IL NUOVO CIMENTO Vol. VIII, N. 9 1 Settembre 1951

Physical Applications of the Resolvent Operators (I). On the Mathematical Formalism of Feynman's Theory of the Positron.

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(ricevuto il 14 Giugno 1951)

Summary. — The resolvent operators of the theory of linear functional equations are applied to the quantum formalism in general and more specially to the Feynman formulation of the hole theory. A generalization of the resolvent operators is given in order to treat problems with time dependent hamiltonians. It is shown that Feynman's formulation amounts to consider divergent waves for the positive kinetic energies and convergent waves for the negative kinetic energies, in the propagation kernel. Expansions of the propagation kernels are derived from the resolvent, without using Feynman's integral equation which leads to difficulties. A relativistically invariant resolvent is defined in the theory of quantized interacting fields. An operator related to the resolvent describes a new kind of collision which can be used in the theory of the ground state of atomic nuclei.

Introduction.

1. – The resolvent operator $(A-\lambda)^{-1}$, λ being a complex numerical parameter and A a linear operator, plays a central part in the mathematical theory of the linear operators and linear functional equations. Stone's theory of the linear transformations in Hilbert space (1) is largely based on the use of the resolvent operator, as had been many other important developments of the linear analysis (2,3). Nervertheless the resolvent operator is seldom used

(1) M. H. Stone: *Linear transformations in Hilbert space* (New York, 1932).

(2) F. Riesz: *Les systèmes d'équations linéaires à une infinité d'inconnues* (Paris, 1913).

(3) E. Hellinger und O. Toeplitz: *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten*, « Encyclopädie der Mathematischen Wissenschaften » vol. 3°, (Leipzig, 1927).

in quantum mechanics, although it is the most adequate instrument for the discussion of many important problems. Recently KATO (4) applied the resolvent operator to the quantum theory of perturbations. The theory of scattering of SOMMERFELD (5) and MEIXNER (6), in which the Green functions of the stationary Schrödinger and Dirac equations play the central part, is actually an application of the resolvent operator to the quantum theory of scattering, since the Green functions considered by SOMMERFELD and MEIXNER are essentialy matrix elements of operators closely related to the resolvent attached to the hamiltonian. The results of SOMMERFELD and MEIXNER can be easily extended to much more general cases than those considered by these authors, once their Green functions are replaced by resolvent operators and the relations between these resolvents and the unitary operators of the motion (contact transformation generated by the hamiltonian) are taken into account. We shall indicate the main points involved in such a generalization in this paper, but the detailed theory will be given elsewhere, the main objective of this paper being the application of the resolvent operators to Feynman's theory of the positron (7).

In sections 2, 3 and 4 we derive the main properties of the resolvent, from the point of view of the application to quantum theory, for the case of time independent hamiltonians. The treatment of the case of time dependent hamiltonians requires an extension of the definition of the resolvent operators, which is given in section 7. The generalized resolvent operators depend on a complex parameter λ and also on a time variable t'. The generalized resolvents are Laplace transforms of the unitary operator of the motion. In the case of a time independent hamiltonian, our generalized resolvents are also time independent and they coincide with the ordinary resolvents. The consideration of the generalized resolvents shows that the resolvents are more closely related to the contact transformations generated by the motion during finite time intervals than to the infinitesimal contact transformations described by the hamiltonian. It would be possible to introduce resolvents even in cases in which there is no hamiltonian.

In the case of a time independent hamiltonian, it is possible to consider a single valued resolvent $(H-\lambda)^{-1}$, by cutting the complex λ-plane along the continuous spectrum of the hamiltonian H. The resolvent $R(\lambda)$ is actually a multiple valued function of λ, which has some similarities with the loga-

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(4) T. KATO: *Prog. Theor. Phys.*, **4**, 514 (1949); **5**, 95 and 207 (1950).

(5) A. SOMMERFELD: *Ann. d. Phys.*, **11**, 257 (1931); *Jahresber. d. Deutsch. Math. Ver.*, **21**, 309 (1913).

(6) J. MEIXNER: *Math. Zeits.*, **36**, 677 (1933); *Zeits. f. Phys.*, **90**, 312 (1934); *Ann. d. Phys.*, **29**, 97 (1937).

(7) R. P. FEYNMAN: *Phys. Rev.*, **76**, 749 (1949).

rithmic function, and which is rendered single valued by the cut in the λ-plane. It is nevertheless preferable to consider two branches $R_{\pm}(\lambda)$ of the multiple valued resolvent, which are Laplace transforms of the unitary operator of the motion. These two branches play an important part in the theory of the scattering as well as in the theory of the positron. In the case of a time dependent hamiltonian, those two branches of the resolvent present themselves directly, whereas the definition of the single-valued resolvent is somewhat artificial. The two branches of the resolvent correspond to different behaviours of the scattered waves, at infinity, in the theory of scattering: waves diverging or converging at infinity (positive or negative sources). The use of the resolvent in Feynman's treatment of the hole theory shows immediately that the waves of negative energy are taken as convergent, whilst the positive energy ones are taken as divergent, in the usual way. The appearance of negative energy particles moving backwards in time corresponds actually to the assimilation of a convergent wave to a divergent one. This interpretation of Feynman's procedure shows that it may be possible to give a plausible interpretation to negative energy particles, even without using the Pauli exclusion principle, by choosing adequate solutions of the wave equations.

The use of the resolvents not only indicates more clearly the essential points in the Feynman theory of the positron but it leads also to a more rigorous mathematical formulation. In Feynman's treatment there is an integral equation for the singular function $K_{+}^{(A)}(x, x')$, which is used to define it in the general case, and also to get its expansion in terms of the external field and the corresponding singular function for free electrons (the general definition of $K_{+}^{(A)}$ can be easily obtained from the second quantization form of the hole theory and is implicitly contained in the analysis given in the appendix of Feynman's paper). It is shown in this paper that Feyman's integral equation is not satisfactory, although it is useful for a perturbation treatment. By means of the resolvents it is possible to get easily expansions for the Feynman singular function. The influence of the closed stationary states on the convergence of the expansion of the singular function is clearly shown by our method of expansion, a point which can not be easily investigated with Feynman's methods.

Instead of Feynman's $K_{+}^{(A)}$ we use an operator $V_F(t, t')$ depending on two time variables, whose matrix elements are simply expressed in terms of $K_{+}^{(A)}(x, x')$. The relativistic invariance of the formalism is not apparent when it is expressed in terms of $V_F(t, t')$, but there are several advantages in using $V_F(t, t')$, which is an effective operator. We show that $V_F(t, t')$ can be expressed directly in terms of the unitary operator of the motion $V(t, t')$, as a kind of complex Stieltjes transform. The Stieltjes transform plays also an important part in the case of time independent hamiltonians, because the resolvent (rendered single valued by the cut in the λ-plane) is a Stieltjes transform of the projection operator $P(E)$ of the continuous spectrum of H.

We shall not discuss the hole theory in second quantization, in this paper. This will be done in a following paper, in which we shall determine the resolvent of the interacting electron-positron and electromagnetic quantized fields. In the case of fields, it is possible to introduce a relativistically invariant resolvent depending on a complex parameter λ and a space-like surface in space-time σ', as it is shown in section 9.

