1-\beta^2\varepsilon(\tilde{\omega}_j))}{(1-\varepsilon(\tilde{\omega}_j))(2-\beta^2\varepsilon(\tilde{\omega}_j))}\right]^2 Rk_0(\tilde{\omega}_j)\exp[-2Rk_0(\tilde{\omega}_j)]\,, \end{aligned} \tag{114a}$$

$$(\sqrt{1-\beta^2}\cong 1)\,.$$

It results from (104) that:

$$(114b)\qquad 2\pi\int_{R_1}^{R}\Omega_1(\varrho)\varrho\,\mathrm{d}\varrho \cong \frac{4\pi^2 n z_1^2 e^4}{mv^2}(1-\beta^2)\sum_j f_j\left(\frac{\omega_j R}{c}\right)^2\frac{\exp[-2Rk_0(\omega_j)]}{Rk_0(\omega_j)},$$

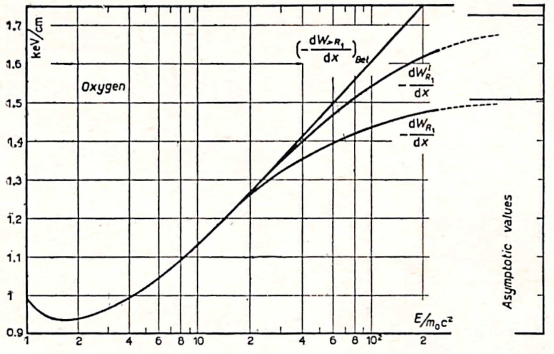

Fig. 1. – Rate of ionization at distances larger than $R_1 = 10^{-8}$ cm.

with $(\sqrt{1-\beta^2}\cong 1)$.

For extremely relativistic energies the correction (111) is roughly equal to $-4\pi n z_1^2 e^4/mc^2$, as a consequence of (112). Therefore we may take in (90):

$$(115)\qquad \begin{cases}\log\dfrac{R}{R'}\approx 1\,,\\[2mm] R'\approx 0.37R\,.\end{cases}$$

The saturation value of the total loss per cm, with transfers limited to E', is roughly:

$$(116)\ \lim_{v\to c}\left(-\frac{\mathrm{d}W^{(E')}}{\mathrm{d}x}\right)\approx\frac{2\pi n z_1^2 e^4}{mc^2}\left[\log\frac{m\gamma^2R^2E'}{2\hbar^2}-2+2\psi(1)-2\,\mathscr{R}\left\{\psi\left(1+\frac{iz_1e^2}{\hbar c}\right)\right\}\right].$$

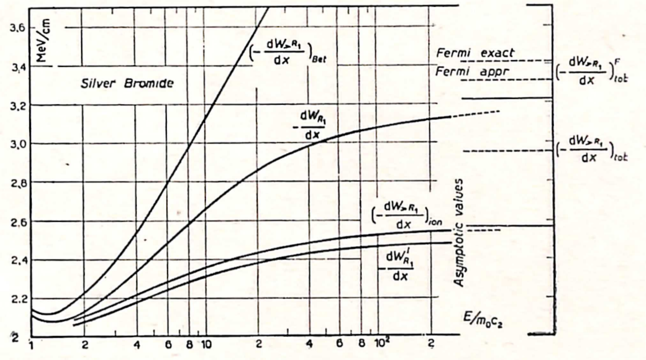

Fig. 2. – Rate of ionization at distances larger than $R_1 = 10^{-8}$ cm.

This value is smaller than that given by the theory on Fermi's lines:

$$(117)\ \lim_{v\to c}\left(-\frac{\mathrm{d}W^{(E')}}{\mathrm{d}x}\right)^{(\mathrm{F})}\cong\frac{2\pi n z_1^2 e^4}{mc^2}\left[\log\frac{4mR^2E'}{2\hbar^2}-1+2\psi(1)-2\mathscr{R}\left\{\psi\left(1+\frac{iz_1e^2}{\hbar c}\right)\right\}\right].$$

The variation of the Bethe loss at distances larger than R_1 $(-\mathrm{d}W_{>R_1}/\mathrm{d}x)_{\mathrm{Bet.}}$, of $-\mathrm{d}W_{R_1}/\mathrm{d}x$ and of $-\mathrm{d}W^{\mathrm{I}}_{R_1}/\mathrm{d}x$ with the energy of the ionizing particle are represented in fig. 1 for oxygen (normal conditions) and in fig. 2 for AgBr crystals. In fig. 2 the variation of the rate of ionisation and excitation at distances larger than R is also included (*).

Discussion of the energy loss due to close collisions.

9. – In the computation of the energy loss at distances less than R, we took into account the interactions of the electrons of this region with the ionizing particle and with the electrons outside C_R, but we neglected the interactions between electrons lying inside C_R. The contribution of the electrons at distances less than R to the field outside C_R has also been neglected in the computation of the loss at distances larger than R. If we would have taken into account the contributions of the electrons inside C_R to the field, both within and outside C_R, by applying classical mechanics and electrodynamics, we would have, of course, been led to the theory on Fermi's lines. In our treatment, the loss at distances between R_1 and R consists of ionization and excitation, whereas in the theory on Fermi's lines the main effect of the electrons between R_1 and R at high relativistic energies is, in most cases, to give rise to the high energy Cerenkov radiation, as we shall now prove. It was shown in references (5) and (6) that the rate of work done by the field on the electrons at distances between ϱ and $\varrho + \mathrm{d}\varrho$, in the theory on Fermi's lines, is

$$(118) \qquad -\frac{\mathrm{d}w_{\mathrm{F}}(\varrho)}{\mathrm{d}x}\,\mathrm{d}\varrho = \frac{4\pi n z_1^2 e^4}{mv^2}\,\varrho\,\mathrm{d}\varrho \sum_j f_j'\overline{\omega}_j^2\left[K_1^2\left(\frac{\overline{\omega}_j\varrho}{v}\right) + K_0^2\left(\frac{\overline{\omega}_j\varrho}{v}\right)\right].$$

with

$$(119) \qquad 2\pi^2\alpha^2 f_j' = a_{\overline{\omega}_j}\overline{\omega}_j\,, \qquad (f_j' \simeq f_j)$$

the notations being the same as in formula (48). The corresponding quantity in the Bethe-Bloch-Möller-Williams theory is

$$(120) \qquad -\frac{\mathrm{d}w_B(\varrho)}{\mathrm{d}x}\,\mathrm{d}\varrho = \frac{4\pi n z_1^2 e^4}{mv^2}\,\varrho\,\mathrm{d}\varrho \sum_j f_j k_0^2(\omega_j)\left[K_1^2(\varrho k_0(\omega_j)) + (1-\beta^2)K_0^2(\varrho k_0(\omega_j))\right],$$

and in ours

$$(121) \qquad -\frac{\mathrm{d}w(\varrho)}{\mathrm{d}x}\,\mathrm{d}\varrho = -\frac{\mathrm{d}w_B(\varrho)}{\mathrm{d}x}\,\mathrm{d}\varrho + 2\pi\Omega(\varrho)\varrho\,\mathrm{d}\varrho\,.$$

(*) In fig. 2 the value obtained with the approximate formula of Halpern and Hall is indicated as Fermi approximate.

