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On the General Theory of Damping in Quantum Mechanics.

M. Schönberg

Centre de Physique Nucléaire de l'Université Libre - Bruxelles

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Summary. — A general form of the equation of Heitler and Peng, valid for time dependent interactions, is given. The relations between the generalized Heitler-Peng and Schwinger theories of damping are examined in detail. A form of the damping theory given previously by the author is generalized to the case of any time independent interaction. It is shown that the generalized form of the Heitler-Peng treatment of the damping may be used to develop a form of stationary perturbation theory applicable to cases in which it is not possible to set up a correspondence between unperturbed and perturbed stationary states.

Introduction.

1. – The first problems of damping treated in quantum theory were those of emission and scattering of light by atoms in discrete levels, solved by Weisskopf and Wigner [1]. Later, another type of damping problem, in which only continuous levels do come in, was treated in particular cases by several authors [2], and a general formalism for this kind of problems was developed by Heitler and Peng [3] and perfected by Pauli [4]. Heitler and Ma [5] and Hamilton [6] have shown that the methods used for con-

(1) V. Weisskopf and E. Wigner: *Zeits. f. Phys.*, **63**, 54 (1930); **65**, 18 (1930).

(2) A. H. Wilson: *Proc. Cam. Phil. Soc.*, **37**, 301 (1941); W. Heitler: *Proc. Cam. Phil. Soc.*, **37**, 291 (1941); A. Sokolov: *Journ. Phys. U.R.S.S.*, **5**, 231 (1941); E. Gora: *Zeits. f. Phys.*, **120**, 121 (1943).

(3) W. Heitler and H. W. Peng: *Proc. Cam. Phil. Soc.*, **38**, 296 (1942).

(4) W. Pauli: *Meson Theory of Nuclear Forces*, (New York, 1946), p. 41.

(5) W. Heitler and S. T. Ma: *Proc. Roy. Irish Acad.*, **52A**, 109 (1949).

(6) J. Hamilton: *Proc. Phys. Soc.*, **62**, 12 (1949).

tinuous levels can be generalized to the case of any kind of level. Recently a treatment of damping in a relativistic covariant form was given by FUKUDA and MIYAZIMA (7) using Schwinger's theory of collisions (8). We have derived (9) an integral equation for the theory of damping, which can be applied both to continuous and discrete levels, from a special treatment of the quantum theory of perturbations. In this paper, we shall discuss the general theory of damping in quantum mechanics as an application of the resolvent operators, using also the generalized definition of the resolvent operator introduced in a previous paper (10), and we shall examine in detail the relations between the different points of view of the aforementioned authors.

The resolvent operators $(H-\lambda)^{-1}$ attached to hermitian operators H, λ being a complex parameter, play a central part in the theory of linear transformations in Hilbert space (11) and are the most adequate tool to treat many problems in quantum theory, as we have indicated in a previous paper (10). It will be shown in this paper that the resolvent operators lead to a very simple and general theory of damping, especially when the generalized resolvents of reference (10) are used. The damping theory deals with effects associated to long intervals of time, which can not be treated conveniently by the usual methods of perturbation theory, because in these methods the evolution of the system is described by series expansions of the wave function which represent the finite evolution of the system as an accumulation of infinitesimal steps. The natural approach to the damping theory is given by formalisms in which the integrated motion over a long interval of time is treated directly, as it is done in the theories of HEITLER and MA (5) and HEITLER and PENG (3), as well as in those of SCHWINGER (8), FUKUDA and MIYAZIMA (7) and SCHÖNBERG (9). The ordinary resolvents $(H-\lambda)^{-1}$ allow to develop in a very simple way a treatment related to that of HEITLER and MA (5), with the advantage of using a simpler integral equation, which is moreover the natural generalization of the equation of HEITLER and PENG (3). The generalized resolvents of reference (10) lead to a treatment of the damping as general as that of SCHWINGER (8) and FUKUDA and MIYAZIMA (7), in a way which may be considered as the generalization of the method of HEITLER and PENG (3).

The main properties of the resolvent operators are discussed in section 2. The generalization of the equation of Heitler and Peng for any kind of time independent hamiltonians (both discrete and continuous levels) is given in section 4. In section 5 the equation of Heitler and Peng is extended to the

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(7) N. FUKUDA and T. MIYAZIMA: *Prog. Theor. Phys.*, 5, 849 (1950).
(8) J. SCHWINGER: *Phys. Rev.*, **74**, 1439 (1948).
(9) M. SCHÖNBERG: *Nuovo Cimento*, **8**, 403 (1951).
(10) M. SCHÖNBERG: *Nuovo Cimento*, **8**, 651 (1951).
(11) M. H. STONE: *Linear Transformations in Hilbert Space* (New York, 1932).

case of a time independent unperturbed hamiltonian with a time dependent perturbation, starting from the integral equation for the generalized resolvents which we have derived ([10]). The treatment of the time dependent perturbations involves linear transformations of operators which can be conveniently handled by the introduction of super-operators (SCHÖNBERG ([12])). The iteration methods of damping theory are replaced by expansions of operators and super-operators in Liouville-Neumann series, analogous to geometrical series. It is shown that the elimination of virtual transitions by the iteration methods amounts to the determination of a certain kind of solutions of the perturbed Schrödinger equation, which do not satisfy the boundary conditions for scattering. The integral equation which generalizes the Heitler-Peng equation is essentially a relation between the satisfactory wave-function and that obtained by the elimination of the virtual transitions.

In section 5 is introduced a width operator which describes the shift and the broadening of the unperturbed levels by the perturbation. It is shown in section 6 that our definition of the widthes is equivalent to that given by HEITLER and MA ([5]), and their integral equation is derived from the theory of the resolvent. The different quantities appearing in the treatment of Heitler and Ma can be easily expressed in terms of the ordinary resolvents.

An alternative treatment of the damping, not involving the resolvent operators, is given in sections 7, 8 and 9. This treatment leads to the formalism of SCHWINGER ([8]) and FUKUDA and MIYAZIMA ([7]), and it shows clearly the relations between this theory and the older forms of the damping theory. It is shown that the passage from the generalized Heitler-Peng formalism to that of Schwinger and Fukuda and Miyazima can be performed simply by a Fourier transformation. This treatment is more general than that of section 4, because it is valid even in the case of a time dependent unperturbed hamiltonian with a time dependent perturbation. SCHWINGER ([8]) expressed the Heisenberg collision operator S in terms of an hermitian operator K (the reaction operator) by a Caylely transformation and introduced an integral equation which leads to an expansion in series of the reaction operator K; the collision operator S is then determined by an integral equation involving the reaction operator K. It is shown that the determination of the reaction operator K corresponds to the determination of the auxiliary perturbed wave function in the generalization of the Heitler-Peng treatment.

The integral equation for the damping, including self-energy corrections, given in reference ([9]) is derived from the formalism of the resolvent operators in section 10. This derivation has the same limitations as the original one ([9]), which is based on the assumption that it is possible to go over from the un-

([12]) M. SCHÖNBERG: *Nuovo Cimento*, 8, 241 (1951).

perturbed eigenstates to the perturbed ones by a rotation in the space of the states of the dynamical system in consideration. In the case of systems with continuous levels, the perturbation methods of reference ([9]) do not allow to determine completely the rotation in the state space and this leaves an arbitrariness in the definition of the self-energies of the states of the continuum. By using the width operator introduced in section 5, it is possible to get rid of the indetermination in the correspondance between perturbed and unperturbed levels of the continuum and also to treat cases in which it is not possible to establish a correspondence between perturbed and unperturbed levels.

In the generalized form of the Heitler-Peng theory of the damping given in this paper, the essential objective is the determination of the resolvent operator for the perturbed hamiltonian. Once the resolvent is determined, the spectral decomposition of the perturbed hamiltonian may be considered as obtained, in particular its discrets levels are the poles of the resolvent. Thus our treatment of the damping can be used to develop a form of perturbation theory in which a correspondence between unperturbed and perturbed eigenstates plays no part at all and which can be applied to problems in which discrete levels are created by the perturbation. This form of perturbation theory allows to get unambiguously the shift of the levels of the continuum, as it is shown in this paper. The detailed treatment of the perturbation theory based on the resolvent will be given elsewhere.

General properties of the resolvents.