By means of the resolvents $R_+(\lambda)$ and $R_-(\lambda)$ it is possible to form an operator $R_s(\lambda) = (1/2)[R_+(\lambda) + R_-(\lambda)]$, which is not a branch of the multiple valued resolvent but which can be used to get special solutions of the Schrödinger equation in the theory of collisions. The kind of collisions described by $R_s(\lambda)$ is altogether different from those considered in the theory of scattering: in these collisions there is a reflecting boundary at infinity, and the solution of the stationary Schrödinger equation is formed by the superposition of outgoing and reflected ingoing waves. This kind of collision can be used in the theory of the nucleus as a rough approximation, the nuclear boundary being assimilated to the reflecting wall at infinity, in order to explain why the mean free path of the nucleons is so big, although the cross-section for non-enclosed nucleons is large. Whereas in classical theory there is a single type of collisions, once the interaction is given, the same is not true in quantum mechanics, because of the influence of the boundary conditions on the nature of the wave-functions. Operators similar to $R_s(\lambda)$ were already introduced by SCHÖNBERG [8] in the quantum theory of perturbations, for real values of λ. The theory of the enclosed collisions will be given elsewhere.

The resolvent operator $R(\lambda)$.

2. – We shall consider a dynamical system whose hamiltonian H has a mixed spectrum, the discrete and continuous parts of the spectrum having no common points. The resolvent operator $R(\lambda)$ will be taken as the inverse of $H-\lambda$, for complex or real values of λ not belonging to the spectrum of H

$$R(\lambda) = (H-\lambda)^{-1}. \tag{1}$$

We shall of course take H hermitian, so that the spectrum will consist only of real values. The discrete eigenvalues will be denoted by E_k and the corresponding eigenvectors of H in functional space by Φ_k. In all cases of physical interest there is an infinite degeneracy of the levels of the continuous

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[8] M. SCHÖNBERG: *Nuovo Cimento*, **8**, 403 (1951).

spectrum and we shall introduce variables θ which characterize the eigenvectors $\Phi_{E,\theta}$ of the continuous spectrum

$$H\Phi_k = E_k\Phi_k\,, \qquad H\Phi_{E,\theta} = E\Phi_{E,\theta}\,. \tag{2}$$

The wave functions, in a representation defined by a complete set of commutable operators ξ_{op}, are functions $\overline{\Phi}_k(\xi)$ and $\overline{\Phi}_{E,\theta}(\xi)$ of the eigenvalues ξ of the ξ_{op}. We shall assume that the eigenfunctions (or eigenvectors) are normalized in the usual way

$$\int \overline{\Phi}^*_{E,\theta}(\xi)\overline{\Phi}_{E',\theta'}(\xi)\,\mathrm{d}\xi = \delta(E-E')\delta(\theta-\theta')\,, \tag{3a}$$

$$\int \overline{\Phi}^*_{E,\theta}(\xi)\overline{\Phi}_k(\xi)\,\mathrm{d}\xi = 0\,, \tag{3b}$$

$$\int \overline{\Phi}^*_k(\xi)\overline{\Phi}_{k'}(\xi)\,\mathrm{d}\xi = \delta_{kk'}\,. \tag{3c}$$

Some of the ξ_{op} may have discrete spectra (spins, isotopic spins, etc.) and the corresponding integrations must be taken as ordinary sums. We treated the θ as continuous variables, in equation (3*a*), but some or all of them may be discrete and the corresponding deltas Kronecker ones, instead of Dirac functions. We shall assume, for the sake of simplicity, that the discrete levels are not degenerate, unless the opposite is specified. Most of our results are independent of that assumption.

We shall now introduce the projection operators P corresponding to the eigenvectors of H

$$\begin{cases} P_k\chi &= (\Phi_k, \chi)\Phi_k\,, \\ P_{E,\theta}\chi &= (\Phi_{E,\theta}, \chi)\Phi_{E,\theta}\,, \end{cases} \tag{4}$$

χ being an arbitrary vector of the functional space in consideration and (Φ, χ) denoting the inner product of the vectors Φ and χ. It is convenient to introduce the projection operators $P(E)$ attached to the linear sub-spaces of the eigenvectors corresponding to the eigenvalues E (see reference 8)

$$P(E) = \int P_{E,\theta}\,\mathrm{d}\theta\,. \tag{5}$$

When the eigenvalue E_k is degenerate, we shall take

$$P_k = \sum_a P_{k,a}\,, \tag{5a}$$

the $P_{k,a}$ being the projection operators corresponding to the various eigenvectors $\Phi_{k,a}$ with the same energy E_k.

We shall use often the following properties of the P

(6)
$$\sum_k P_k + \int_{\text{sp.}} P(E)\,\mathrm{d}E = 1\,,$$

(7)
$$\begin{cases} P_k P_{k'} = P_k \delta_{kk'}\,, & P(E)P(E') = \delta(E-E')P(E)\,, \\ P_k P_{E,\theta} = 0\,, & P_{E,\theta}P_{E',\theta'} = P_{E,\theta}\delta(E-E')\delta(\theta-\theta')\,. \end{cases}$$

The θ may be considered as eigenvalues of operators θ_{op} commutable with H. The spectral decomposition of H is

(8)
$$H = \sum_k E_k P_k + \int_{\text{sp.}} E P(E)\,\mathrm{d}E\,.$$

The spectral decomposition of an arbitrary function of H is

(9)
$$F(H) = \sum_k F(E_k) P_k + \int_{\text{sp.}} (F) E P(E)\,\mathrm{d}E\,.$$

In particular, we have for the resolvent $R(\lambda)$

(10)
$$R(\lambda) = \sum_k (E_k - \lambda)^{-1} P_k + \int_{\text{sp.}} (E-\lambda)^{-1} P(E)\,\mathrm{d}E\,.$$

The matrix elements of $R(\lambda)$ are functions of λ with simple poles at the discrete eigenvalues of H. We may say that $R(\lambda)$ has simple poles at the points E_k of the complex λ-plane.

It results from equations (9) and (10) that

(11)
$$F(H) = \frac{1}{2\pi i}\left(\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} - \int_{-\infty-i\eta}^{+\infty-i\eta}\right) F(\lambda) R(\lambda)\,\mathrm{d}\lambda\,,$$
$$(\varepsilon, \eta > 0)\,,$$

provided the function $F(\lambda)$ is regular in the belt of the λ-plane between the straight-lines $i\varepsilon \pm \infty$ and $-i\eta \pm \infty$. In particular we have

(12)
$$\exp[-iHt] = \frac{1}{2\pi i}\left(\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon} - \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon}\right) \exp[-i\lambda t] R(\lambda)\,\mathrm{d}\lambda\,.$$

We shall assume that the units are chosen in such a way that the Planck constant has the value 2π, and we shall introduce the following notation

(13)
$$V(t) = \exp[-iHt]\,, \qquad (\hbar = 1)$$

$V(t)$ is the unitary operator of the movement

$$\psi(t) = V(t - t_0)\psi(t_0) \; ; \tag{14}$$

$\psi(t)$ being the state vector of the dynamical system in consideration

$$i \frac{\mathrm{d}}{\mathrm{d}t} \psi(t) = \boldsymbol{H}\psi(t) \, . \tag{15}$$