The total loss by ionization and excitation in the interval of distances $R_1 - R$, in the theory on Fermi's lines, is:

$$(122)\quad \int_{R_1}^{R} -\frac{\mathrm{d}w_F(\varrho)}{\mathrm{d}x}\,\mathrm{d}\varrho = \frac{4\pi n z_1^2 e^4}{mv^2}\sum_j f_j'\left[\frac{\overline{\omega}_j R_1}{v} K_1\left(\frac{\overline{\omega}_j R_1}{v}\right) K_0\left(\frac{\overline{\omega}_j R_1}{v}\right) - \frac{\overline{\omega}_j R}{v} K_1\left(\frac{\overline{\omega}_j R}{v}\right) K_0\left(\frac{\overline{\omega}_j R}{v}\right)\right] \cong \frac{4\pi n z_1^2 e^4}{mv^2}\log\frac{2\hbar v}{\gamma R_1 I_z}\,.$$

At extremely relativistic energies the rate of emission of Cerenkov radiation per cm is given by (76) and is larger than the value (122) for most elements in a condensed state and even more in the gaseous state. The Cerenkov radiation is due to the interaction of the ionizing particle with electrons at distances larger than atomic dimensions and the high energy bands, which account for the largest contribution in that theory, arise largely from interactions with electrons at distances less than R. (This is shown by the fact that, in our treatment, in which the electrons at distances less than R are not taken into account in this connection, the high energy bands are practically suppressed). Therefore, we may conclude that the main effect of the electrons in the interval of distance $R_1 - R$, at high relativistic energies, in the theory on Fermi's lines, is to give rise to the high energy Cerenkov radiation, in most cases.

In our treatment the quantity corresponding to (122) is $-\mathrm{d}W_{R_1}^{\mathrm{I}}/\mathrm{d}x$. It results from (83) and (115) that:

$$(123)\qquad \lim_{v\to c}\left(-\frac{\mathrm{d}W_{R_1}^{\mathrm{I}}}{\mathrm{d}x}\right) \approx \frac{4\pi n z_1^2 e^4}{mc^2}\left(\log\frac{R}{R_1} - 1\right).$$

This value is larger than that given by (122) in all cases, excepting condensed media of very low atomic number.

We have already considered in the Introduction reasons for not taking into account the field of the electrons in the interval of distances $R - R_1$, in the computation of the loss. We shall now examine this point in more detail. The influence of the field created by the electrons in the loss of energy is not of the same nature as that of the Coulomb force of the nuclei. It is an effect of the field due to the acceleration of the electrons and the coherence of the contributions of the different electrons plays an essential part. In order to see whether the interference of the fields of the electrons belonging to different atoms may be of importance, we must consider the times of collision. We (shall denote by $T_c(\varrho)$

$$(124)\qquad T_c(\varrho) = \frac{2\varrho}{v}\sqrt{1-\beta^2}$$

the time of collision defined, as usually, with the Lorentz transformed Coulomb field of the ionizing particle. The important contributions to the coherent fields of the electrons are due to particles whose phase differences correspond to time lags τ of the order of $T_c(\varrho)$. It results from the uncertainty principle that

$$\Delta\tau\Delta\omega \geqslant 1\,, \tag{125}$$

and, since we need τ of the order of $T_c(\varrho)$, we must have

$$\Delta\tau \ll T_c(\varrho)\,, \tag{126}$$

hence:

$$\Delta\omega \gg \frac{v}{2\varrho\sqrt{1-\beta^2}}\,. \tag{127}$$

Important coherence effects can only occur in frequency intervals of the order of $\bar{\omega}_j - \omega_j$. It was shown in references (5) and (6) that:

$$\bar{\omega}_j - \omega_j \approx \frac{4\pi^2\alpha^2 f_j}{3\omega_j} = \frac{\hbar c^2}{3R^2}\frac{f_j}{\hbar\omega_j}\,. \tag{128}$$

In order to have important coherence effects the uncertainty $\Delta\omega$ must satisfy the condition

$$\Delta\omega \ll \frac{\hbar c^2}{R^2 I_z}\,, \tag{129}$$

which requires that:

$$\frac{\varrho}{R} \gg \frac{R I_z}{\hbar c}\frac{\beta}{\sqrt{1-\beta^2}} \approx \frac{Z}{137}\frac{R}{R_1}\frac{\beta}{\sqrt{1-\beta^2}}\,. \qquad \left(I_z \approx \frac{Ze^2}{R_1}\right) \tag{130}$$

Since the minimum of ionization corresponds to $\sqrt{1-\beta^2} \approx 1/3$, the condition (130) cannot be satisfied, after the minimum, at distances less than R. The condition (126) is certainly too stringent, since the polarization effects lead to an increase of the times of collision. The preceding considerations should therefore not be valid when the polarization effects are important, i.e. at distances larger than R. Moreover the small frequencies become very important. In most of the interval $R_1 - R$ the considerations leading to (130) are valid and show that the contribution of the electrons of that interval to the field may be neglected. The applicability of the classical treatment of sections 2, 3 and 4 to the atoms at distances larger than R depends essentially on the modification of the times of collision due to the strong polarization effects. We shall examine this point in the following section.

10. – The value (124) of the time of collision $T_c(\varrho)$ follows immediately from the formula:

(131) $$E_\varrho^{(c)} = \frac{z_1 e(1-\beta^2)\varrho}{[\varrho^2(1-\beta^2)+(x-vt)^2]^{3/2}}\,.$$

We cannot apply the same method to the case of a dispersive medium, because we do not have any more a formula corresponding to (131). There is, however, another method of estimation of $T_c(\varrho)$ based on the Fourier integral (33). It is well known that the duration of an electromagnetic pulse, whose spectrum covers an interval $\Delta\omega$, is of the order of $2\pi/\Delta\omega$. Equation (33) shows that the width $\Delta\omega$ at the distance ϱ is determined by the condition

(132) $$\varrho k_0(\Delta\omega) \approx 1\,, \qquad \Delta\omega \approx \frac{v}{\varrho\sqrt{1-\beta^2}}\,,$$

which leads to (124). In the case of the Fermi field, the function $K_1(\varrho k_0(\omega))$ is replaced by $K_1(\varrho k(\omega))$:

(133) $$E_\varrho^{(F)} = \frac{z_1 e}{\pi v}\int_{-\infty}^{+\infty} \varepsilon^{-1}(\omega)k(\omega)K_1(\varrho k(\omega)) \exp\left[i\omega\left(\frac{x}{v}-t\right)\right] d\omega\,.$$

When ω is much larger than the oscillator frequencies ω_j, we have:

(134) $$\varepsilon(\omega) \cong 1 - \frac{4\pi^2\alpha^2}{\omega^2} = 1 - \frac{c^2}{\omega^2 R^2}\,, \qquad (\omega \gg \omega_j).$$

Hence

(135) $$\varrho k(\omega) \cong \sqrt{\frac{\varrho^2}{R^2} + \varrho^2 k_0^2(\omega)}\,, \qquad (\omega \gg \omega_j),$$

and at distances larger than R the contributions of the large frequencies to $E_\varrho^{(F)}$ become small, since

(136) $$\varrho k(\omega) > 1 \qquad (\omega \gg \omega_j,\ \varrho > R)\,.$$

The preceding considerations show that the distance R behaves as a radius of action in the theory on Fermi's lines, as it was shown in an elementary way by A. BOHR [4].