2. – We shall consider dynamical systems whose hamiltonians $H(t)$ will in general be time dependent, although some of the most important problems lead to time independent hamiltonians. $H(t)$ will be called the perturbed hamiltonian, it is the sum of the unperturbed hamiltonian $H_0(t)$ and of the perturbation $H'(t)$

$$H(t) = H_0(t) + H'(t) . \tag{1}$$

The contact transformation generated by the movement during the time interval $t'-t$ will be described by the unitary operator $V(t, t')$ defined by the equations

$$i\frac{\mathrm{d}}{\mathrm{d}t}V(t,t') = H(t)V(t,t') , \qquad V(t',t') = 1 \; ; \qquad (\hbar = 1). \tag{2}$$

$V(t, t')$ transforms the state $\psi(t')$ at the time t' into the state $\psi(t)$ at the time t

$$\psi(t) = V(t,t')\psi(t') , \tag{3}$$

the ψ being vectors in a convenient functional space. The state $\psi(t)$ is determined by the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H(t)\psi(t)\,, \tag{4}$$

and its initial value $\psi(t')$.

We may treat t as a parameter and consider $V(t,t')$ as a function of t' determined by the equations

$$-i\frac{d}{dt'}V(t,t') = V(t,t')H(t')\,, \qquad V(t,t) = 1\,. \tag{5}$$

The contact transformation generated by $H_0(t)$ will be denoted by $V_0(t,t')$

$$i\frac{d}{dt}V_0(t,t') = H_0(t)V_0(t,t')\,, \qquad V_0(t',t') = 1\,. \tag{6}$$

In the important particular case of a time independent H_0, the operator $V_0(t,t')$ depends only on the difference $t-t'$

$$V_0(t,t') = \exp[-i(t-t')H_0]\,. \tag{7}$$

It is convenient to introduce a special notation for $V_0(t,0)$

$$V_0(t) = V_0(t,0)\,. \tag{8}$$

The resolvent operator R is defined by means of the $\bar{R}_\pm(\lambda,t')$

$$\bar{R}_+(\lambda,t') = i\int_{t'}^{\infty}\exp[i\lambda t]V(t,t')\,dt \qquad Im\lambda > A_+\,, \tag{9a}$$

$$\bar{R}_-(\lambda,t') = -i\int_{-\infty}^{t'}\exp[i\lambda t]V(t,t')\,dt \qquad Im\lambda < A_-\,. \tag{9b}$$

We shall assume that the convergence limits A_+ and A_- satisfy the following conditions

$$A_+ < +\infty\,, \qquad A_- > -\infty\,, \tag{10}$$

and we shall take

$$R(\lambda,t') = \exp[-i\lambda t']\bar{R}_+(\lambda,t') \qquad Im\lambda > A_+\,, \tag{11a}$$

$$R(\lambda,t') = \exp[-i\lambda t']\bar{R}_-(\lambda,t') \qquad Im\lambda < A_-\,. \tag{11b}$$

When

$$A_+ \leqslant 0\,, \qquad A_- \geqslant 0\,, \tag{12}$$

it is possible to define $\bar{R}_\pm(\lambda,t')$ and $R(\lambda,t')$ for real values of λ. The **Fourier** transform of $V(t, t')$ will be denoted by $F(\omega, t')$

(13) $$V(t, t') = \int_{-\infty}^{+\infty} \exp[-i\omega t] F(\omega, t')\, d\omega \,.$$

It results from equations (9) that

(14a) $$V(t, t')\eta(t - t') = \frac{1}{2\pi i}\int_{-\infty+i\alpha}^{+\infty+i\alpha} \exp[-i\lambda t]\overline{R}_{+}(\lambda, t')\, d\lambda \,, \qquad (\alpha > A_{+}) \,,$$

(14b) $$V(t, t')\eta(t' - t) = -\frac{1}{2\pi i}\int_{-\infty+i\beta}^{+\infty+i\beta} \exp[-i\lambda t]\overline{R}_{-}(\lambda, t')\, d\lambda \,, \qquad (\beta < A_{-}) \,,$$

with

(15) $$\eta(u) = \begin{cases} 1 & u > 0 \\ 1/2 & u = 0 \\ 0 & u < 0 \,. \end{cases}$$

Hence, at the points of the real axis where it is possible to define the $\overline{R}_{\pm}$

(16) $$\overline{R}_{+}(\omega, t') - \overline{R}_{-}(\omega, t') = 2\pi i F(\omega, t') \,,$$

and the resolvent is discontinuous at these points

(17) $$R(\omega + i0, t') - R(\omega - i0, t') = 2\pi i F(\omega, t') \exp[-i\omega t'] \,,$$

unless the Fourier coefficient F vanishes for the corresponding circular frequencies.

In the case of a time independent H

(18) $$R(\lambda, t') = (H - \lambda)^{-1} = R(\lambda) \,, \qquad (Im\lambda \neq 0)$$

so that our generalized resolvent goes over into the ordinary resolvent $R(\lambda)$. The resolvent is now regular at the points of the real axis which do not belong to the spectrum of H, and it has simple poles at the discrete eigenvalues of H. At the points which belong to the continuous spectrum of H the resolvent is discontinuous. All these properties follow immediately from (17) and the spectral decomposition of H and $V(t, t')$

(19) $$H = \sum_{k,a} P_{k,a} E_k + \int_{sp.} P(E) E\, dE$$

(20) $$V(t, t') = \exp[-i(t - t')H] = \sum_{k,a} P_{k,a} \exp[-i(t - t')E_k] + \int_{sp.} P(E) \exp[-i(t - t')E]\, dE .$$

the $P_{k,a}$ being the projection operators corresponding to the discrete eigenstates $\Phi_{k,a}$ and the $P(E)$ being the projection operators on the linear variety of the functional space corresponding to the eigenstates $\Phi_{E,\theta}$ of energy E

$$(21)\quad \begin{cases} P_{k,a}\Phi = (\Phi_{k,a}, \Phi)\Phi_{k,a} & (\Phi_{k,a}, \Phi_{k',a'}) = \delta_{kk'}\delta_{aa'} \\ P(E)\Phi = \int (\Phi_{E,\theta}, \Phi)\Phi_{E,\theta}\, d\theta\,, & (\Phi_{E,\theta}, \Phi_{E',\theta'}) = \delta(E-E')\,\delta(\theta-\theta')\,. \end{cases}$$

A more detailed discussion of the case of a time independent H is given in reference 10. In the case in consideration, it is convenient to introduce the operators $R_{\pm}(\lambda)$. $R_{+}(\lambda)$ is defined in the upper part of the complex λ-plane by the equation.

$$(22a)\qquad R_{+}(\lambda) = R(\lambda)\,, \qquad (Im\lambda > 0)\,,$$

and prolonged by analytical continuation to the lower part of the λ-plane across the continuous spectrum of H. $R_{-}(\lambda)$ is taken as $R(\lambda)$ in the lower part of the λ-plane

$$(22b)\qquad R_{-}(\lambda) = R(\lambda)\,, \qquad (Im\lambda < 0)\,,$$

and prolonged by analytical continuation to the upper part of the λ-plane across the continuous spectrum of H. $R(\lambda)$ is actually a branch of a multiple valued operator, isolated by a cut along the continuous spectrum of H. We get from (16) and (22)

$$(23)\qquad R_{+}(\lambda) - R_{-}(\lambda) = 2\pi i P(\lambda)\,.$$

We shall denote by $\overline{R}_{\pm,0}(\lambda, t')$ and $R_0(\lambda, t')$ the operators $\overline{R}_{\pm}$ and R corresponding to the unperturbed hamiltonian $H_0(t)$. In the case of a time independent H_0, we shall also consider the $R_{\pm\,0}(\lambda)$ and $R_0(\lambda)$.

Generalization of the equation of Heitler and Peng.