$V(t)$ is determined by the equations

$$i \frac{\mathrm{d}}{\mathrm{d}t} V(t) = \boldsymbol{H}V(t) \, , \qquad V(0) = 1 \, . \tag{16}$$

We shall now simplify (12) by taking into account the identities

$$\int\limits_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \exp\left[-i\lambda t\right] R(\lambda)\, \mathrm{d}\lambda = 0 \, , \qquad (t < 0) \, , \tag{17a}$$

$$\int\limits_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \exp\left[-i\lambda t\right] R(\lambda)\, \mathrm{d}\lambda = 0 \, , \qquad (t > 0) \, . \tag{17b}$$

To establish (17a), we close the integration path with a half-circle of infinite radius with center at the origin, lying in the upper part of the complex λ-plane, and apply the Cauchy residue theorem. We get (17b) in a similar way, by closing the integration path with a half-circle in the lower part of the λ-plane. It results from (12) and (17) that

$$\begin{cases} V(t) = \dfrac{1}{2\pi i} \displaystyle\int\limits_{-\infty + i\varepsilon}^{+\infty + i\varepsilon} \exp\left[-i\lambda t\right] R(\lambda)\, \mathrm{d}\lambda \, , & (t > 0) \, , \\ V(t) = -\dfrac{1}{2\pi i} \displaystyle\int\limits_{-\infty - i\varepsilon}^{+\infty - i\varepsilon} \exp\left[-i\lambda t\right] R(\lambda)\, \mathrm{d}\lambda \, , & (t < 0) \, . \end{cases} \tag{18}$$

The operators

$$V_{\pm}(t) = \frac{1}{2\pi i} \int\limits_{\pm i\varepsilon - \infty}^{\pm i\varepsilon + \infty} \exp\left[-i\lambda t\right] R(\lambda)\, \mathrm{d}\lambda \, , \tag{19}$$

are solutions of the inhomogeneous equation

$$i \frac{\mathrm{d}}{\mathrm{d}t} V_{\pm}(t) - \boldsymbol{H} V_{\pm}(t) = i\delta(t) \, , \tag{20}$$

since

$$V_{\pm}(t) = \pm\,\eta(\pm\,t)V(t)\,, \tag{21}$$

with

$$\eta(t) = \begin{cases} 1 & t > 0\,, \\ 0 & t < 0\,. \end{cases} \tag{22}$$

It is easily seen that

$$R(\lambda) = \begin{cases} i\int\limits_0^{\infty} \exp\,[i\lambda t]V(t)\,\mathrm{d}t & (Im\lambda > 0)\,, \\ -i\int\limits_{-\infty}^{0} \exp\,[i\lambda t]V(t)\,\mathrm{d}t & (Im\lambda < 0)\,, \end{cases} \tag{23}$$

hence

$$R(\lambda) = i\int\limits_{-\infty}^{+\infty} \exp\,[i\lambda t]V_{+}(t)\,\mathrm{d}t\,, \qquad (Im\lambda > 0) \tag{24a}$$

$$R(\lambda) = i\int\limits_{-\infty}^{+\infty} \exp\,[i\lambda t]V_{-}(t)\,\mathrm{d}t\,, \qquad (Im\lambda < 0) \tag{24b}$$

and

$$R(E + i0) + R(E - i0) = 2i\int\limits_{-\infty}^{+\infty} \exp\,[iEt]V_s(t)\,\mathrm{d}t\,, \qquad (ImE = 0) \tag{24c}$$

with

$$V_s(t) = \frac{1}{2}\,[V_{+}(t) + V_{-}(t)]\,. \tag{25}$$

We have

$$V_{+}(0+) = -V_{-}(0-) = 1\,, \qquad V_{+}(0-) = V_{-}(0+) = 0\,, \tag{26}$$

hence

$$V_{+}(0+) + V_{+}(0-) = 1\,, \qquad V_{-}(0+) + V_{-}(0-) = -1\,, \tag{27}$$

and

$$V_{+}(0) = \frac{1}{2}\,, \qquad V_{-}(0) = -\frac{1}{2}\,, \qquad V_s(0) = 0\,. \tag{28}$$

Equation (21) will also be valid for $t = 0$, if we take

$$\eta(0) = \frac{1}{2} . \tag{29}$$

The relation between V_s and V is

$$V_s(t) = \frac{1}{2}\,\varepsilon(t) V(t) , \tag{30}$$

with

$$\varepsilon(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 . \end{cases} \tag{31}$$

We shall now discuss the nature of the singularity of $R(\lambda)$ at the points of the real axis belonging to the continuous spectrum of H. Let us consider the operator p.v.$(H - E)^{-1}$, E being real

(32a) p.v. $(H - E)^{-1} = R(E)$ (when E does not belong to the spectrum of H)

$$\text{p.v.}\ (H - E_k)^{-1} = \sum_{l \neq k} (E_l - E_k)^{-1} P_l + \int_{\text{sp.}} (E - E_k)^{-1} P(E)\, \mathrm{d}E , \tag{32b}$$

$$\text{p.v.}\ (H - E)^{-1} = \sum_{k} (E_k - E)^{-1} P_k + \text{p.v.} \int_{\text{sp.}} (E' - E) P(E')\, \mathrm{d}E' , \tag{32c}$$

(E belonging to the continuous spectrum of H). In the right hand side of (32c) p.v. denotes the Cauchy principal value of the integral. In order to compute $R(E + i0)$, E being a point of the continuous spectrum, we shall indent the integration path in (10) at the point E with an infinitesimal half-circle centered at E and lying in the lower part of the complex E-plane. Thus we get

$$R(E + i0) = \text{p.v.}\ (H - E)^{-1} + i\pi P(E) . \tag{33a}$$

By indenting with an infinitesimal half-circle lying in the upper part of the E-plane, we get

$$R(E - i0) = \text{p.v.}\ (H - E)^{-1} - i\pi P(E) , \tag{33b}$$

hence

$$R(E + i0) - R(E - i0) = 2\pi i P(E) . \tag{34}$$

The operator $R(\lambda)$ has a discontinuity $2\pi i P(E)$ at any point E of the continuous spectrum of H. The operator $R(\lambda)$ may be considered as a branch of a multiple valued operator function of a complex variable λ, which is obtained by cutting the complex λ-plane along the continuous spectrum of H, provided the continuous spectrum does not stretch from $-\infty$ to $+\infty$.