It results from (33) that:

(137) $$\lim_{v\to c} E_\varrho^{(c)} = \frac{z_1 e}{\pi c \varrho}\int_{-\infty}^{+\infty} \exp\left[i\omega\left(\frac{x}{c}-t\right)\right] d\omega = \frac{2z_1 e}{\varrho}\,\delta(x-ct)\,.$$

This formula shows clearly that the collision time T_c vanishes in the limit of $v = c$; the frequency range that gives important contributions to the field is now infinite. In the case of the Fermi field

$$(138) \qquad \lim_{v \to c} E_\varrho^{(F)} = \frac{z_1 e}{\pi c} \int_{-\infty}^{+\infty} \varepsilon^{-1}(\omega) k_c(\omega) K_1(\varrho k_c(\omega)) \exp\left[i\omega\left(\frac{x}{c} - t\right)\right] d\omega,$$

and the transverse component of the electric field is not delta-like in the limit of $v = c$. This could also be seen by taking into account that the range of frequencies giving important contributions to the field does not tend to ∞, as a consequence of (136).

In the case of E^{II}, (133) is replaced by (28). It results from (22) that:

$$(139) \qquad B(\omega) \cong 1 \,, \qquad (\omega \gg \omega_j).$$

The contribution of the very high frequencies is not larger than in the case of the Fermi field.

Summary of the formulas.

11. - We shall now summarize the main formulas. We shall not take into account the corrections corresponding to the case in which the ionizing particle is an electron, as we did already in section 7 (*). The total rate of loss per cm $-\mathrm{d}W/\mathrm{d}x$ is

$$(140) \qquad -\frac{\mathrm{d}W}{\mathrm{d}x} = -\frac{\mathrm{d}W_{<R}}{\mathrm{d}x} - \frac{\mathrm{d}W_{>R}}{\mathrm{d}x},$$

$-\mathrm{d}W_{<R}/\mathrm{d}x$ and $-\mathrm{d}W_{>R}/\mathrm{d}x$ being the contributions of distances smaller than R and larger than R, respectively, with

$$(141) \qquad R = \sqrt{\frac{mc^2}{4\pi n e^2}}\,, \qquad \begin{array}{l}(m\text{: mass of the electron})\\ (n\text{: number of electrons per cm}^3)\,.\end{array}$$

The loss at distances less than R is the sum of two terms:

$$(142) \qquad -\frac{\mathrm{d}W_{<R}}{\mathrm{d}x} = -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x} - \frac{\mathrm{d}W^{\text{I}}_{R_1}}{\mathrm{d}x}, \qquad (R_1 = 10^{-8}\ \text{cm}).$$

(*) Our formulas are nevertheless applicable to fast electrons when high energy transfers may be disregarded (see MÖLLER ([16])).

$-\mathrm{d}W_{<R_1}/\mathrm{d}x$ is given by a modified form of the Bethe formula:

$$(143)\quad -\frac{\mathrm{d}W_{<R_1}}{\mathrm{d}x}=\frac{2\pi n z_1^2 e^4}{mv^2}\left[\log\frac{2mv^2E_{\max}}{I_z^2(1-\beta^2)}-2\beta^2-\sum_j f_j k_0^2(\omega_j)U(\omega_j,R_1)\right],$$

$$(143a)\quad \frac{z_1e^2}{\hbar v}\ll 1\,,$$

(v = velocity of the ionizing particle) ($z_1 e$ = charge of the ionizing particle).

The ω and f are the oscillator frequencies (circular) and percentuals, respectively. I_z is the mean ionization potential of the atoms of the ionized medium. The function U is:

$$(144)\quad U(\omega,\varrho)=\varrho^2\left[\beta^2\{K_0^2(\varrho k_0(\omega))-K_1^2(\varrho k_0(\omega))\}+\frac{2}{\varrho k_0(\omega)}K_0(\varrho k_0(\omega))K_1(\varrho k_0(\omega))\right].$$

The K are modified Hankel functions and

$$(144a)\quad k_0(\omega)=\frac{\omega}{v}\sqrt{1-\beta^2}\,.$$

$-\mathrm{d}W^{\mathrm{I}}_{R_1}/\mathrm{d}x$ is the contribution of the distances between R_1 and R:

$$(145)\quad -\frac{\mathrm{d}W^{\mathrm{I}}_{R_1}}{\mathrm{d}x}=\frac{2\pi n z_1^2 e^4}{mv^2}\sum_j f_j k_0^2(\omega_j)[U(\omega_j,R_1)-U(\omega_j,R)]+2\pi\int_{R_1}^{R}\Omega(\varrho)\varrho\,\mathrm{d}\varrho\,.$$

The last term in the right hand side of (145) is the correction to the loss at distances less than R due to the polarization at distances larger than R:

$$(146)\quad \Omega(\varrho)=\Omega_1(\varrho)+\Omega_2(\varrho)\,,$$

$$(147)\quad 2\pi\int_{R_1}^{R}\Omega_1(\varrho)\varrho\,\mathrm{d}\varrho\cong-\frac{z_1^2e^2R^2}{v^2c^2}(1-\beta^2)\sum_j\overline{\overline{\omega}}_j^3[K_0(Rk_0(\overline{\overline{\omega}}_j))I_0(Rk_0(\overline{\overline{\omega}}_j))+$$
$$+K_1(Rk_0(\overline{\overline{\omega}}_j))I_1(Rk_0(\overline{\overline{\omega}}_j))-2(\beta^2Rk_0(\overline{\overline{\omega}}_j))^{-1}K_0(Rk_0(\overline{\overline{\omega}}_j))I_1(Rk_0(\overline{\overline{\omega}}_j))+$$
$$+\{Rk_0(\overline{\overline{\omega}}_j)I_0(Rk_0(\overline{\overline{\omega}}_j))\}^{-1}I_1(Rk_0(\overline{\overline{\omega}}_j))]\frac{K_0(Rk_0(\overline{\overline{\omega}}_j))}{I_0(Rk_0(\overline{\overline{\omega}}_j))}\operatorname*{Res}_{\overline{\overline{\omega}}_j}\varepsilon(\omega)\,.$$

$\varepsilon(\omega)$ is the dielectric constant of the medium for waves of circular frequency ω:

$$(147a)\quad \overline{\overline{\omega}}_j\cong\omega_j\,,\quad(\overline{\overline{\omega}}_j=\text{pole of }\varepsilon(\omega))\,,\quad \operatorname*{Res}_{\overline{\overline{\omega}}_j}\varepsilon(\omega)\cong-\frac{2\pi ne^2}{m\omega_j}f_j\,.$$

The I are modified Bessel functions

(147*b*) $$\lim_{v\to c} \Omega_1(\varrho) = 0 \, .$$

The correction due to Ω_1 is less important than that due to Ω_2.