3. – We shall consider first the case of a time independent unperturbed hamiltonian H_0 and a time independent perturbation H'. It follows from the definition (18) of $R(\lambda)$ that

$$(24)\qquad (H-\lambda)R(\lambda) = [(H_0-\lambda) + H']R(\lambda) = 1\,,$$

hence

$$(25)\qquad (H_0-\lambda)R(\lambda) = 1 - H'R(\lambda)\,,$$

and

$$(26)\qquad R(\lambda) = R_0(\lambda)[1 - H'R(\lambda)]\,,$$

so that

$$R_{\pm}(\lambda) = R_{\pm,0}(\lambda)[1 - H'R_{\pm}(\lambda)] . \tag{27}$$

$R_{\pm,0}(\lambda)$ being the analogue of $R_{\pm}(\lambda)$ for the unperturbed hamiltonian H_0. We shall introduce now the operator $R_{s,0}(\lambda)$

$$R_{s,0}(\lambda) = \frac{1}{2}[R_{+,0}(\lambda) + R_{-,0}(\lambda)] . \tag{28}$$

Taking into account the analogue of (23) for H_0, we get

$$R_{\pm,0}(\lambda) = R_{s,0}(\lambda) \pm i\pi P_0(\lambda) , \tag{29}$$

$P_0(\lambda)$ being the analogue of $P(\lambda)$. Equation (27) can be written as

$$R_{\pm}(\lambda) = R_{\pm,0}(\lambda) - R_{s,0}(\lambda)H'R_{\pm}(\lambda) \mp i\pi P_0(\lambda)H'R_{\pm}(\lambda) , \tag{30}$$

hence

$$[1 + R_{s,0}(\lambda)H']R_{\pm}(\lambda) = R_{\pm,0}(\lambda) \mp i\pi P_0(\lambda)H'R_{\pm}(\lambda) , \tag{31}$$

and

$$R_{\pm}(\lambda) = [1 + R_{s,0}(\lambda)H']^{-1}R_{\pm,0}(\lambda) \mp i\pi[1 + R_{s,0}(\lambda)H']^{-1}P_0(\lambda)H'R_{\pm}(\lambda) . \tag{32}$$

The operator $[1 + R_{s,0}(\lambda)H']^{-1}$ will be expanded in a Liouville-Neumann series

$$[1 + R_{s,0}(\lambda)H']^{-1} = \sum_{n=0}^{\infty}(-1)^n[R_{s,0}(\lambda)H']^n , \tag{33}$$

so that

$$R_{\pm}(\lambda) = \left\{ \sum_{n=0}^{\infty}(-1)^n[R_{s,0}(\lambda)H']^n \right\} R_{\pm,0}(\lambda) \mp \mp i\pi \left\{ \sum_{n=0}^{\infty}(-1)^n[R_{s,0}(\lambda)H']^n \right\} P_0(\lambda)H'R_{\pm}(\lambda) . \tag{34}$$

In order to get a generalization of the integral equation of Heitler and Peng, we shall introduce the operator $U'(\lambda)$

$$U'(\lambda) = H'R_{+}(\lambda)(H_0 - \lambda) . \tag{35}$$

It follows from (34) that $U'(\lambda)$ satisfies the equation

$$U'(\lambda) = \left\{ \sum_{n=0}^{\infty}(-1)^n[H'R_{s,0}(\lambda)]^n \right\} H' - - i\pi \left\{ \sum_{n=0}^{\infty}(-1)^n[H'R_{s,0}(\lambda)]^n \right\} H'P_0(\lambda)U'(\lambda) . \tag{36}$$

This equation may be considered as a generalization of the equation of HEITLER and PENG ([3]). The Liouville-Neumann expansion (33) corresponds to the iteration procedure of Heitler and Peng, in the improved form due to PAULI ([4]). We get from (36), for real values E of the parameter λ, an integral equation for the matrix elements $\langle E, \theta \mid U'(E) \mid E', \theta' \rangle$

$$(37) \quad \langle E, \theta \mid U'(E) \mid E', \theta' \rangle = \sum_{n=0}^{\infty} (-1)^n \langle E, \theta \mid [H' R_{s,0}(E)]^n H' \mid E', \theta' \rangle - \\ - i\pi \int \langle E, \theta \mid \sum_{n=0}^{\infty} (-1)^n [H' R_{s,0}(E)]^n H' \mid E, \theta'' \rangle \langle E, \theta'' \mid U'(E) \mid E', \theta' \rangle \, d\theta'' .$$

We have shown ([10]) that on the continuous spectrum of H_0

$$(38) \qquad R_{s,0}(E) = \text{p.v.}\ (H_0 - E)^{-1} ,$$

p.v. $(H_0 - E)^{-1}$ being the operator defined by the following equations

$$(39a) \quad \text{p.v.}\ (H_0 - E)^{-1} = (H_0 - E)^{-1} \quad (E \text{ not belonging to the spectrum of } H_0),$$

$$(39b) \quad \text{p. v.}\ (H_0 - E_{0,l})^{-1} = \sum_{l' \neq l} P_{0,l'} (E_{0,l'} - E_{0,l})^{-1} + \\ + \int_{\text{sp.}} P_0(E')(E' - E_{0,l})^{-1} \, dE' \quad (E_{0,l} \text{ being a discrete eigenvalue of } H_0),$$

$$(39c) \quad \text{p.v.}\ (H_0 - E_0)^{-1} = \sum_{l'} P_{0,l'} (E_{0,l'} - E_0)^{-1} + \\ + \text{p.v.} \int_{\text{sp.}} P_0(E')(E' - E_0)^{-1} \, dE' \quad (E_0 \text{ belonging to the cont. spectrum of } H_0) .$$

In order to get the physical meaning of equation (34), let us take into account that

$$(40) \qquad (H_0 - \lambda) R_{s,0}(\lambda) = 1 ,$$

hence

$$(41) \qquad (H_0 - \lambda)[R_{s,0}(\lambda) H']^n = H' [R_{s,0}(\lambda) H']^{n-1} ,$$

so that we have

$$(42) \quad (H_0 - \lambda) \left\{ \sum_{n=0}^{\infty} (-1)^n [R_{s,0}(\lambda) H']^n \right\} = - H' \left\{ \sum_{n=0}^{\infty} (-1)^n [R_{s,0}(\lambda) H']^n \right\} + (H_0 - \lambda).$$

Let E be a real number included in the continuous spectra of H and H_0, and

$\Phi_E^{(0)}$ an eigenstate of H_0 corresponding to the eigenvalue E. It follows from (42) that the operator $\sum_{n=0}^{\infty}(-1)^n[R_{s,0}(E)H']^n$ transforms $\Phi_E^{(0)}$ into an eigenstate of H

$$(H-E)\left\{\sum_{n=0}^{\infty}(-1)^n[R_{s,0}(E)H']^n\right\}\Phi_E^{(0)}=0\,. \tag{43}$$

These eigenfunctions of H have already been discussed in reference (9). In a collision problem $\Phi_E^{(0)}$ can be taken as the eigenstate of the incoming free particles. It was shown in reference (9) that $\sum_{n=0}^{\infty}(-1)^n[R_{s,0}(E)H']^n\Phi_E^{(0)}$ does not satisfy the correct boundary conditions for scattering. It results from the preceding considerations that the first step of the Heitler-Peng theory of damping consists in the determination of an insatisfactory perturbed eigenfunction by means of an expansion in series, the second step consisting in the determination of the satisfactory eigenfunction by means of an integral equation involving the insatisfactory one.

Equation (34) may ben considered as the fundamental equation of the damping theory, instead of the more complicated equation (36). We see that the integral equation of the damping theory, in the Heitler-Peng form, is essentially an equation for the determination of the branch R_+ of the resolvent. This point of view shows an interesting relation between the theory of damping and the form of the theory of collisions developed by SOMMERFELD (13) and MEIXNER (14), first indicated in reference (10): The matrix elements of R_+ are Green functions of the stationary Schrödinger equation. This results at once from the relation

$$(H-E)R_+(E)=1\,, \tag{44}$$

by taking the matrix elements of both sides

$$\int\langle E',\theta'\,|\,H-E\,|\,E''',\theta'''\rangle\langle E''',\theta'''\,|\,R_+(E)\,|\,E'',\theta''\rangle\,\mathrm{d}E'''\,\mathrm{d}\theta'''= \tag{45}$$

$$=\delta(E'-E'')\,\delta(\theta'-\theta'')\,.$$

4. – We shall now discuss the case of a time dependent perturbation, using the generalized resolvents. In reference (10) we have given an integral equation

(13) A. SOMMERFELD: *Ann. d. Phys.*, **11**, 257 (1931).