It results from equation (34) that

$$(35)\qquad \frac{1}{2\pi i}\left(\int_{E'+i\varepsilon}^{E''+i\varepsilon} - \int_{E'-i\varepsilon}^{E''-i\varepsilon}\right) F(\lambda)R(\lambda)\,\mathrm{d}\lambda = \sum_{E'<E_k<E''} F(E_k)P_k + \int_{E'\leqslant E\leqslant E''} F(E)P(R)\,\mathrm{d}E$$

$$(\varepsilon \text{ infinitesimal}),$$

E' and E'' being two real numbers not belonging to the discrete spectrum of H, and $F(\lambda)$ a function which is regular in a region of the λ-plane including the segment $E'E''$ of the real axis. Equation (35) leads to a generalization of (11)

$$(36)\qquad \frac{1}{2\pi i}\left(\int_{E'+i\varepsilon}^{E''+i\varepsilon} - \int_{E'-i\eta}^{E''-i\eta}\right) F(\lambda)R(\lambda)\,\mathrm{d}\lambda = \sum_{E'<E_k<E''} F(E_k)P_k + \int_{E'\leqslant E\leqslant E''} F(E)P(E)\,\mathrm{d}E\,,$$

$$(\varepsilon, \eta > 0)\,;$$

ε and η must be infinitesimal, in order that the segments $(E'+i\varepsilon, E''+i\varepsilon)$ and $(E'-i\eta, E''-i\eta)$ be assimilable to a closed path.

In the case of a pure continuous spectrum, $R(\lambda)$ may be considered as a Stieltjes transform of $P(E)$. Equation (34) corresponds to the Stieltjes formula for the inversion of the Stieltjes transformation. Equations (23) show that $R(\lambda)$ is a Laplace transform of $V(t)$ and equations (18) give actually the inversion of that Laplace transformation.

Equation (11) gives a generalization for operators of the Cauchy formula

$$(37)\qquad F(\lambda) = \frac{1}{2\pi i}\int_C \frac{F(z)}{z-\lambda}\,\mathrm{d}z\,,$$

C being a closed path described in the positive sense and containing the point λ.

The operators $R_+(\lambda)$ and $R_-(\lambda)$.

3. – We shall now consider the resolvent $R(\lambda)$ as a multiple valued operator and define two of its branches $R_+(\lambda)$ and $R_-(\lambda)$. Let us introduce a cut in the λ-plane along the continuous spectrum of H and define a single valued operator $R(\lambda)$, as we did in section 2. We shall take

$$(38)\qquad \begin{cases} R_+(\lambda) = R(\lambda)\,, & Im\lambda < 0\,, \\ R_-(\lambda) = R(\lambda)\,, & Im\lambda < 0\,, \end{cases}$$

and define $R_+(\lambda)$ in the lower part of the λ-plane, and $R_-(\lambda)$ in its upper part, by analytical continuation across the continuous spectrum of H. Assuming that the $P(E)$ can be prolongated outside the real axis and that the operators $P(\lambda)$ thus obtained are single valued, we will get well defined operators $R_+(\lambda)$ and $R_-(\lambda)$

$$(39)\qquad \begin{cases} R_+(\lambda) = R(\lambda) + 2\pi i P(\lambda)\,, & (Im\lambda < 0)\,, \\ R_-(\lambda) = R(\lambda) - 2\pi i P(\lambda)\,, & (Im\lambda > 0)\,, \end{cases}$$

We have everywhere in the λ-plane

$$(40)\qquad R_+(\lambda) - R_-(\lambda) = 2\pi i P(\lambda)\,.$$

It follows from (38) and (39) that

$$(41)\qquad (H - \lambda) R_\pm(\lambda) = 1\,,$$

hence

$$(42)\qquad \int \langle \xi' \mid H - \lambda \mid \xi \rangle \, \mathrm{d}\xi \, \langle \xi \mid R_\pm(\lambda) \mid \xi'' \rangle = \delta(\xi' - \xi'')\,.$$

Equation (42) shows that the matrix elements of $R_\pm(\lambda)$ are Green functions of the stationary Schrödinger equation. In the special case of a particle moving in a field of force described by a potential $U(\boldsymbol{x})$, the Green functions $\langle \boldsymbol{x} \mid R_+(\lambda) \mid \boldsymbol{x}_0 \rangle$ and $\langle \boldsymbol{x} \mid R_-(\lambda) \mid \boldsymbol{x}_0 \rangle$ correspond respectively to a positive or a negative source (outgoing and ingoing waves) situated at the point $\boldsymbol{x}_0$. Indeed, the asymptotic value of $\langle \boldsymbol{x} \mid R_+(\lambda) \mid \boldsymbol{x}_0 \rangle$ can be computed by using the first equation (23)

$$(43)\qquad \langle \boldsymbol{x} \mid R_+(\lambda) \mid \boldsymbol{x}_0 \rangle_{\text{asy.}} \sim i \int_0^\infty \exp[i\lambda t] \left\langle \boldsymbol{x} \left| \exp\left[-\frac{it\boldsymbol{p}^2}{2m}\right] \right| \boldsymbol{x}_0 \right\rangle \mathrm{d}t\,,$$

m being the mass of the particle and $\boldsymbol{p}^2$ the operator representing the square of the momentum of the particle. Using well known formulae we get

$$(44)\qquad \left\langle \boldsymbol{x} \left| \exp\left[-\frac{it\boldsymbol{p}^2}{2m}\right] \right| \boldsymbol{x}_0 \right\rangle = \exp\left[\frac{it}{2m}\Delta\right] \delta(\boldsymbol{x} - \boldsymbol{x}_0) =$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \exp\left[-\frac{it}{2m}\varrho^2\right] \exp\left[i(\boldsymbol{\rho} \cdot \{\boldsymbol{x} - \boldsymbol{x}_0\})\right] \mathrm{d}\boldsymbol{\rho}\,,$$

hence

$$(45)\qquad \langle \boldsymbol{x} \mid R_+(\lambda) \mid \boldsymbol{x}_0\rangle_{\text{asy.}} \sim \frac{2m}{(2\pi)^3}\int_{-\infty}^{+\infty} \frac{\exp\left[i(\boldsymbol{\rho}\cdot\{\boldsymbol{x}-\boldsymbol{x}_0\})\right]}{\varrho^2-k^2}\,\mathrm{d}\boldsymbol{\rho} = \frac{m}{2\pi}\,\frac{\exp\left(ik\mid \boldsymbol{x}-\boldsymbol{x}_0\mid\right)}{\mid \boldsymbol{x}-\boldsymbol{x}_0\mid}\qquad\left(\lambda=\frac{k^2}{2m}\right).$$

In a similar way we get

$$(46)\qquad \langle \boldsymbol{x} \mid R_-(\lambda) \mid \boldsymbol{x}_0\rangle_{\text{asy.}} \sim \frac{m}{2\pi}\,\frac{\exp\left[-ik\mid \boldsymbol{x}-\boldsymbol{x}_0\mid\right]}{\mid \boldsymbol{x}-\boldsymbol{x}_0\mid}\,.$$

Taking the source at infinity, we get the wave functions which describe the scattering (incoming plane wave plus diffused waves) as the matrix elements $\langle \boldsymbol{x} \mid R_+(E) \mid -\boldsymbol{k}\,\infty\rangle$ (Sommerfeld's treatment of the scattering (5)), $\boldsymbol{k}$ being the momentum of the incident particles and $E=k^2/2m$ their energy.