(148) $$2\pi \int_{R_1}^{R} \Omega_2(\varrho)\varrho \, \mathrm{d}\varrho \cong -\frac{8\pi n z_1^2 e^4}{mv^2} \sum_j f_j \left(\frac{\varepsilon(\tilde{\omega}_j)}{1-\varepsilon(\tilde{\omega}_j)}\right)^2 \left(\frac{1-\beta^2\varepsilon(\tilde{\omega}_j)}{2-\beta^2\varepsilon(\tilde{\omega}_j)}\right)^2 \times$$
$$\times \left[\beta^2\{I_1^2(Rk_0(\tilde{\omega}_j)) - I_0^2(Rk_0(\tilde{\omega}_j))\} + 2(Rk_0(\tilde{\omega}_j))^{-1} I_0(Rk_0(\tilde{\omega}_j)) I_1(Rk_0(\tilde{\omega}_j))\right] \times$$
$$\times \left[I_0(Rk_0(\tilde{\omega}_j)) I_1(Rk_0(\tilde{\omega}_j))\right]^{-2} .$$

($\tilde{\omega}_j$-pole of $B(\omega)$ close to ω_j).

(148*a*) $$[B(\omega)]^{-1} = R\left[K_1(Rk(\omega))I_0(Rk_0(\omega))k(\omega) + \right.$$
$$\left. + \frac{k^2(\omega)}{\varepsilon(\omega)k_0(\omega)} K_0(Rk(\omega))I_1(Rk_0(\omega))\right] ,$$

(148*b*) $$k^2(\omega) = \frac{\omega^2}{v^2}(1-\beta^2\varepsilon(\omega)) \, , \qquad \left(-\frac{\pi}{2} \leqslant \arg k(\omega) < \frac{\pi}{2} , \ \omega \geqslant 0\right).$$

$-\mathrm{d}W_{<R}/\mathrm{d}x$ corresponds to direct ionization and excitation. The loss due to the emission of Cerenkov radiation is included in $-\mathrm{d}W_{>R}/\mathrm{d}x$:

(149) $$-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x} = -\frac{\mathrm{d}W_{\text{Cer}}}{\mathrm{d}x} - \frac{2z_1e^2}{v^2} \sum_j \left[I_1(Rk_0(\tilde{\omega}_j)) \frac{\tilde{\omega}_j R}{v\sqrt{1-\beta^2}}\right]^{-1} \tilde{\omega}_j \left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)} ,$$

(149*a*) $$-\frac{\mathrm{d}W_{\text{Cer}}}{\mathrm{d}x} = -\frac{z_1^2 e^2}{v^2} \int_{\left(\substack{1-\beta^2\varepsilon<0 \\ \omega>0}\right)} |B(\omega)|^2 \left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) \omega \, \mathrm{d}\omega \, ,$$

(149*b*) $$D(\omega) = [K_1(Rk(\omega))Rk(\omega)B(\omega)]^{-1} ,$$

(149*c*) $$\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)} \cong \frac{4\pi n e^2 f_j}{n\tilde{\omega}_j I_0(Rk_0(\tilde{\omega}_j))} \, \frac{1-\beta^2\varepsilon(\tilde{\omega}_j)}{2-\beta^2\varepsilon(\tilde{\omega}_j)} \, \frac{\varepsilon(\tilde{\omega}_j)}{(1-\varepsilon(\tilde{\omega}_j))^2} ,$$

(150) $$\lim_{v\to c} \left(-\frac{\mathrm{d}W_{>R}}{\mathrm{d}x}\right) \cong \frac{2\pi n z_1^2 e^4}{mc^2} \, .$$

All the preceding formulas correspond to the case of a real dielectric constant and are not valid when there are conduction electrons. The case of a medium with conduction electrons presents special difficulties. Nevertheless some of the formulas derived in the preceding sections are still valid in this case. For instance, we get from (46) an expression of $-\mathrm{d}W_{>R}/\mathrm{d}x$ which is general:

$$(151)\quad -\frac{\mathrm{d}W_{>R}}{\mathrm{d}x}=\frac{2z_1^2e^2R}{\pi v^2}\mathcal{R}\int_0^\infty |B(\omega)|^2\left(\frac{1}{\varepsilon(\omega)}-\beta^2\right)k^*(\omega)K_1(Rk^*(\omega))K_0(Rk(\omega))i\omega\,\mathrm{d}\omega.$$

The expression (95) of $\Omega(\varrho)$ is also general and allows us to get a general formula for $\int_{R_1}^{R}\Omega(\varrho)\varrho\,\mathrm{d}\varrho$:

$$(152)\quad 2\pi\int_{R_1}^{R}\Omega(\varrho)\varrho\,\mathrm{d}\varrho\cong\frac{z_1^2e^2R^2}{2v^2c^2}(1-\beta^2)\mathcal{R}\left[\frac{i\pi}{2}\int_0^\infty\varepsilon(\omega)\,|\,C(\omega)\,|^2\{I_1^2(Rk_0(\omega)-I_0^2(Rk(\omega))+\right.$$

$$+2(\beta^2Rk_0(\omega))^{-1}I_0(Rk_0(\omega))I_1(Rk_0(\omega))\}\omega^3\,\mathrm{d}\omega-$$

$$-\int_0^\infty\varepsilon(\omega)(C(\omega)+C(-\omega))\{K_0(Rk_0(\omega))I_0(Rk_0(\omega))+K_1(Rk_0(\omega))I_1(Rk_0(\omega))-$$

$$\left.-2(\beta^2Rk_0(\omega))^{-1}K_0(Rk_0(\omega))I_1(Rk_0(\omega))\}\omega^3\,\mathrm{d}\omega\right],$$

$$(152a)\qquad C(\omega)=\frac{2}{i\pi}\left[\frac{k^2(\omega)}{k_0^2(\omega)}\varepsilon^{-1}(\omega)B(\omega)K_0(Rk(\omega))-K_0(Rk_0(\omega))\right]\{I_0(Rk_0(\omega))\}^{-1},$$

Discussion of some experimental results.

12. – We shall examine only some of the existing experimental results in this paper, a more systematic discussion will be published later. We shall not consider here the experimental results regarding only the effects of the polarization on the stopping power, since our main interest is in the relative contributions of ionization-excitation and Cerenkov radiation to the stopping power.