(14) J. MEIXNER: *Math. Zeits.*, **36**, 677 (1933); *Zeits. f. Phys.*, **90**, 312 (1934); *Ann. d. Phys.*, **29**, 97 (1937).

for the operators $\overline{R}_{\pm}(\lambda, t')$

$$(46a)\qquad \int_{-\infty}^{+\infty} L(\omega)\overline{R}_{+}(\lambda-\omega, t')\,d\omega - \lambda\overline{R}_{+}(\lambda, t') = \exp[i\lambda t'], \qquad (Im\lambda > 0),$$

$$(46b)\qquad \int_{-\infty}^{+\infty} L(\omega)\overline{R}_{-}(\lambda-\omega, t')\,d\omega - \lambda\overline{R}_{-}(\lambda, t') = \exp[i\lambda t'], \qquad (Im\lambda < 0);$$

with

$$(47)\qquad L(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp[i\omega t]H(t)\,dt.$$

In the case of a time independent H_0 we have

$$(48)\qquad L(\omega) = H_0\delta(\omega) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[i\omega t]H'(t)\,dt = H_0\delta(\omega) + L'(\omega),$$

and equations (46) can be written as

$$(49)\qquad \overline{R}_{\pm}(\lambda, t') = \overline{R}_{\pm,0}(\lambda)\left[\exp[i\lambda t'] - \int_{-\infty}^{+\infty} L'(\omega)\overline{R}_{\pm}(\lambda-\omega, t')\,d\omega\right].$$

This equation is a generalization of (27). In order to derive the damping equation, we shall introduce a super-operator $\mathcal{R}$ which transforms any operator $C(\lambda)$ into another

$$(50)\qquad \mathcal{R}\cdot C(\lambda) = R_{s,0}(\lambda)\int_{-\infty}^{+\infty} L'(\omega)C(\lambda-\omega)\,d\omega.$$

Equation (49) can be easily transformed into the following one

$$(51)\qquad \overline{R}_{\pm}(\lambda, t') = Z_{\pm}(\lambda, t') - \mathcal{R}\cdot\overline{R}_{\pm}(\lambda, t'),$$

with

$$(52)\qquad Z_{\pm}(\lambda, t') = R_{\pm,0}(\lambda)\exp[i\lambda t'] \mp i\pi P_0(\lambda)\int_{-\infty}^{+\infty} L'(\omega)\overline{R}_{\pm}(\lambda-\omega, t')\,d\omega.$$

We get from (51)

$$(53)\qquad \overline{R}_{\pm}(\lambda, t') = [1+\mathcal{R}]^{-1}\cdot Z_{\pm}(\lambda, t').$$

We shall assume that the super-operator $[1+\mathcal{R}]^{-1}$ can be expanded into for a Liouville-Neumann series

$$[1+\mathcal{K}]^{-1}=\sum_{n=0}^{\infty}(-1)^n\mathcal{K}^n\,. \tag{54}$$

Introducing into (53) the expression (52) of $Z_\pm(\lambda, t')$, we get the generalization of equation (34)

$$\begin{aligned}\overline{R}_\pm(\lambda, t') &= [1+\mathcal{K}]^{-1}\cdot \overline{R}_{\pm,0}(\lambda, t') \mp \\ &\mp i\pi[1+\mathcal{K}]^{-1}\cdot P_0(\lambda)\int_{-\infty}^{+\infty} L'(\omega)\overline{R}_\pm(\lambda-\omega, t')\,\mathrm{d}\omega\,.\end{aligned} \tag{55}$$

In the case of a time independent H' we have

$$\left\{\begin{aligned} &L'(\omega)=H'\delta(\omega)\,, \qquad \mathcal{K}\cdot C(\lambda)=R_{s,0}(\lambda)H'C(\lambda)\\ &[1+\mathcal{K}]^{-1}\cdot C(\lambda)=[1+R_{s,0}(\lambda)H']^{-1}C(\lambda)\,.\end{aligned}\right. \tag{56}$$

The width operator $\Gamma_+(\lambda)$.

5. – In order to define the width of an unperturbed state we shall need a definition of the diagonal part of an operator A with respect to H_0. The definition of the diagonal part $A_{\text{d.}}$ is quite simple in the case in which H_0 has a purely discrete spectrum, but it presents certain difficulties when there are continuous levels. We shall use the following general definition of $A_{\text{d.}}$ given in reference 9

$$A_{\text{d.}}=A-\frac{i}{2}\int_{-\infty}^{+\infty}\exp[iH_0\tau][A, H_0]\exp[-iH_0\tau]\frac{\tau}{|\tau|}\,\mathrm{d}\tau\,. \tag{57}$$

It is easily seen that

$$A_{\text{d.}}=\sum_l P_l^{(0)}AP_l^{(0)}+\lim_{\tau\to\infty}\int_{\text{sp.}}\mathrm{d}E\int_{\text{sp.}}\mathrm{d}E'P_0(E)AP_0(E')\cos[(E-E')\tau]\,, \tag{58}$$

by taking into account the spectral decomposition of H_0 and $\exp[iH_0\tau]$,

$$H_0=\sum_l P_l^{(0)}E_l^{(0)}+\int_{\text{sp.}}P_0(E)E\,\mathrm{d}E\,, \tag{59}$$

$$\exp[iH_0\tau]=\sum_l P_l^{(0)}\exp[iE_l^{(0)}\tau]+\int_{\text{sp.}}P_0(E)\exp[iE\tau]\,\mathrm{d}\tau\,. \tag{60}$$

The difference between A and $A_{\text{d.}}$ will be called the non diagonal part of A

and denoted by $A_{\text{n.d.}}$,

$$A = A_{\text{d.}} + A_{\text{n.d.}} . \tag{61}$$

We shall put

$$R_+(\lambda)_{\text{d.}} = \left[H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)\right]^{-1} . \tag{62}$$

It follows from (14*a*) that

$$\eta(t-t')V(t,t')_{\text{d.}} = \frac{1}{2\pi i}\int_{-\infty+ia}^{+\infty+ia} \frac{\exp[-i\lambda(t-t')]\,d\lambda}{H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)} , \tag{63}$$

hence

$$\langle l\,|\,V(t,t')\,|\,l\rangle\eta(t-t') = \langle l\,|\,V(t,t')_{\text{d.}}\,|\,l\rangle\eta(t-t') = \tag{64}$$

$$= \frac{1}{2\pi i}\int_{-\infty+ia}^{+\infty+ia} \frac{\exp[-i\lambda(t-t')]\,d\lambda}{E_l^{(0)} - \lambda - \frac{i}{2}\langle l\,|\,\Gamma_+(\lambda)\,|\,l\rangle} .$$

The integral in (64) can be evaluated by residues, closing the integration path with a half-circle at infinity in the lower part of the λ-plane. We shall assume that there is a single pole ε_l defined by the equation

$$\varepsilon_l = E_l^{(0)} - \frac{i}{2}\langle l\,|\,\Gamma_+(\varepsilon_l)\,|\,l\rangle . \tag{65}$$

Thus we get

$$\langle l\,|\,V(t,t')\,|\,l\rangle\eta(t-t') = \exp[-iE_l^{(0)}(t-t')]\exp\left[-\frac{1}{2}(t-t')\langle l\,|\,\Gamma^+(\varepsilon_l)\,|\,l\rangle\right] . \tag{66}$$

The real part of $\langle l\,|\,\Gamma_+(\varepsilon_l)\,|\,l\rangle$ is the width of the unperturbed level. The imaginary part of $1/2\langle l\,|\,\Gamma_+(\varepsilon_l)\,|\,l\rangle$ gives the displacement of the unperturbed level due to the perturbation H'. $\Gamma_+(\lambda)$ will be called the width operator.