The operator

$$(47)\qquad R_s(\lambda)=\frac{1}{2}\left[R_+(\lambda)+R_-(\lambda)\right]$$

is also a solution of equation (41), and its matrix elements are also Green functions of the stationary Schrödinger equation. In the case of a particle moving in a field of force, the matrix elements of $R_s(\lambda)$ are the Green functions corresponding to point sources, with a reflecting wall at infinity

$$(48)\qquad \langle \boldsymbol{x} \mid R_s(\lambda) \mid \boldsymbol{x}_0\rangle_{\text{asy.}} \sim \frac{m}{2\pi}\,\frac{\cos\left(k\mid \boldsymbol{x}-\boldsymbol{x}_0\mid\right)}{\mid \boldsymbol{x}-\boldsymbol{x}_0\mid}\,.$$

In many cases $P(\lambda)$ is an integral function of λ, so that both $R_+(\lambda)$ and $R_-(\lambda)$ have simple poles at the points of the discrete spectrum of H. It results from (40) that

$$(49)\qquad \langle \xi' \mid R_+(E) \mid \xi''\rangle - \langle \xi' \mid R_-(E) \mid \xi''\rangle = 2\pi i\int \mathrm{d}\theta\, \overline{\Phi}_{E,\theta}(\xi')\overline{\Phi}^*_{E,\theta}(\xi'')\,,$$

since

$$(50)\qquad \langle \xi' \mid P_{E,\theta} \mid \xi''\rangle = \overline{\Phi}_{E,\theta}(\xi')\overline{\Phi}^*_{E,\theta}(\xi'')\,.$$

When the eigenvalues are not degenerate (49) becomes

$$(51)\qquad \langle \xi' \mid R_+(E) \mid \xi''\rangle - \langle \xi' \mid R_-(E) \mid \xi''\rangle = 2\pi i\overline{\Phi}_E(\xi')\overline{\Phi}^*_E(\xi'')\,.$$

In this case, we can express the normalized eigenfunction $\overline{\Phi}_E(\xi')$ in terms of the matrix elements of the operators $R_\pm(E)$, by keeping the ξ'' constant

$$(52)\qquad \overline{\Phi}_E(\xi') = \left[\langle \xi' \mid R_+(E) \mid \xi''\rangle - \langle \xi' \mid R_-(E) \mid \xi''\rangle\right](2\pi i\overline{\Phi}^*_E(\xi''))^{-1}\,.$$

The difference of the matrix elements of R_+ and R_-, in (52), is a non normalized eigenfunction. It results from (10) that

$$P_k = -\operatorname*{Residue}_{\lambda=E_k} R(\lambda)\,, \tag{53}$$

hence

$$\sum_a \overline{\Phi}_{k,a}(\xi')\overline{\Phi}^*_{k,a}(\xi'') = -\operatorname*{Residue}_{\lambda=E_k} \langle \xi' \mid R(\lambda) \mid \xi'' \rangle\,. \tag{54}$$

In the particular case of a non degenerate discrete level, the residue of the matrix element $\langle \xi' \mid R(\lambda) \mid \xi'' \rangle$ is a non normalized eigenfunction of energy E_k, considered as a function of ξ', since

$$\overline{\Phi}_k(\xi') = -\operatorname*{Residue}_{\lambda=E_k} \langle \xi' \mid R(\lambda) \mid \xi'' \rangle\, (\overline{\Phi}^*_k(\xi''))^{-1}\,. \tag{55}$$

4. – Equation (40) is obviously a generalization of (34), since

$$R(E+i0) = R_+(E)\,, \qquad R(E-i0) = R_-(E)\,. \tag{56}$$

In order to get a generalization of equations (33), we shall use the following relation

$$R_s(E) = \text{p.v.}\ (H-E)^{-1}\,, \tag{57}$$

which is a consequence of the definition of R_s

$$\begin{aligned} R_s(E) &= \frac{1}{2}\lim_{\varepsilon\to 0}\,[R(E+i\varepsilon)+R(E-i\varepsilon)] = \sum_k (E_k-E)^{-1}P_k + \\ &\quad + \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\text{sp.}}\left(\frac{1}{E'-E-i\varepsilon}+\frac{1}{E'-E+i\varepsilon}\right)P(E')\,\mathrm{d}E' = \\ &= \text{p.v.}\int_{\text{sp.}}\frac{P(E')}{E'-E}\,\mathrm{d}E' + \sum_k (E_k-E)^{-1}P_k = \text{p.v.}\ (H-E)^{-1}\,. \end{aligned} \tag{58}$$

The generalization of equations (33) is given by the following ones

$$R_+(\lambda) = R_s(\lambda) + i\pi P(\lambda)\,, \qquad R_-(\lambda) = R_s(\lambda) - i\pi P(\lambda)\,, \tag{59}$$

which are equivalent to (40)-(47).

Equations (33) were derived by deformation of the integration path in equation (10). It is also possible to get them in a different way

$$R(E+i0)-R(E-i0)=\lim_{\varepsilon\to 0}[R(E+i\varepsilon)-R(E-i\varepsilon)]= \tag{60}$$

$$=\lim_{\varepsilon\to 0}\sum_k \frac{2i\varepsilon P_k}{(E_k-E)^2+\varepsilon^2}+\lim_{\varepsilon\to 0}\int_{\text{sp.}}\frac{2i\varepsilon}{(E'-E)^2+\varepsilon^2}P(E')\,\mathrm{d}E'=$$

$$=2\pi i\sum_k P_k\delta(E-E_k)+2\pi iP(E)=2\pi iP(E)\,,$$

taking into account the well known relation

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{(x-a)^2+\varepsilon^2}=\pi\delta(x-a)\,. \tag{61}$$

It results from (9) that

$$V(t)=\sum_k \exp[-iE_kt]P_k+\int_{\text{sp.}}\exp[-iE't]P(E')\,\mathrm{d}E'\,, \tag{62}$$

$$V_s(t)=\frac{1}{2}\,\varepsilon(t)\left[\sum_k \exp[-iE_kt]Pk+\int_{\text{sp.}}\exp[-iE't]P(E')\,\mathrm{d}E'\right], \tag{63}$$

hence

$$\int_{-\infty}^{+\infty}\exp[iEt]V(t)\mathrm{d}t=2\pi P(E)\,, \tag{64}$$

$$\int_{-\infty}^{+\infty}i\exp[iEt]V_s(t)\,\mathrm{d}t=\text{p.v.}\,(H-E)^{-1}\,. \tag{65}$$

Equation (64) is equivalent to (34), because

$$\int_{-\infty}^{+\infty}i\exp[iEt]V(t)\,\mathrm{d}t=R(E+i0)-R(E-i0)\,, \tag{66}$$

as a consequence of (23). We get from (65) and (58) a formula equivalent to (24c)

$$R_s(E)=i\int_{-\infty}^{+\infty}\exp[iEt]V_s(t)\,\mathrm{d}t\,. \tag{67}$$

We get from (24a) and (24b) formulae corresponding to (67)

$$R_\pm(E)=i\int_{-\infty}^{+\infty}\exp[iEt]V_\pm(t)\,\mathrm{d}t\,. \tag{68}$$

The inverse formulae can be obtained easily from (19) by a deformation of the integration pathes

$$V_{\pm}(t) = \frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{+\infty} \exp\,[-\,iEt]\;R_{\pm}(E)\,\mathrm{d}E \pm \frac{1}{2}\sum_k \exp\,[-\,iE_k t]P_k\,. \tag{69}$$

Application of the operator $R(\lambda)$ to the Feynman theory of the positron.