The most important experimental results, from the point of view we are assuming, are those of GOSH, JONES and WILSON [14] for mesons in oxygen. In this case the increase of the ionization after the minimum is considerable, and the difficulty of explaining it with the theory on Fermi's lines is obvious. Indeed, all the relativistic increase in that theory is due to the emission of Cerenkov radiation and the absorption of the radiation within about 2 mm is required to explain the experimental value of the increase. The present theory leads to an increase of ionization of about 30%, by neglecting the ionization due to the absorption of Cerenkov radiation. (The loss at distances larger than R is almost entirely due to Cerenkov radiation, as shown by the discussion of section 5). This value agrees satisfactorily with the experimental one. It is important to remark that the theoretical value of the total increase is obtained by comparing the loss at the minimum with the loss at saturation, and is not affected by the uncertainty on the values of the oscillator frequencies ω_j and oscillator percentuals f_j, since in the region of the minimum of ionization the rate of loss in a gas depends only on the mean ionization potential and the value at saturation does not depend on the ω_j and f_j. The mean ionization potential was taken as 89.2 eV for oxygen, i.e. the experimental value obtained by LIVINGSTON and BETHE ([18]) for air multiplied by 8/7.22.

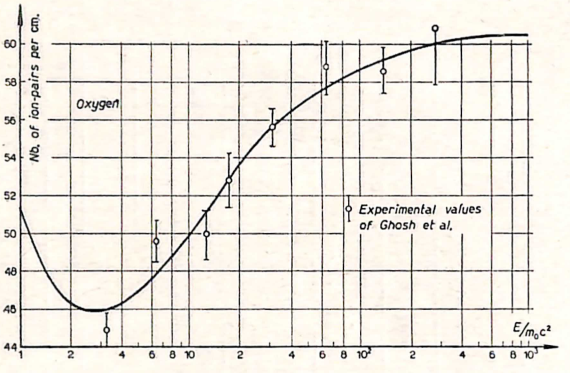

Fig. 3. – Total rate of ionization (transfers less than 1 000 eV).

The variation of the rate of ionization for oxygen at normal conditions is shown in fig. 3, which corresponds to the values of the ν_j and f_j, given in table I. There is some uncertainty in these values, but the effect on the rate of loss is not considerable. The agreement with the experimental data is satisfactory, by taking as 34.5 eV the average energy necessary to create an ion pair. The transfers in close collisions were cut at 1 000 eV.

In the case of silver and bromine, the results were found not to be very sensitive to the choice of the ω_j and f_j. The two sets of values A and B of table I led to results differing always by less than 0.5%. The mean ionization potentials of Ag and Br were taken as 470 eV and 367.5 eV respectively, by interpolating the data of MATHER and SEGRÉ ([19]) and taking into account those of BAKKER and SEGRÉ ([20]). The energy transfers in close collisions were cut at 5 000 eV (see JANSSENS and HUYBRECHTS ([21])). The results are insensitive to the choice of this cut-off value. By neglecting the absorption of Cerenkov radiation, the increase in ionization from the minimum to the plateau in AgBr

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([18]) M. S. LIVINGSTON and H. BETHE: *Rev. Mod. Phys.*, **9**, 246 (1937).
([19]) R. MATHER and E. SEGRÉ: *Phys. Rev.*, **84**, 191 (1951).
([20]) C. J. BAKKER and E. SEGRÉ: *Phys. Rev.*, **81**, 489 (1951).
([21]) P. JANSSENS and M. HUYBRECHTS: *Bull. Cent. Phys. Nucl. Brux.*, n. 27 (1951).

crystals was found to be 3.8%. The total rate of loss, including Cerenkov radiation, with transfers in close collisions cut at 5 000 eV is of 5.96 MeV/cm at saturation, the corresponding value in the theory on Fermi's lines being 6.42 MeV/cm. The value 3.8% seems too low to explain the observed increase of grain density of about 7%. Moreover the saturation of the grain density occurs faster than that of the total rate of ionization. The discrepancy between the theoretical results and experimental data can be considerably atte-

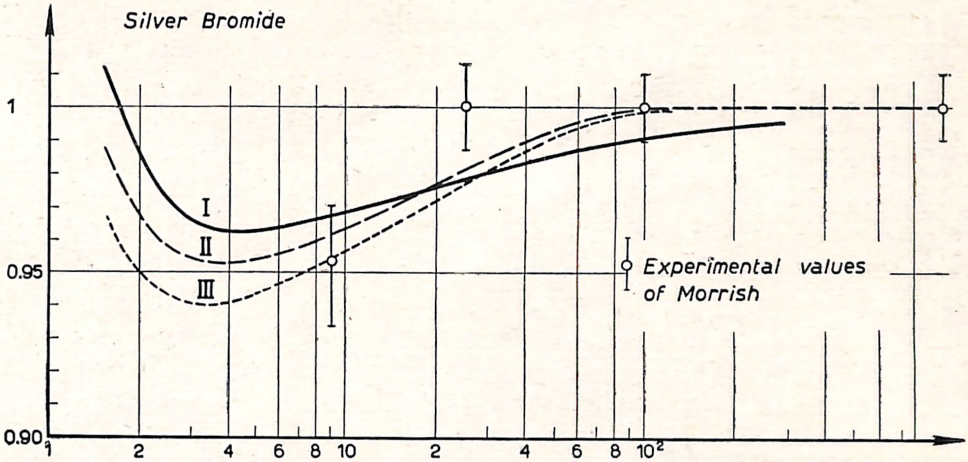

Fig. 4. – Total rate of ionization (transfers less than 5 000 eV). Curve I: Contributions of all the shells. Curve II: Contribution of the outer shell ($j = 1$) neglected. Curve III: Contributions of the outer shell and one third of that of the second shell ($j = 2$) neglected.

nuated by taking into account that small energy transfers are probably not sufficient to develop a grain of AgBr. By neglecting the contribution of the electrons of the outer shell, the increase from minimum to saturation becomes 4.8%, and an increase of 6% is obtained by neglecting entirely the contribution of the outer shell and 33% of the contribution of the next shell; this procedure leads also to a faster saturation. The theoretical results and the experimental values of MORRISH (²²) are compared in fig. 4 (*). The agreement seems satisfactory, when the small energy transfers are neglected, although the saturation of the grain density is still faster than that of the reduced theoretical loss. The eventual existence of an instrumental cut-off of the very small transfers in the photographic emulsion can

(²²) A. H. MORRISH: *Phil. Mag.*, 43, 533 (1952).

(*) The normalizations of the three curves in fig. 4 are of course different, in order to get the same saturation plateau.

TABLE I.

j	Silver Bromide				Oxygen		
	Shells	$82\, f_j$	ν_j A	ν_j B	Shells	$8\, f_j$	ν_j
1	N Br	8.4	1.025	1.502	L 2p	4.2	54.31
2	N Ag	20.4	3.33	3.33	L 2s	2.0	108.62
3	M Br	18.0	7.18	6.00	K	1.8	727.10
4	M Ag	18.0	10.62	10.62			
5	L Br	7.2	33.50	33.50			
6	L Ag	7.2	72.45	72.45			
7	K Br	1.4	277.80	277.80			
8	K Ag	1.4	526.60	526.60			
a	$11.76 \cdot 10^{15}\ s^{-1}$				$0.1861 \cdot 10^{15}\ s^{-1}$		
R	$4.065 \cdot 10^{-7}$ cm				$2.562 \cdot 10^{-5}$ cm		
I_z	Ag 470 eV		Br 367.5 eV		89.2 eV		

$$\nu_j = \frac{\omega_j}{2\pi\alpha}.$$

be put in evidence by accurate measurements in the region before the minimum of ionization, where the polarization effects are very small. Such an effect would make the effective mean ionization potential for the developing of grains in the photographic emulsion higher than the true mean ionization potential. The effective mean ionization potential for the developing of grains may eventually depend on the average size of the grains and other conditions affecting their sensitivity, as well as on the processing of the plate.