We shall consider now the non diagonal part of $R_+(\lambda)$. It follows from (27) that

$$R_+(\lambda)_{\text{n.d.}} = -R_{+,0}(\lambda)[H'R_+(\lambda)]_{\text{n.d.}} . \tag{67}$$

It is convenient to introduce the operator $U(\lambda)$

$$U(\lambda) = [H'R_+(\lambda)]_{\text{n.d.}}\left(H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)\right) . \tag{68}$$

Equation (67) can be written as

$$R_+(\lambda)_{\text{n.d.}} = - R_{+,0}(\lambda)\, U(\lambda) R_+(\lambda)_{\text{d.}} \,. \tag{69}$$

It follows from (14*a*) that

$$\eta(t-t')\, V(t,t')_{\text{n.d.}} = \frac{1}{2\pi i} \int\limits_{-\infty+ia}^{+\infty+ia} \exp\left[-i\lambda(t-t')\right] R_+(\lambda)_{\text{n.d.}}\, \mathrm{d}\lambda \,, \tag{70}$$

hence

$$\eta(t-t')\, V(t,t')_{\text{n.d.}} = -\frac{1}{2\pi i} \int\limits_{-\infty+ia}^{+\infty+ia} R_{+,0}(\lambda)\, U(\lambda)\, \frac{\exp\left[-i\lambda(t-t')\right] \mathrm{d}\lambda}{H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)} \,, \tag{71}$$

so that we get by deformation of the integration path into the real axis

$$\eta(t-t')\, \langle l \,|\, V(t,t') \,|\, l' \rangle = \tag{72}$$

$$= \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \left[\text{p.v.}\, (E_l^{(0)} - \lambda)^{-1} + i\pi\, \delta(E_l^{(0)} - \lambda)\right] \frac{\exp\left[-i\lambda(t-t')\right] \langle l \,|\, U(\lambda) \,|\, l' \rangle}{\lambda - E_{l'}^{(0)} + \frac{i}{2} \langle l \,|\, \Gamma_+(\lambda) \,|\, l \rangle}\, \mathrm{d}\lambda \,.$$

The formulae for the matrix elements of $V(t, t')$ that we have derived are essentially the same given by HEITLER and MA (5).

The diagonal part of an operator A with respect to an operator B does not commute necessarily with B, as it is easily seen by considering the diagonal part of a momentum p with respect to the corresponding coordinate q, which coincides with p. Therefore, we cannot be a priori sure that the matrix elements of $\Gamma_+(\lambda)$ are of the following form

$$\langle E, \theta \,|\, \Gamma_+(\lambda) \,|\, E', \theta' \rangle = G_+(\theta, \theta'; \lambda)\, \delta(E - E') \,, \tag{73}$$

as it would appear from an uncritical analogy with the case of an H_0 with a purely discrete spectrum.

The equation of Heitler and Ma.

6. – We shall now derive the equation of HEITLER and MA from our formalism. It follows from (69) that:

$$R_+(\lambda) = R_+(\lambda)_{\text{d.}} - R_{+,0}(\lambda)\, U(\lambda) R_+(\lambda)_{\text{d.}} \,, \tag{74}$$

hence

$$H' R_+(\lambda) \left(H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)\right) = - H' - H' R_{+,0}(\lambda)\, U(\lambda). \tag{75}$$

Taking the non diagonal parts of both sides of (75), we get

$$(76)\qquad U(\lambda) = H'_{\text{n.d.}} - [H'R_{+,0}(\lambda)U(\lambda)]_{\text{n.d.}} = $$
$$= H'_{\text{n.d.}} - [H'R_{s,0}(\lambda)U(\lambda)]_{\text{n.d.}} - i\pi[H'P_0(\lambda)U(\lambda)]_{\text{n.d.}} .$$

Let us introduce the super-operator $\mathcal{A}$ defined by the following equation

$$(77)\qquad \mathcal{A}\cdot C(\lambda) = [H'R_{s,0}(\lambda)C(\lambda)]_{\text{n.d.}} .$$

We get from (76) and (77)

$$(78)\qquad U(\lambda) + \mathcal{A}\cdot U(\lambda) = [H' - i\pi H'P_0(\lambda)U(\lambda)]_{\text{n.d.}} ,$$

hence

$$(79)\qquad U(\lambda) = [1+\mathcal{A}]^{-1}\cdot[H' - i\pi H'P_0(\lambda)U(\lambda)]_{\text{n.d.}} .$$

The equation of Heitler and Ma follows from (79) by expanding the super-operator $[1+\mathcal{A}]^{-1}$ in a Liouville-Neumann series

$$(80)\qquad U(\lambda) = \left[\sum_{n=0}^{\infty}(-1)^n\mathcal{A}^n\right]\cdot H'_{\text{n.d.}} - i\pi\left[\sum_{n=0}^{\infty}(-1)^n\mathcal{A}^n\right]\cdot[H'P_0(\lambda)U(\lambda)]_{\text{n.d.}} .$$

The definition of $U(E)$ for real values of E which corresponds to the treatment of Heitler and Ma is given by the following equation

$$(81)\qquad \omega(E) = U(E)[\text{p.v.}\,(H-E)^{-1} + i\pi P(E)]_{\text{d.}} ,$$

with

$$(82)\qquad \omega(E) = -\int_0^{\infty}\exp[-i(H_0-E)t]\,\frac{\mathrm{d}}{\mathrm{d}t}\{\exp[iH_0t]\exp[-iHt]\}_{\text{n.d.}}\,\mathrm{d}t .$$

We shall replace E by a complex parameter λ and transform (82) by partial integration. Thus we get

$$(83)\qquad \omega(\lambda) = -(H_0-\lambda)R_+(\lambda)_{\text{n.d.}} , \qquad (Im\lambda > 0) .$$

The definition (81) of $U(E)$ can be extended to complex values of the parameter by taking

$$(84)\qquad \omega(\lambda) = U(\lambda)R_+(\lambda)_{\text{d.}} .$$

It follows from (84) and (83) that

$$85)\qquad R_+(\lambda)_{\text{n.d.}} = -R_{+,0}(\lambda)U(\lambda)R_+(\lambda)_{\text{d.}} .$$

This equation coincides with (69), therefore the two definitions of $U(\lambda)$ are equivalent.

The definition of the width operator $\Gamma_+(\lambda)$ which corresponds to that given by HEITLER and MA is the following one

$$-\frac{i}{2}\Gamma_+(\lambda) = [H' - H'R_{+,0}(\lambda)U(\lambda)]_{\text{d.}} \,. \tag{86}$$

In order to derive (86) we shall start from the equation

$$R_+(\lambda)_{\text{d.}} = R_{+,0}(\lambda) - R_{+,0}(\lambda)[H'R_+(\lambda)]_{\text{d.}} = \\ = R_{+,0}(\lambda) - R_{+,0}(\lambda)[H' - H'R_{+,0}(\lambda)U(\lambda)]_{\text{d.}}R_+(\lambda)_{\text{d.}} \,, \tag{87}$$

which follows from (26) and (69). It results from (87) that

$$R_+(\lambda)_{\text{d}} = \{[H_0 - \lambda + H' - H'R_{+,0}(\lambda)U(\lambda)]_{\text{d.}}\}^{-1} \,, \tag{88}$$

hence

$$\left(H_0 - \lambda - \frac{i}{2}\Gamma_+(\lambda)\right) = [H_0 - \lambda + H' - H'R_{+,0}(\lambda)U(\lambda)]_{\text{d.}} \,. \tag{89}$$

Equation (86) follows immediately from (89).

The preceding considerations show that the two operators U and Γ_+ can be defined easily in terms of the more fundamental one R_+, and also that the equation of Heitler and Ma is a consequence of the equations satisfied by the resolvent, which are simpler.

Alternative treatment of the damping effects.

7. – We shall now discuss the damping with a different method, which will lead us to the theory of SCHWINGER [8], FUKUDA and MIYAZIMA [7]. Let us introduce the operators $U_\pm(t, t')$ and $U_s(t, t')$

$$U_+(t, t') = \eta(t - t')V(t, t') \tag{90a}$$

$$U_-(t, t') = -\eta(t' - t)V(t, t') \tag{90b}$$

$$U_s(t, t') = \frac{1}{2}[U_+(t.\, t') + U_-(t, t')] \,. \tag{90c}$$

The corresponding operators for the unperturbed hamiltonian will be denoted by $U_{\pm,0}(t, t')$ and $U_{s,0}(t, t')$.