5. – Let H be the hamiltonian of an electron in a time independent electromagnetic field (relativistic theory). We shall assume that there are discrete levels in the interval $(0 - mc^2)$ and a continuous spectrum covering the intervals $(-\infty - -mc^2)$ and $(mc^2 - \infty)$, m being the rest-mass of the electron. In Feynman's treatment of the hole theory, the operator $V_+(t)$ is replaced by another operator $V_F(t)$

$$V_F(t) = \eta(t)\left[\int_{mc^2}^{\infty} \exp\,[-\,iEt]P(E)\,\mathrm{d}E + \sum_k \exp\,[-\,iE_k t]P_k\right] - \\ -\,\eta(-t)\int_{-\infty}^{-mc^2} \exp\,[-\,iEt]P(E)\mathrm{d}E\,, \tag{70}$$

which satisfies the same differential equation as $V_+(t)$

$$i\,\frac{\mathrm{d}}{\mathrm{d}t}\,V_F(t) - HV_F(t) = i\delta(t)\,. \tag{71}$$

It is easily seen that $V_F(t)$ can be represented by a contour integral involving $R(\lambda)$

$$V_F(t) = \frac{1}{2\pi i}\int_C \exp\,[-\,i\lambda t]R(\lambda)\,\mathrm{d}\lambda\,. \tag{72}$$

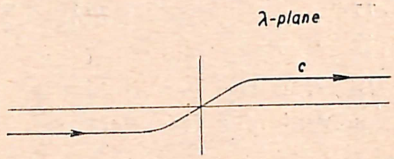

Fig. 1.

C being the path represented in fig. 1, when 0 is not a singular point of $R(\lambda)$. Formula (72) is a generalization of that given by Feynman for the case of free electrons. To prove the equivalence of (72) and (70), we close the integration path with an infinite half circle in the lower or the upper part of the λ-plane, for $t > 0$ and $t < 0$ respectively, and apply the Cauchy residue theorem.

It results from (72) that

$$V_F(t) = \frac{1}{2}\,\eta(t) \sum_k \exp[-iE_kt]P_k + \tag{73}$$

$$+ \frac{1}{2\pi i}\,\text{p.v.} \int_0^\infty \exp[-iEt]R_+(E)\,\mathrm{d}E + \frac{1}{2\pi i}\int_{-\infty}^0 \exp[-iEt]R_-(E)\,\mathrm{d}E\,.$$

Since the operators R_+ and R_- correspond respectively to divergent and convergent waves, we see that in Feynman's prescription the waves of negative energy are taken as convergent and those of positive energy as divergent. Thus we get an intuitive picture, instead of Feynman's idea of positrons as electrons moving backwards in time, which is mathematically expressed by the Fourier expansion (70).

Feynman's relativistic singular function $K_+^{(A)}(x, x')$ is easily expressed in terms of the matrix elements of $V_F(t-t')$

$$K_+^{(A)}(x, x') = \langle \boldsymbol{x} \,|\, V_F(t-t') \,|\, \boldsymbol{x}' \rangle \beta\,, \tag{74}$$

β being the well known operator of the Dirac theory of the electron. We get from (71) and (74) a differential equation for $K_+^{(A)}$ given by Feynman

$$\left[\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - ieA_\mu(x)\right) + im\right] K_+^{(A)}(x, x') = \delta_4(x - x')\,, \tag{75}$$

e being the charge of the electron (absolute value) and the A_μ the covariant components to the potentials. We are now assuming that $\hbar = 1$, $c = 1$ and

$$\gamma_0 = \beta\,, \quad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha}\,, \quad g_{11} = g_{22} = g_{33} = -g_{00} = -1\,. \tag{76}$$

Equation (72) allows us to get easily an expansion of $V_F(t)$ by means of the following expansion of $R(\lambda)$

$$R(\lambda) = \sum_{n=0}^{\infty} (-1)^n\,[R_0(\lambda)H']^n\,R_0(\lambda)\,, \tag{77}$$

H_0 being the hamiltonian of a free electron, $H' = H - H_0$ and $R_0(\lambda)$ the resolvent of a free electron

$$R_0(\lambda) = (H_0 - \lambda)^{-1} = \frac{(\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m + \lambda)}{p^2 + m^2 - \lambda^2}\,. \tag{78}$$

In order to derive (77) we shall use the following identity

$$R(\lambda) = (H_0 + H' - \lambda)^{-1} = [(H_0 - \lambda)\{1 + R_0(\lambda)H'\}]^{-1} = \tag{79}$$

$$= [1 + R_0(\lambda)H']^{-1}R_0(\lambda)\,,$$

and expand the operator $[1 + R_0(\lambda)H']^{-1}$ in a Liouville-Neumann series

(80) $$[1 + R_0(\lambda)H']^{-1} = \sum_{n=0}^{\infty} (-1)^n [R_0(\lambda)H']^n .$$

Introducing (77) into (72), we get an expansion of $V_F(t)$ in powers of H'

(81) $$V_F(t) = \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} (-1)^n \exp[-i\lambda t][R_0(\lambda)H']^n R_0(\lambda)\, d\lambda .$$

The expansion of the matrix elements of V_F in the momentum representation is

(82) $$\langle \boldsymbol{p} \,|\, V_F(t) \,|\, \boldsymbol{p}' \rangle = \frac{1}{2\pi i} \int_C d\lambda \exp[-i\lambda t] \sum_{n=0}^{\infty} (-1)^n \times$$
$$\times \int_{-\infty}^{+\infty} \frac{\langle \boldsymbol{p} \,|\, H' \,|\, \boldsymbol{p}_1 \rangle \langle \boldsymbol{p}_1 \,|\, H' \,|\, \boldsymbol{p}_2 \rangle \ldots \langle \boldsymbol{p}_{n-1} \,|\, H' \,|\, \boldsymbol{p}' \rangle}{(p^2 + m^2 - \lambda^2)(p_1^2 + m^2 - \lambda^2) \ldots (p'^2 + m^2 - \lambda^2)} \times$$
$$\times [(\boldsymbol{\alpha} \cdot \boldsymbol{p}) + \beta m + \lambda][(\boldsymbol{\alpha} \cdot \boldsymbol{p}_1) + \beta m + \lambda] \ldots \times$$
$$\times [(\boldsymbol{\alpha} \cdot \boldsymbol{p}_{n-1}) + \beta m + \lambda][(\boldsymbol{\alpha} \cdot \boldsymbol{p}') + \beta m + \lambda] \prod_{j=1}^{n-1} d\boldsymbol{p}_j .$$

The integration with respect to λ can be performed by the residue method. Thus we get

(83) $$\langle \boldsymbol{p} \,|\, V_F(t) \,|\, \boldsymbol{p}' \rangle = \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{+\infty} \prod_{j=1}^{n-1} d\boldsymbol{p}_j \langle \boldsymbol{p} \,|\, H' \,|\, \boldsymbol{p}_1 \rangle \ldots \langle \boldsymbol{p}_{n-1} \,|\, H' \,|\, \boldsymbol{p}' \rangle \times$$
$$\times \left[\sum_q \exp[-iE_q |t|] \frac{H_0(q) + \varepsilon(t) E_q}{2E_q} \prod_{q' \neq q} \frac{H_0(q') + \varepsilon(t) E_q}{q'^2 - q^2} \right.,$$
$$(q, q' = p, p_1, \ldots, p')$$