Unpublished results of VOYVODIC (*) for π-mesons and electrons in the photographic plate show also an increase of about 6-7% and the agreement with the present theory seems quite satisfactory by assuming the existence of the instrumental effect discussed in connection with the MORRISH data. The results of L. JAUNEAU and F. HUG-BOUSSER [23] show the existence of an increase of the grain density after the minimum in the photographic emulsion, but the scatter of the results in different series of measurements appears to be considerable and a precise evaluation of the increase after the minimum is difficult.

The results for the photographic emulsion can probably be improved by

(*) Private communication.

[23] L. JAUNEAU and F. HUG-BOUSSER: *Journ. de Phys.* (in press).

using a more accurate expression of the dielectric constant, in which the effect of a shell is not assimilated to that of a single oscillators (see WICK [2]).

Note added in proof. — Dr. A. H. MORRISH has called our attention to the paper of MC DIARMID (*Phys. Rev.*, **84**, 851 (1951)), whose results agree rather better with ours with respect to the beginning of the saturation in photographic emulsions.

APPENDIX I.

Let us denote by $F(\omega)$ the quantity within the square brackets in (40):

$$F(\omega) = \frac{2\varepsilon(\omega)}{Rk_c(\omega)} \frac{K_1(Rk_c(\omega))}{K_0(Rk_c(\omega))} + 1 . \tag{1}$$

It was shown in references (5) and (6) that the ω corresponding to real values or poles of the function $\varepsilon(\omega)$ defined by (42) have negative imaginary parts, when the damping constants are not 0. $F(\omega)$ has therefore no other singularities in the quadrant of the positive real and positive imaginary axis besides that at the point $\omega = 0$. In order to show that $F(\omega)$ has no zeros in this region, we shall consider the closed contour Γ formed by an infinitesimal quarter of circle centred at the origin, the positive real and positive imaginary axes and a quarter of circle centred at the origin with infinite radius. $F(\omega)$ is real and positive on the positive imaginary axis and on the infinite quarter of circle, and the variation of its argument along the infinitesimal quarter of circle and the positive real axis is equal to zero. Since the variation of the argument of $F(\omega)$ along the closed contour Γ is equal to zero and $F(\omega)$ has no poles, either on the contour or inside it, when the damping constants in (42) are not 0, it follows from the argument principle that it has no zeros in that region.

$F(\omega)$ is related to the function $D(\omega)$ defined by (52):

$$F(\omega) = \frac{2\varepsilon(\omega)}{Rk_c(\omega)} \frac{K_1(Rk_c(\omega))}{K_0(Rk_c(\omega))} D_c(\omega) , \tag{2}$$

$$D_c(\omega) = \lim_{v\to c} D(\omega) = \frac{Rk_c(\omega)}{2\varepsilon(\omega)} \frac{K_0(Rk_c(\omega))}{K_1(Rk_c(\omega))} + 1 . \tag{3}$$

We have shown that $D(\omega)$ has zeros in the neighbourhoods of the ω_j, whose imaginary parts are negative when the damping constants are finite. It results from (2) and (3) that, in this case, $F(\omega)$ has also zeros in the neighbourhoods of the ω_j, with negative imaginary parts.

We shall now prove that, in the case of a real $\varepsilon(\omega)$, the function $D(\omega)$ has no zeros in the Cerenkov bands. It follows from (52) that, at a zero $\tilde{\omega}$ of $D(\omega)$:

$$\text{(4)} \qquad \frac{K_0(Rk(\tilde{\omega}))}{K_1(Rk(\tilde{\omega}))} = -\frac{I_0(Rk_0(\tilde{\omega}))}{I_1(Rk_0(\tilde{\omega}))}\frac{k_0(\tilde{\omega})}{k(\tilde{\omega})}\varepsilon(\tilde{\omega}) .$$

If $\tilde{\omega}$ would lie in a Cerenkov band, we would have:

$$\text{(5)} \qquad \arg\left[\frac{K_0(-iR\,|\,k(\tilde{\omega})\,|\,)}{K_1(-iR\,|\,k(\tilde{\omega})\,|\,)}\right] = \arg\left[-i\,\frac{H_0^{(1)}(R\,|\,k(\tilde{\omega})\,|\,)}{H_1^{(1)}(R\,|\,k(\tilde{\omega})\,|\,)}\right] = -\frac{\pi}{2} .$$

It follows from (5) that:

$$\text{(6)} \qquad \arg\left[\frac{J_0(R\,|\,k(\tilde{\omega})\,|\,) + iN_0(R\,|\,k(\tilde{\omega})\,|\,)}{J_1(R\,|\,k(\tilde{\omega})\,|\,) + iN_1(R\,|\,k(\tilde{\omega})\,|\,)}\right] = 0 .$$

Since $J\,(u)$ and $J_1(u)$ do not vanish simultaneously for any real value of u and the same happens with $N_0(u)$ and $N_1(u)$, equation (6) will only be satisfied when $k(\tilde{\omega}) = 0$. It is therefore not compatible with (4).

Appendix II.

It results from (22) that:

$$\text{(1)} \quad \left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega)K_0(Rk(\omega)) - \left(\frac{1}{\varepsilon(-\omega)} - \beta^2\right) B(-\omega)K_0(Rk(-\omega)) =$$

$$= B(\omega)B(-\omega)\left[B^{-1}(-\omega)\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)K_0(Rk(\omega)) - B^{-1}(\omega)\left(\frac{1}{\varepsilon(-\omega)} - \beta^2\right)K_0(Rk(-\omega)\right] =$$

$$= RI_0(Rk_0(\omega))B(\omega)B(-\omega)\left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)K_0(Rk(\omega))K_1(Rk(-\omega)k(-\omega) -\right.$$

$$\left. - \left(\frac{1}{\varepsilon(-\omega)} - \beta^2\right)K_0(Rk(-\omega))K_1(Rk(\omega))k(\omega)\right] ,$$

hence

$$\text{(2)} \quad \mathcal{R}\left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)B(\omega)K_0(Rk(\omega))i\omega\right] =$$

$$= RI_0(Rk_0(\omega))\,|\,B(\omega)\,|^2\,\mathcal{R}\left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)K_0(Rk(\omega))K_1(Rk(-\omega))k(-\omega)i\omega\right].$$

This equation shows that (39) is equivalent to (46) for $\varrho = R$.