It is easily seen that $V(t, t')$ satisfies the following integral equation

$$V(t, t') = V_0(t, t') - i\int_{t'}^{t} V_0(t, \tau)H'(\tau)V(\tau, t')\,d\tau \,. \tag{91}$$

By differentiating both sides of (91) with respect to t we get the first equation (2). The second equation (2) is also obviously satisfied by the solution of (91). From (91) follows an integral equation for $U_\pm(t, t')$.

$$U_\pm(t, t') = U_{\pm,0}(t, t') - i \int_{-\infty}^{+\infty} U_{\pm,0}(t, \tau) H'(\tau) U_\pm(\tau, t')\, d\tau , \tag{92}$$

$$U_\pm(t, t') = U_{\pm,0}(t, t') \mp \frac{i}{2} \int_{-\infty}^{+\infty} V_0(t, \tau) H'(\tau) U_\pm(\tau, t')\, d\tau - \\ - i \int_{-\infty}^{+\infty} U_{s,0}(t, t') H'(\tau) U_\pm(\tau, t')\, d\tau , \tag{93}$$

hence

$$U_s(t, t') = U_{s,0}(t, t') - \frac{i}{2} \int_{-\infty}^{+\infty} V_0(t, \tau) H'(\tau) V(\tau, t')\, d\tau - \\ - i \int_{-\infty}^{+\infty} U_{s,0}(t, t') H'(\tau) U_s(\tau, t')\, d\tau , \tag{94}$$

$U_\pm(t, t')$ and $U_s(t, t')$ satisfy the differential equation

$$i \frac{d}{dt} U(t, t') = H(t) U(t, t') - i\delta(t - t') . \tag{95}$$

Let us introduce the super-operator $\mathcal{U}$ defined by the following equation

$$\mathcal{U} \cdot C(t) = - i \int_{-\infty}^{+\infty} U_{s,0}(t, \tau) H'(\tau) C(\tau)\, d\tau . \tag{96}$$

It follows from (93) that

$$U_\pm(t, t') = [1 - \mathcal{U}]^{-1} \cdot \left\{ U_{\pm,0}(t, t') \mp \frac{i}{2} \int_{-\infty}^{+\infty} V_0(t, \tau) H'(\tau) U_\pm(\tau, t')\, d\tau \right\} , \tag{97}$$

hence

$$U_\pm(t, t') = \sum_{n=0}^{\infty} \mathcal{U}^n \cdot \left\{ U_{\pm,0}(t, t') \mp \frac{i}{2} \int_{-\infty}^{+\infty} V_0(t, \tau) H'(\tau) U_\pm(\tau, t')\, d\tau \right\} . \tag{98}$$

This equation is related to (34), because in the case of a time independent hamiltonian

$$R_{\pm}(\lambda) = i \int_{-\infty}^{+\infty} \exp\left[-i\lambda(t-t')\right] U_{\pm}(t,t')\, dt \,. \tag{99}$$

The decomposition of $U_{\pm,0}(t,t')$ into $U_{s,0}(t,t')$ and $\pm i/2\, V_0(t,t')$ corresponds to the decomposition of $R_{\pm,0}(\lambda)$ into $R_{s,0}(\lambda)$ and $\pm i\pi P_0(\lambda)$. The integral equation (98) is the fundamental equation in the present treatment of damping. It is interesting to notice that (98) is more general than both (34) and (55), since it is valid even when H_0 and H' depend on t.

8. – Let us consider the super–operator $\mathcal{W}$

$$\mathcal{W} \cdot C(t) = -\frac{i}{2} \int_{-\infty}^{+\infty} \varepsilon(t-\tau) \left\{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \right\} C(\tau)\, d\tau \,. \tag{100}$$

$V_0(t)$ being the operator defined by (8) and

$$\varepsilon(t-\tau) = \frac{t-\tau}{|t-\tau|} \,. \tag{101}$$

Taking into account that

$$V_0(t,t') = V_0(t) V_0^{-1}(t') \,, \tag{102}$$

we get the following relation between $\mathcal{U}$ and $\mathcal{W}$

$$\mathcal{U} \cdot C(t) = V_0(t) \left\{ \mathcal{W} \cdot V_0^{-1}(t) C(t) \right\} . \tag{103}$$

It follows from (103) that

$$\mathcal{U}^n \cdot C(t) = V_0(t) [\mathcal{W}^n \cdot V_0^{-1}(t) C(t)] \,, \tag{104}$$

hence

$$[1-\mathcal{U}]^{-1} \cdot C(t) = V_0(t) \left\{ [1-\mathcal{W}]^{-1} \cdot V^{-1}(t) C(t) \right\} . \tag{105}$$

It follows from (97) that

$$V(t,t') = [1-\mathcal{U}]^{-1} \cdot V_0(t,t') - -\frac{i}{2} \int_{-\infty}^{+\infty} \left\{ [1-\mathcal{U}]^{-1} \cdot V_0(t,\tau) \right\} H'(\tau) V(\tau,t') \varepsilon(\tau - t')\, d\tau \,. \tag{106}$$

This equation can be taken as the integral equation of the damping, instead of (98). Let us introduce the operator $\overline{V}(t,t')$

$$\overline{V}(t,t') = [1-\mathcal{U}]^{-1} \cdot V_0(t,t') . \tag{107}$$

Equation (106) can be written as

$$V(t, t') = \overline{V}(t, t') - \frac{i}{2}\int_{-\infty}^{+\infty} \overline{V}(t, \tau)H'(\tau)V(\tau, t')\varepsilon(\tau - t')\,d\tau\,. \tag{108}$$

It follows from (105) that

$$\overline{V}(t, t') = V_0(t)\{[1 - \mathcal{W}]^{-1}\cdot V_0^{-1}(t)V_0(t)\}V_0^{-1}(t') = V_0(t)\overline{W}(t)V_0^{-1}(t')\,, \tag{109}$$

with

$$\overline{W}(t) = [1 - \mathcal{W}]^{-1}\cdot 1(t)\,, \tag{110}$$

$1(t)$ is the operator 1, with the indication that it is considered as a function of t. Taking into account the identity

$$[1 - \mathcal{W}]^{-1} = 1 + \mathcal{W}[1 - \mathcal{W}]^{-1}\,, \tag{111}$$

we see that $\overline{W}(t)$ is the solution of the integral equation of SCHWINGER (8)

$$\overline{W}(t) = 1 - \frac{i}{2}\int_{-\infty}^{+\infty} \varepsilon(t - \tau)\{V_0^{-1}(\tau)H'(\tau)V_0(\tau)\}\overline{W}(\tau)\,d\tau\,. \tag{112}$$

Therefore $\overline{V}(t, t')$ is the solution of the integral equation

$$\overline{V}(t, t') = V_0(t, t') - \frac{i}{2}\int_{-\infty}^{+\infty} V_0(t, \tau)H'(\tau)\overline{V}(\tau, t')\varepsilon(t - \tau)\,d\tau\,. \tag{113}$$

It follows from (113) that

$$i\frac{d}{dt}\overline{V}(t, t') = H(t)\overline{V}(t, t')\,. \tag{114}$$

The preceding results show that, in the present treatment of the damping, a solution $\overline{V}(t, t')$ of the differential equation for $V(t, t')$ is obtained by an expansion in series and then $V(t, t')$ is determined as the solution of the integral equation (108), which involves $\overline{V}(t, t')$. There is a very close analogy with the procedure of HEITLER and PENG, in which the correct eigenfunction is determined by an integral equation involving an insatisfactory wave function obtained by an expansion in series.

We have already obtained Schwinger's integral equation (112), (strictly speaking a more general one, since Schwinger considered only time independent hamiltonians), now we shall derive the fundamental equation of the damping treatment of FUKUDA and MIYAZIMA (7). Let us introduce the operator $W(t, t')$

$$V(t, t') = V_0(t)W(t, t')V_0^{-1}(t')\,. \tag{115}$$

The Heisenberg collision operator S is $W(\infty, -\infty)$

$$S = W(\infty, -\infty) . \tag{116}$$

From (91) we get an integral equation for $W(t, t')$

$$W(t, t') = 1 - i \int_{t'}^{t} \{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \} \, W(\tau, t') \mathrm{d}\tau , \tag{117}$$

which can be transformed into

$$W(t, t') = 1 - \frac{i}{2} \int_{-\infty}^{+\infty} \{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \} \, W(\tau, t') \varepsilon(\tau - t') \, \mathrm{d}\tau - \\ - \frac{i}{2} \int_{-\infty}^{+\infty} \{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \} \, W(\tau, t') \varepsilon(t - \tau) \, \mathrm{d}\tau . \tag{118}$$

It results from (118) and (112) that

$$W(t, t') = \overline{W}(t) \left[1 - \frac{i}{2} \int_{-\infty}^{+\infty} \{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \} \, W(\tau, t') \varepsilon(\tau - t') \, \mathrm{d}\tau \right] , \tag{119}$$

hence

$$S = W(\infty, -\infty) = \overline{W}(\infty) \left[1 - \frac{i}{2} \int_{-\infty}^{+\infty} \{ V_0^{-1}(\tau) H'(\tau) V(\tau) \} \, W(\tau, -\infty) \, \mathrm{d}\tau \right] . \tag{120}$$