(84) $$E_q = \sqrt{q^2 + m^2}, \qquad H_0(q) = (\boldsymbol{\alpha} \cdot \boldsymbol{q}) + \beta m .$$

Thus the use of the resolvent allows us to get easily an expansion of V_F. The validity of that expansion depends on the convergence of the Liouville-Neumann series (80), which is justified when the operator $[R_0(\lambda)H']$ is small compared to 1. It is easily seen that, when there are discrete levels in the spectrum of H, the use of the Liouville-Neumann series may be insatisfactory, because

(85) $$[1 + R_0(\lambda)H']^{-1} = [R_0(\lambda)(H - \lambda)]^{-1} = R(\lambda)(H_0 - \lambda) ,$$

and the operator in the right hand side is very large compared to 1 in the neighbourhood of the discrete eigenvalues of H.

FEYNMAN (7) obtained an expansion of $K_+^{(A)}(x', x'')$ from the integral equation

$$(86)\qquad K_+^{(A)}(x', x'') = K_+^{(0)}(x', x'') + ie\int_{-\infty}^{+\infty} K_+^{(0)}(x', x''')\gamma^\mu A_\mu(x''')K_+^{(A)}(x''', x'')\,\mathrm{d}x''',$$

$K_+^{(0)}(x', x'')$ being the $K_+^{(A)}$ for free electrons. This equation can be written even in the case of a time dependent electromagnetic field, and it was used by FEYNMAN to define $K_+^{(A)}(x', x'')$ in this general case. In the case of a time independent field, (86) is equivalent to the following equation,

$$(87)\qquad V_F(t) = V_F^{(0)}(t) - i\int_{-\infty}^{+\infty} V_F^{(0)}(t-\tau)H'V_F(\tau)\,\mathrm{d}\tau\,,$$

$V_F^{(0)}$ being the operator V_F for free electrons

$$(88)\qquad V_F^{(0)}(t) = \frac{1}{2\pi i}\int_C \exp[-i\lambda t]R_0(\lambda)\,\mathrm{d}\lambda\,.$$

Instead of (87), we may consider the following integral equation

$$(87a)\qquad V_F(t) = V_F^{(0)}(t) - i\int_{-\infty}^{+\infty} V_F(t-\tau)H'V_F^{(0)}(\tau)\,\mathrm{d}\tau\,,$$

which leads also to Feynman's expansion of the operator $V_F(t)$.

A formal solution of (87) can be obtained by the application of the method of successive approximations (Liouville-Neumann expansion), taking $V_F^{(0)}$ as the zero order approximation

$$(89)\qquad V_F(t) = V_F^{(0)}(t) +$$
$$+\sum_{n=1}^{\infty}(-i)^n\int_{-\infty}^{+\infty} V_F^{(0)}(t-\tau_1)H'V_F^{(0)}(\tau_1-\tau_2)H' \dots V_F^{(0)}(\tau_{n-1}-\tau_n)H'V_F^{(0)}(\tau_n)\prod_{j=1}^{n}\mathrm{d}\tau_j\,.$$

This formula gives the Feynman expansion of $V_F(t)$, which is however not strictly equivalent to (81), as we shall see. Let us take new integration variables in (89)

$$(90)\qquad t_j = \tau_j - \tau_{j+1}\,,\qquad (j = 1, 2, \dots, n-1)\,,\qquad t_n = \tau_n\,.$$

Each term in the expansion of $V_F(t)$ becomes now a convolution integral

$$V_F(t) = V_F^{(0)}(t) + \tag{91}$$
$$+ \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{+\infty} V_F^{(0)}(t - \sum_{j=1}^{n} t_j) H' V_F^{(0)}(t_1) \dots H' V_F^{(0)}(t_n) \prod_{l=1}^{n} \mathrm{d}t_l \,.$$

Let us introduce the Fourier integral expansion of $V_F^{(0)}(t)$

$$V_F^{(0)}(t) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \exp[-i\lambda t] \mathcal{R}_0(\lambda) \, \mathrm{d}\lambda \,, \tag{92}$$

$$\mathcal{R}_0(\lambda) = \begin{cases} R_0(\lambda + i0) = R_{+,0}(\lambda)\,, & \lambda > 0 \\ R_0(\lambda - i0) = R_{-,0}(\lambda)\,, & \lambda < 0\,. \end{cases} \tag{92a}$$

We get from (91) and the well known properties of the Fourier transforms of convolution integrals the following expansion

$$V_F(t) = V_F^{(0)}(t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2\pi i} \int_{-\infty}^{+\infty} \exp[-i\lambda t] \mathcal{R}_0(\lambda) H']^n{}_0(\lambda) \, \mathrm{d}\lambda \,. \tag{93}$$

We can get immediately (93) from (81) by deforming the integration path C into the real axis, when such a deformation is permitted. We have seen that the expansion (77) of $R(\lambda)$ may not converge in the neighbourhood of the real axis, even when H' is a small perturbation. In such cases the deformation of the integration path C into the real axis is obviously not permitted and there is no equivalence between Feynman's expansion and (81). From the practical point of view, expansions of V_F are useful only to compute corrections of small order in H', so that only the first terms of (81) and (89) are of interest and both expansions lead to the same results.

Strictly speaking, Feynman's integral equation is in general not true. Indeed, we get from (71) and the corresponding equation for $V_F^{(0)}(t)$, for $T > |t|$

$$-i \int_{-T}^{+T} V_F^{(0)}(t-\tau) H' V_F(\tau) \, \mathrm{d}\tau = -i \int_{-T}^{+T} V_F^{(0)}(t-\tau)(H - H_0) V_F(\tau) \, \mathrm{d}\tau = \tag{94}$$

$$= \int_{-T}^{+T} V_F^{(0)}(t-\tau) \left[\frac{\mathrm{d}}{\mathrm{d}\tau} V_F(\tau) - \delta(\tau)\right] \mathrm{d}\tau + \int_{-T}^{+T} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} V_F^{(0)}(t-\tau) + \delta(t-\tau)\right] V_F(\tau) \, \mathrm{d}\tau =$$

$$= V_F(t) - V_F^{(0)}(t) + \int_{-T}^{+T} \frac{\mathrm{d}}{\mathrm{d}\tau} [V_F^{(0)}(t-\tau) V_F(\tau)] \, \mathrm{d}\tau \,.$$

For the validity of Feynman's equation it is necessary and sufficient that

$$\lim_{T\to\infty} [V_F^{(0)}(-T)V_F(T)] = \lim_{T\to\infty} [V_F^{(0)}(T)V_F(-T)], \tag{95}$$

but this condition is not fulfilled in general. We shall examine this point in section 6.