In the Cerenkov bands $\arg k(\omega) = -\pi/2$, therefore:

(3) $$k(-\omega) = \exp[i\pi]k(\omega)\,, \qquad (\omega > 0,\; 1 - \beta^2\varepsilon(\omega) < 0).$$

By taking into account the well known formulas

(4) $$K_0(\exp[i\pi]u) = K_0(u) - i\pi I_0(u)\,, \qquad K_1(\exp[i\pi]u) = -K_1(u) - i\pi I_1(u)\,,$$

and (21), we can simplify (2) in the Cerenkov bands:

(5) $$\mathcal{R}\left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega) K_0(Rk(\omega)) i\omega\right] = -\frac{\pi}{2}\,\omega\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) I_0(Rk_0(\omega))\,|B(\omega)|^2\,, \qquad (\omega > 0,\; 1 - \beta^2\varepsilon(\omega) < 0).$$

It results from (5) that the first term in the right hand side of (62) is equal to the right hand side of (59).

When the damping constants are finite, we can obtain regular analytic functions in a strip around the real positive ω-axis starting from the functions $k(\omega)$ and $B(\omega)$ defined on that axis by (16) and (22), with

(6) $$k_0(\omega) = \frac{\omega}{v}\sqrt{1 - \beta^2}\,,$$

$k(-\omega)$ and $B(-\omega)$, as defined by (16) and (22), are not obtained by analytical continuation of the regular analytic functions $k(\omega)$ and $B(\omega)$. We shall introduce the analytic functions $\bar{k}(\omega)$ and $\bar{B}(\omega)$ defined by the equations:

(7) $$\bar{k}(\omega) = k^*(\omega)\,, \qquad \bar{B}(\omega) = B^*(\omega)\,, \qquad (\omega > 0).$$

It results from the well known symmetry principle of Riemann and Schwarz (*) that, with a suitable choice of the branches:

(8) $$k(\omega) = (\bar{k}(\omega^*))^*\,, \qquad B(\omega) = (\bar{B}(\omega^*))^*\,.$$

By introducing the function $G(\omega)$

(9) $$G(\omega) = \omega\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega) K_0(Rk(\omega))[1 - RI_0(Rk_0(\omega))\bar{B}(\omega)K_1(R\bar{k}(\omega))\bar{k}(\omega)]\,,$$

equation (1) can be written as:

(10) $$\mathcal{I}m\,G(\omega) = 0 \quad \text{for} \quad \omega > 0\,.$$

(*) See, for instance, Z. NEHARI: *Conformal Mapping* (New York, 1952) p. 184.

Hence we have

$$G(\omega^*) = (G(\omega))^* \,, \tag{11}$$

as a consequence of the symmetry principle. It results from (11) that:

$$\operatorname*{Res}_{\tilde{\omega}'_j + i\tilde{\eta}_j} G(\omega) = -\{\operatorname*{Res}_{\tilde{\omega}'_j - i\tilde{\eta}_j} G(\omega)\}^* \,. \tag{12}$$

This equation can be obtained by comparing the values of $\int_\Gamma G(\omega)\,d\omega$ and $\int_{\Gamma^*} G(\omega)\,d\omega$, Γ being a circle with sufficiently small radius centred at the point $\tilde{\omega}'_j + i\tilde{\eta}_j$ and Γ^* the circle with the same radius centred at $\tilde{\omega}'_j - i\tilde{\eta}_j$, both circles being described in the same sense. By introducing the expression (9) of G into (12), we get:

$$\begin{aligned}
&\operatorname*{Res}_{\tilde{\omega}'_j + i\tilde{\eta}_j} \left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega)\bar{B}(\omega) I_0(Rk_0(\omega)) K_0(Rk(\omega)) K_1(R\bar{k}(\omega)) R\bar{k}(\omega)\omega\right] + \\
&+ \left\{\operatorname*{Res}_{\tilde{\omega}'_j - i\tilde{\eta}_j} \left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega)\bar{B}(\omega) I_0(Rk_0(\omega)) K_0(Rk(\omega)) K_1(R\bar{k}(\omega)) R\bar{k}(\omega)\omega\right]\right\} = \\
&= \operatorname*{Res}_{\tilde{\omega}'_j - i\tilde{\eta}_j} \left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B(\omega) K_0(Rk(\omega))\omega\right] .
\end{aligned} \tag{13}$$

In the limit of zero damping constants, equation (13) becomes:

$$\begin{aligned}
&\operatorname*{Res}_{\tilde{\omega}_j} \left[\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B^2(\omega) I_0(Rk(\omega)) K_0(Rk(\omega)) K_1((\omega)) Rk(\omega)\omega\right] = \\
&= \left(\frac{1}{\varepsilon(\tilde{\omega}_j)} - \beta^2\right) K_0(Rk(\tilde{\omega}_j))\tilde{\omega}_j \operatorname*{Res}_{\tilde{\omega}_j} B(\omega) .
\end{aligned} \tag{14}$$

It is easily seen that for any analytic symmetric function $H(k(\omega), \bar{k}(\omega);\ \omega)$

$$H(k(\omega), \bar{k}(\omega);\ \omega) = H(\bar{k}(\omega), k(\omega);\ \omega) \,, \tag{15}$$

regular in neighbourhood of the poles $\tilde{\omega}'_j \pm i\tilde{\eta}_j$ of $G(\omega)$ and taking real values for positive ω, we have:

$$\begin{aligned}
&\operatorname*{Res}_{\tilde{\omega}'_j - i\tilde{\eta}_j} \left[H(k(\omega), \bar{k}(\omega);\ \omega) B(\omega) K_0(Rk(\omega)) \left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)\omega\right] = \\
&= \operatorname*{Res}_{\tilde{\omega}'_j + i\tilde{\eta}_j} L(\omega) + \operatorname*{Res}_{\tilde{\omega}'_j - i\tilde{\eta}_j} L(\omega) \,,
\end{aligned} \tag{16}$$

with:

$$(17)\qquad L(\omega) = H(k(\omega),\, \bar{k}(\omega);\, \omega)\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right)\times$$
$$\times\, B(\omega)\bar{B}(\omega)I_0(Rk_0(\omega))K_0(R\bar{k}(\omega))K_1(R\bar{k}(\omega))R\bar{k}(\omega)\omega\,.$$

In the limit of zero damping constants, we get:

$$(18)\qquad H(k(\tilde{\omega}_j),\, k(\tilde{\omega}_j);\, \tilde{\omega}_j)K_0(Rk(\tilde{\omega}_j))\left(\frac{1}{\varepsilon(\tilde{\omega}_j)} - \beta^2\right)\tilde{\omega}_j \operatorname*{Res}_{\tilde{\omega}_j} B(\omega) =$$
$$= \operatorname*{Res}_{\tilde{\omega}_j}\left[H(k(\omega),\, k(\omega);\, \omega)\left(\frac{1}{\varepsilon(\omega)} - \beta^2\right) B^2(\omega)I_0(Rk_0(\omega))K_0(Rk(\omega))K_1(Rk(\omega))Rk(\omega)\omega\right].$$

This equation shows how it is possible to transform the expression of the loss at distances larger than R by direct ionization-excitation obtained from the equation (58) of section 4 into that given by (62) of section 5, which involves only the residue of $[D(\omega)]^{-1}$.