It follows from (117) that

$$S = 1 - i \int_{-\infty}^{+\infty} \{ V_0^{-1}(\tau) H'(\tau) V_0(\tau) \} \, W(\tau, -\infty) \, \mathrm{d}\tau , \tag{121}$$

therefore (120) may be written as

$$S = \frac{1}{2} \, \overline{W}(\infty) [1 + S] . \tag{122}$$

By putting

$$S = 1 - iT , \qquad \overline{W}(\infty) = 1 - \frac{i}{2} K , \tag{123}$$

(122) goes over into the basic equation of the FUKUDA and MIYAZIMA treatment of the damping (7)

$$T = K - \frac{i}{2} KT . \tag{124}$$

FUKUDA and MIYAZIMA use the series expansion of $\overline{W}(\infty)$ obtained from (112) by the application of the method of successive approximations. This expansion can be obtained immediately from (110)

$$\overline{W}(t) = \sum_{n=0}^{\infty} \mathcal{W}^n \cdot 1(t) . \tag{125}$$

9. – We shall now examine in detail the relations between the two treatments of the damping. The super-operators $\mathcal{U}$ and $\mathcal{K}$ are related by the following equation

$$\int_{-\infty}^{+\infty} \exp[i\lambda t]\mathcal{U} \cdot A(t)\,\mathrm{d}t = -\mathcal{K} \cdot \int_{-\infty}^{+\infty} A(t) \exp[i\lambda t]\,\mathrm{d}t , \qquad (Im\lambda = 0) . \tag{126}$$

Taking into account that

$$R_{s,0}(\lambda) = i \exp[-i\lambda\tau] \int_{-\infty}^{+\infty} \exp[i\lambda t] U_{s,0}(t, \tau)\,\mathrm{d}\tau , \qquad (Im\lambda = 0) , \tag{127}$$

we get from (96), (48) and (50)

$$\begin{aligned} \int_{-\infty}^{+\infty} \exp[i\lambda t]\, \mathcal{U} \cdot A(t)\,\mathrm{d}t &= -i \int_{-\infty}^{+\infty} \exp[i\lambda t]\,\mathrm{d}t \int_{-\infty}^{+\infty} U_{s,0}(t, \tau) H'(\tau) A(\tau)\,\mathrm{d}\tau = \\ &= -R_{s,0}(\lambda) \int_{-\infty}^{+\infty} \exp[i\lambda\tau] H'(\tau) A(\tau)\,\mathrm{d}\tau = \\ &= -R_{s,0}(\lambda) \int_{-\infty}^{+\infty} L'(\omega)\,\mathrm{d}\omega \int_{-\infty}^{+\infty} \exp[i(\lambda-\omega)\tau] A(\tau)\,\mathrm{d}\tau = \\ &= -\mathcal{K} \cdot \int_{-\infty}^{+\infty} A(\tau) \exp[i\lambda\tau]\,\mathrm{d}\tau . \end{aligned} \tag{128}$$

It follows from (126) that

$$\int_{-\infty}^{+\infty} \exp[i\lambda t]\mathcal{U}^n \cdot A(t)\,\mathrm{d}t = (-1)^n \mathcal{K}^n \cdot \int_{-\infty}^{+\infty} A(t) \exp[i\lambda t]\,\mathrm{d}t , \tag{129}$$

and

$$\int_{-\infty}^{+\infty} \exp[i\lambda t][1 - \mathcal{U}]^{-1} \cdot A(t)\,\mathrm{d}t = [1 + \mathcal{K}]^{-1} \cdot \int_{-\infty}^{+\infty} A(t) \exp[i\lambda t]\,\mathrm{d}t . \tag{130}$$

Equation (130) shows that we can go over from (97) to (55) by a Fourier transformation with respect to t. It follows from (130) that when λ belongs to the continuous spectrum of H_0

$$\int_{-\infty}^{+\infty} \overline{V}(t, t') \exp[i\lambda t]\, dt = 2\pi[1 + \mathcal{K}]^{-1} \cdot P_0(\lambda) \exp[i\lambda t']. \tag{131}$$

Since

$$\overline{V}(t, t') = \overline{V}(t) V_0^{-1}(t'), \qquad \overline{V}(t) = V_0(t)\overline{W}(t), \tag{132}$$

we get from (131)

$$\int_{-\infty}^{+\infty} \overline{V}(t) \exp[i\lambda t]\, dt = 2\pi[1 + \mathcal{K}]^{-1} \cdot P_0(\lambda). \tag{133}$$

Generalization of Schönberg's perturbation methods.

10. – We have given a treatment of the damping (9) related to the perturbation theory, which unifies the Heitler-Peng and Schwinger theories and which allows to take into account self-energies. That treatment can be applied only when it is possible to establish a correspondence between the eigenstates of H and H_0

$$\begin{cases} \Phi_{\overline{E}_\theta} \to \Phi^{(0)}_{E,\theta}, & \overline{E}_\theta \to (E, \theta) \\ \Phi_l \to \Phi^{(0)}_l, & \overline{E}_l \to E^{(0)}_l. \end{cases} \tag{134}$$

We shall assume that H depends on a parameter μ, such that H tends to H_0 when μ tends to zero

$$\lim_{\mu\to 0} H = H_0, \tag{135}$$

and we shall assume that the correspondence between perturbed and unperturbed eigenstates is such that

$$\begin{cases} \lim\limits_{\mu\to 0} \Phi_{\overline{E}_\theta} = \Phi^{(0)}_{E,\theta}, & \lim\limits_{\mu\to 0} \overline{E}_\theta = E \\ \lim\limits_{\mu\to 0} \Phi_l = \Phi^{(0)}_l, & \lim\limits_{\mu\to 0} \overline{E}_l = E^{(0)}_l \end{cases} \tag{136}$$

In the perturbation treatment of reference (9) the operator J plays a central part

$$J = \sum_l P^{(0)}_l \overline{E}_l + \int P^{(0)}_{E,\theta} \overline{E}_\theta \, dE\, d\theta, \tag{137}$$

$P_l^{(0)}$ and $P_{E,\theta}^{(0)}$ being the projection operators corresponding to the eigenstates $\Phi_l^{(0)}$ and $\Phi_{E,\theta}^{(0)}$. The hamiltonian H is splitted into J and I

(138) $$H = J + I \, .$$

In the perturbation treatment in consideration, J is essentially the unperturbed hamiltonian and I the perturbation. The perturbed eigenstates Φ can be obtained from the $\Phi^{(0)}$ by means of operators Ω such that

(139) $$\Omega^{-1} H \Omega = J \, ,$$

(140) $$\Phi_l = \Omega \Phi_l^{(0)} \, , \qquad \Phi_{\overline{E_\theta}} = \Omega \Phi_{E,\theta}^{(0)} \, .$$

We have shown in reference (9) that the operators Ω satisfy the integral equation

(141) $$\Omega = \Omega_{\overline{\text{d.}}} + \frac{i}{2} \int_{-}^{+\infty} \exp\,[iJ\tau] I \Omega \exp\,[-\,iJ\tau]\varepsilon(\tau)\mathrm{d}\tau \, ,$$

$\Omega_{\overline{\text{d.}}}$ being the diagonal part of Ω with respect to J. It follows from (141) that Ω is determined, once $\Omega_{\overline{\text{d.}}}$ is given.