Appendix III.

We shall now prove formula (109) of section 8. In order to get an approximate value of $\mathrm{d}D/\mathrm{d}\omega$ at the zero $\tilde{\omega}_j$, we shall start from the formula (52) of section 4 and we shall neglect the variation of the functions I, as well as the variation of the ratio of the two K functions. We may write:

$$(1)\qquad D(\omega) = I_0(Rk_0(\omega)) + \frac{K_0(Rk(\omega))}{K_1(Rk(\omega))}\,\frac{\sqrt{1-\beta^2\varepsilon(\omega)}}{\sqrt{1-\beta^2\varepsilon(\omega)}}\,I_1(Rk_0(\omega))\,.$$

With the aforementioned approximations we get:

$$(2)\qquad \frac{\mathrm{d}D}{\mathrm{d}\omega} \cong \frac{K_0(Rk(\tilde{\omega}_j))}{K_1(Rk(\tilde{\omega}_j))}\,\frac{I_1(Rk_0(\tilde{\omega}_j))}{\sqrt{1-\beta^2}}\,\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\varepsilon^{-1}(\omega)\sqrt{1-\beta^2\varepsilon(\omega)}\right].$$

Since

$$(3)\qquad \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\varepsilon^{-1}(\omega)\sqrt{1-\beta^2\varepsilon(\omega)}\right] = -\frac{\sqrt{1-\beta^2\varepsilon(\omega)}}{2\varepsilon(\omega)}\,\frac{2-\beta^2\varepsilon(\omega)}{1-\beta\varepsilon(\omega)}\,\varepsilon^{-1}(\omega)\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\,,$$

and

$$(4)\qquad \varepsilon^{-1}(\tilde{\omega}_j)\sqrt{1-\beta^2\varepsilon(\tilde{\omega}_j)} = -\sqrt{1-\beta^2}\,\frac{K_1(Rk(\tilde{\omega}_j))}{K_0(Rk(\tilde{\omega}_j))}\,\frac{I_0(Rk_0(\tilde{\omega}_j))}{I_1(Rk_0(\tilde{\omega}_j))}\,,$$

we have:

$$\left\{\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\varepsilon^{-1}(\omega)\sqrt{1-\beta^2\varepsilon(\omega)}\right]\right\}_{(\omega=\tilde{\omega}_j)} = \tag{5}$$

$$=\frac{1}{2}\sqrt{1-\beta^2}\,\frac{K_1(Rk(\tilde{\omega}_j))}{K_0(Rk(\tilde{\omega}_j))}\,\frac{I_0(Rk_0(\tilde{\omega}_j))}{I_1(Rk_0(\tilde{\omega}_j))}\,\frac{2-\beta^2\varepsilon(\tilde{\omega}_j)}{1-\beta^2\varepsilon(\tilde{\omega}_j)}\,\varepsilon^{-1}(\tilde{\omega}_j)\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right)_{(\omega=\tilde{\omega}_j)}.$$

It results from (2) and (5) that:

$$\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)_{(\omega=\tilde{\omega}_j)} \cong \frac{1}{2}\,I_0(Rk_0(\tilde{\omega}_j))\,\frac{2-\beta^2\varepsilon(\tilde{\omega}_j)}{1-\beta^2\varepsilon(\tilde{\omega}_j)}\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega}\right)_{(\omega=\tilde{\omega}_j)}\varepsilon^{-1}(\tilde{\omega}_j)\,. \tag{6}$$

It results from formula (50) of section 4, combined with formula (119) of section 9, that in the neighbourhood of $\bar{\omega}_j$ we have:

$$\varepsilon^{-1}(\omega) \cong 1 + \frac{4\pi^2\alpha^2 f_j}{\omega^2-\tilde{\omega}_j^2}\,. \tag{7}$$

We get from (7)

$$\varepsilon^{-1}(\omega)\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega} = -\,\varepsilon(\omega)\,\frac{\mathrm{d}}{\mathrm{d}\omega}\,\varepsilon^{-1}(\omega) \cong \frac{\omega(\varepsilon(\omega)-1)^2}{2\pi^2\alpha^2 f_j\varepsilon(\omega)}\,, \tag{8}$$

hence

$$\left(\frac{\mathrm{d}D}{\mathrm{d}\omega}\right)^{-1}_{(\omega=\tilde{\omega}_j)} \cong \frac{4\pi ne^2 f_j}{m\tilde{\omega}_j I_0(Rk_0(\tilde{\omega}_j))}\,\frac{1-\beta^2\varepsilon(\tilde{\omega}_j)}{2-\beta^2\varepsilon(\tilde{\omega}_j)}\,\frac{\varepsilon(\tilde{\omega}_j)}{(\varepsilon(\tilde{\omega}_j)-1)^2}\,. \tag{9}$$

This is formula (109) of section 8.

It is interesting to remark that formula (66) of section 5 can be obtained immediately from (9). This is important because (66), for extremely relativistic energies, gives a good approximation when $\tilde{\omega}_j$ differs little from $\bar{\bar{\omega}}_j$, whereas (7) is only satisfactory when ω differs little from $\bar{\omega}_j$. Formula (9) gives therefore a good approximation when $\tilde{\omega}_j$ is close to either $\bar{\bar{\omega}}_j$ or $\bar{\omega}_j$, i.e. for an entire interval which contains $\tilde{\omega}_j$, at least for extremely relativistic energies.

RIASSUNTO (*)

Si formula una teoria della perdita d'energia di una particella carica, in cui il meccanismo della perdita è differente da quello della teoria di Fermi. La nostra teoria conduce a una distribuzione della perdita tra l'eccitazione per ionizzazione e l'emis-

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(*) *Traduzione a cura della Redazione.*

sione di radiazione Cerenkov che differisce notevolmente da quelle delle teorie del tipo di quella di Fermi, per quanto il potere di frenamento non sia essenzialmente diverso, essendo soltanto nella nostra teoria alquanto minore che in quelle. Questo trattamento conduce a un aumento della ionizzazione diretta per parametri d'urto maggiori delle dimensioni atomiche dopo il minimo relativistico, l'aumento essendo considerevole nel caso dei gas. L'effetto di saturazione di Fermi della perdita a distanze superiori alle dimensioni atomiche esiste anche nella nostra teoria, per quanto in virtù di un differente meccanismo. L'aumento relativistico della perdita nelle interazioni a distanza è dovuto largamente a un aumento dei raggi d'azione per la ionizzazione e l'eccitazione (meccanismo di Bohr-Williams), ma è dovuto anche a un contributo della radiazione di Cerenkov (meccanismo di Fermi). La saturazione è dovuta a una limitazione dell'aumento dei raggi d'azione dovuta alla polarizzazione del mezzo e alla saturazione dell'emissione di radiazione Cerenkov. La presente teoria conduce a una modificazione della formula di Frank e Tamm per il tasso d'emissione della radiazione Cerenkov, l'emissione di radiazione Cerenkov di elevata energia essendo considerevolmente ridotta.