Let us introduce the operators $V_J(t)$ and $R_{\pm,J}(\lambda)$

(142) $$V_J(t) = \exp\,[-\,iJt] = \Omega^{-1} \exp\,[-\,iHt]\Omega \, ,$$

(143a) $$R_{+,J}(\lambda) = i \int_0^\infty \exp\,[i\lambda t] V_J(t)\mathrm{d}t = \Omega^{-1} R_+(\lambda)\Omega \, , \qquad (Im\lambda > 0) \, ,$$

(143b) $$R_{-,J}(\lambda) = -\,i \int_{-\infty}^0 \exp\,[i\lambda t] V_J(t)\mathrm{d}t = \Omega^{-1} R_-(\lambda)\Omega \, , \qquad (Im\lambda < 0) \, ,$$

and the operators $R_J(\lambda)$ and $R_{s,J}(\lambda)$

(144) $$R_J(\lambda) = (J - \lambda)^{-1} = \Omega^{-1} R(\lambda)\Omega \, ,$$

(145) $$R_{s,J}(\lambda) = \frac{i}{2}\,[R_{+,J}(\lambda) + R_{-,J}(\lambda)] = \Omega^{-1} R_s(\lambda)\Omega \, .$$

We have

(146) $$R_{+,J}(\lambda) - R_{-,J}(\lambda) = 2\pi i \Omega^{-1} P(\lambda)\Omega \, .$$

Let us denote by $P_J(\lambda)$ the operator $\Omega^{-1}P(\lambda)\Omega$

(147) $$P_J(\lambda) = \Omega^{-1} P(\lambda)\Omega \, .$$

We have obviously

(148) $$J P_J(\lambda) = \lambda P_J(\lambda) \, ,$$

and

$$P_J(\overline{E}) = \int_{\text{s.p.}} P^{(0)}_{E,\theta}\,\delta(\overline{E} - \overline{E}_\theta)\,\mathrm{d}E\,\mathrm{d}\theta\,. \tag{149}$$

In the same way as we derived (32), we can obtain the following equation

$$R_\pm(\lambda) = [1 + R_{s,J}(\lambda)I]^{-1}R_{\pm,J}(\lambda) \mp i\pi[1 + R_{s,J}(\lambda)I]^{-1}P_J(\lambda)IR_\pm(\lambda)\,. \tag{150}$$

The operator $U'_J(\lambda)$, analogous to $U(\lambda)$

$$U'_J(\lambda) = IR_+(\lambda)(J - \lambda)\,, \tag{151}$$

satisfies an equation analogous to (36)

$$U'_J(\lambda) = [1 + IR_{s,J}(\lambda)]^{-1}I - i\pi[1 + IR_{s,J}(\lambda)]^{-1}IP_J(\lambda)\,U'_J(\lambda)\,, \tag{152}$$

hence

$$\langle E, \theta \mid U'_J(\overline{E}) \mid E', \theta'\rangle = \langle E, \theta \mid [1 + IR_{s,J}(\overline{E})]^{-1}I \mid E', \theta'\rangle - \tag{153}$$

$$- i\pi \int \langle E, \theta \mid [1 + IR_{s,J}(\overline{E})]^{-1}I \mid E'', \theta''\rangle\, \delta(\overline{E} - \overline{E}''_{\theta''})\,\langle E'', \theta'' \mid U'_J(\overline{E}) \mid E', \theta'\rangle \mathrm{d}E''\,\mathrm{d}\theta''.$$

This is the integral equation given in reference ([9]), which takes into account the self-energy corrections. The essential difference between (153) and (37) is clearly shown by the $\delta(\overline{E} - \overline{E}''_{\theta''})$ which replaces $\delta(E - E'')$, so that the shell of the energies in (153) is defined by the shifted energy $\overline{E}$.

There is an equation analogous to (43)

$$(H - \overline{E}_\theta)\,\{\,[1 + R_{s,J}(\overline{E}_\theta)I]^{-1}\Phi^{(0)}_{E,\theta}\,\} = 0\,. \tag{154}$$

The operator $[1 + R_{s,J}(\overline{E}_\theta)I]^{-1}$ transforms the unperturbed eigenstate $\Phi^{(0)}_{E,\theta}$ of energy E into an eigenstate of H of energy $\overline{E}_\theta$. This eigenstate of H is not yet the satisfactory one, which must be obtained by solving equation (153) or equation (150). It is easily seen that

$$(H - \overline{E}_l)\,\{\,[1 + R_{s,J}(\overline{E}_l)I]^{-1}\Phi^{(0)}_l\,\} = 0\,, \tag{155}$$

therefore the operator Ω_1

$$\Omega_1 = \sum_l [1 + R_{s,J}(\overline{E}_l)I]^{-1}P^{(0)}_l + \int_{\text{sp.}H} [1 + R_{s,J}(\overline{E})I]^{-1}P_J(\overline{E})\mathrm{d}\overline{E}\,, \tag{156}$$

transforms the $\Phi^{(0)}$ into the corresponding Φ, hence Ω_1 is an Ω.

11. – The treatment of damping given in reference ([9]) has two serious limitations:

a) It is not applicable to problems in which is involved a level of H or H_0 which cannot be put in correspondence with a level of H_0 or H, respectively.

b) There is an arbitrariness in the way in which the correspondence is set up between states of the continuous spectra of H_0 and H, as it has already been shown in reference (9), so that it is not possible to give a unique definition of the basic operator J without introducing some special criterium which does not follow from the perturbation method used. In the usual applications of perturbation methods to the states of the continuum, it is assumed that the corresponding states have the same energy (see P. A. M. DIRAC: *The Principles of Quantum Mechanics*, chap. VIII, Oxford, 1947). This procedure is not satisfactory in problems involving self-energies, in which it is necessary to introduce shifts of the levels of the continuum.

We have seen that the damping theory allows us to get the shift of the discrete levels by means of the width operator $\Gamma_+(\lambda)$. When $\langle l \,|\, \Gamma_+(\varepsilon_l) \,|\, l\rangle$ is imaginary (vanishing real part), with ε_l defined by (65), the level $E_l^{(0)}$ of H_0 is shifted to a level $\overline{E}_l = \varepsilon_l$ of H

(157) $$\overline{E}_l = E_l^{(0)} - \frac{i}{2}\langle l \,|\, \Gamma_+(\varepsilon_l) \,|\, l\rangle = \varepsilon_l\,.$$

We may apply the same procedure to determine the shifts of the levels of the continuum. Let us introduce the numerical function $G_+(E, \theta; \lambda)$

(158) $$G_+(E, \theta; \lambda) = \lim_{\tau\to\infty} \int_{\text{sp.}} \langle E, \theta \,|\, \Gamma_+(\lambda) \,|\, E', \theta'\rangle \cos[(E - E')\tau] \prod_\theta \cos[(\theta - \theta')\tau]\, \mathrm{d}E'\, \mathrm{d}\theta'\,.$$

The limit in (158) is of a generalized kind, such that $\lim\limits_{\tau\to\infty} \cos \tau x = 0$ for $x \neq 0$ (see HARDY: *Divergent Series*, pag. 12, Oxford, 1949). When the width operator Γ_+ is diagonal in the $(H_0, \theta_{\text{op}})$ representation, we have (*)

(159) $$\Gamma_+(\lambda) = G_+(H_0, \theta_{\text{op}}, \lambda)\,,$$

because in this case

(160) $$\langle E, \theta \,|\, \Gamma_+(\lambda) \,|\, E, \theta'\rangle = G_+(E, \theta; \lambda)\, \delta(E - E')\,(\theta - \theta')\,.$$

We shall assume that the equation

(161) $$\varepsilon_\theta = E - \frac{i}{2} G_+(E, \theta; \varepsilon_\theta)\,,$$

has a single root in the neighbourhood of E, and we shall take

(162) $$E_\theta = Re\, \varepsilon_\theta\,.$$

(*) $G_+(H_0, \theta_{\text{op.}}; \lambda)$ is the diagonal part of $\Gamma_+(\lambda)$ with respect to both H_0 and the $\theta_{\text{op.}}$.

It results from the preceding considerations that it is satisfactory to take always

$$J = \sum_l P_l^{(0)} Re\varepsilon_l + \int_{\text{sp.}} P_{E,\theta}^{(+)} Re\varepsilon_\theta \, dE \, d\theta \, . \tag{163}$$

Thus we can get rid of the inconveniences of the treatment given in reference (9). In the general case we are considering now, it is no more possible to introduce operators Ω, because the spectra of H and J do not coincide always, but it is still possible to define V_J, $R_{\pm,J}$, $R_{s,J}$ and P_J by means of the difference $R_{+,J} - R_{-,J}$. Equations (150) and (153) are still valid.

RIASSUNTO (*)

Si dà un'espressione generale per l'equazione di Heitler e Peng. Si esaminano dettagliatamente le relazioni tra le teorie generalizzate di Heitler e Peng e di Schwinger sullo smorzamento. Si generalizza una forma della teoria dello smorzamento data dall'autore in un precedente lavoro per il caso di un'interazione qualsiasi indipendente dal tempo. Si mostra che la forma generalizzata della trattazione dello smorzamento di Heitler e Peng può essere usata per sviluppare una forma di teoria delle perturbazioni stazionarie applicabile a casi in cui non sia possibile stabilire una corrispondenza tra stati stazionari perturbati e imperturbati.

(*) *Traduzione a cura della Redazione*