Feynman's integral equation corresponds to the following one for $V_\pm(t)$

$$V_\pm(t) = V_{\pm,0}(t) - i\int_{-\infty}^{+\infty} V_{\pm,0}(t-\tau)H'V_\pm(\tau)\,\mathrm{d}\tau, \tag{96}$$

$$V_{\pm,0}(t) = \pm\,\eta(\pm\,t)\exp[-iH_0t] = \pm\,\eta(\pm\,t)V_0(t). \tag{97}$$

Equation (96) is an immediate consequence of the following one

$$V(t) = V_0(t) - i\int_0^t V_0(t-\tau)H'V(\tau)\,\mathrm{d}\tau, \tag{98}$$

which can be easily checked by derivation with respect to t. The condition which corresponds to (95) in the case of $V_\pm(t)$ is

$$\lim_{T\to\infty} [V_{\pm,0}(-T)V_\pm(T)] = \lim_{T\to\infty} [V_{\pm,0}(T)V_\pm(-T)]. \tag{99}$$

This condition is fulfilled, because both sides of (99) are equal to 0.

We shall see that it is possible to introduce quite easily the operator corresponding to $V_F(t)$ when the electromagnetic field acting on the electrons is time dependent, and that there is no need of Feynman's integral equation to derive an expansion for Feynman's operator, even in this more general situation.

6. – There is a very simple relation between the operators $V_F(t)$ and $V(t)$

$$V_F(t) = \frac{1}{2}\,\varepsilon(t)V(t) + \frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{+\infty} V(t-\tau)\,\frac{\mathrm{d}\tau}{\tau}. \tag{100}$$

Indeed, we have

$$\begin{aligned}\frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{+\infty} V(t-\tau)\,\frac{\mathrm{d}\tau}{\tau} &= \frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\tau}{\tau}\,[\sum_k \exp[-iE_k(t-\tau)]P_k + \\ &\quad + \int_{\text{sp.}} \exp[-iE(t-\tau)]P(E)\,\mathrm{d}E] = \\ &= \frac{1}{2}\sum_k \varepsilon(E_k)\exp[-iE_kt]P_k + \frac{1}{2}\int_{\text{sp.}} \varepsilon(E)\exp[-iEt]P(E)\,\mathrm{d}E,\end{aligned} \tag{101}$$

since

$$\frac{1}{2\pi i}\,\text{p.v.}\int\limits_{-\infty}^{+\infty}\exp\,[i\alpha\tau]\,\frac{\mathrm{d}\tau}{\tau}=\frac{1}{2}\,\varepsilon(\alpha)\,. \tag{102}$$

By taking into account that the discrete levels are positive, we get

$$\frac{1}{2\pi i}\,\text{p.v.}\int\limits_{-\infty}^{+\infty}V(t-\tau)\,\frac{\mathrm{d}\tau}{\tau}=\frac{1}{2}\,V(t)-\int\limits_{E<0}\exp\,[-iEt]P(E)\,\mathrm{d}E, \tag{103}$$

hence

$$\frac{1}{2}\,\varepsilon(t)V(t)+\frac{1}{2\pi i}\,\text{p.v.}\int\limits_{-\infty}^{+\infty}V(t-\tau)\,\frac{\mathrm{d}\tau}{\tau}= \\ =\eta(t)V(t)-\int\limits_{E<0}\exp\,[-iEt]P(E)\,\mathrm{d}E=V_F(t)\,. \tag{104}$$

By taking as integration variable in (100) $\lambda=t-\tau$, we get

$$V_F(t)=\frac{1}{2}\,\varepsilon(t)V(t)+\frac{1}{2\pi i}\,\text{p.v.}\int\limits_{-\infty}^{+\infty}V(\lambda)\,\frac{\mathrm{d}\lambda}{t-\lambda}\,, \tag{105}$$

therefore

$$\left\{\begin{aligned} V_F(t)&=\frac{1}{2\pi i}\int\limits_{-\infty+i\eta}^{+\infty+i\eta}V(\lambda)\,\frac{\mathrm{d}\lambda}{t-\lambda}\,, &&(t>0)\\ & &&(\eta>0)\\ V_F(t)&=\frac{1}{2\pi i}\int\limits_{-\infty-i\eta}^{+\infty-i\eta}V(\lambda)\,\frac{\mathrm{d}\lambda}{t-\lambda}\,, &&(t<0)\,. \end{aligned}\right. \tag{106}$$

By a suitable deformation of the integration pathes in equations (106), we see that

$$V_F(t)=\frac{1}{2\pi i}\int\limits_{C'}V(\lambda)\,\frac{\mathrm{d}\lambda}{t-\lambda}\,, \tag{107}$$

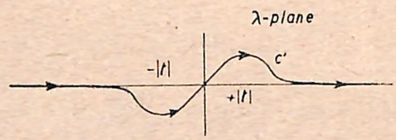

Fig. 2.

C' being the path represented in fig. 2.

It results from (105) that the condition (95) can be replaced by the following one

$$\lim_{T\to\infty}[V_0(-T)V(T)]=\lim_{T\to\infty}[V_0(T)V(-T)]\,. \tag{108}$$

Let us introduce the operator $U(t, t')$

$$U(t, t') = V_0(-t)V(t-t')V_0(t') \,. \tag{109}$$

It is well known that the Heisenberg collision operator S is given by $U(\infty, -\infty)$

$$S = U(\infty, -\infty) = U(\infty, 0)U(0, -\infty) = U(\infty, 0)U^{-1}(-\infty, 0) \,. \tag{110}$$

Since (108) means that

$$U(\infty, 0) = U(-\infty, 0) \,, \tag{111}$$

we should have for the Dirac equation with any time independent electromagnetic field

$$S = 1 \,, \tag{112}$$

and this is obviously not true.

The generalized resolvent operator $R(\lambda, t')$.

7. – We shall consider now the more general case of a time dependent hamiltonian $H(t)$. The unitary operator of the motion $V(t, t')$ is defined by the equations

$$i\frac{\mathrm{d}}{\mathrm{d}t}V(t, t') = H(t)V(t, t') \,, \qquad V(t', t') = 1 \,, \tag{113}$$

or by the equivalent ones

$$-i\frac{\mathrm{d}}{\mathrm{d}t'}V(t, t') = V(t, t')H(t') \,, \qquad V(t, t) = 1 \,, \tag{114}$$

which are not a trivial consequence of (113), because $V(t, t')$ is no more a function of the difference $t-t'$, as in the case of a time independent hamiltonian. Instead of the $V_\pm(t)$ we have now the $V_\pm(t, t')$

$$V_\pm(t, t') = \pm\,\eta(\pm\,(t-t'))V(t, t') \,, \tag{115}$$

which satisfy the differential equations

$$i\frac{\mathrm{d}}{\mathrm{d}t}V_\pm(t, t') - H(t)V_\pm(t, t') = i\delta(t-t') \,, \tag{116a}$$

$$-i\frac{\mathrm{d}}{\mathrm{d}t'}V_\pm(t, t') - V_\pm(t, t')H(t') = i\delta(t-t') \,. \tag{116b}$$

We shall denote by $\overline{R}_{\pm}(\lambda, t')$ the Laplace transforms of the $V_{\pm}(t, t')$

$$\overline{R}_{\pm}(\lambda, t') = i \int_{-\infty}^{+\infty} \exp[i\lambda t] V_{\pm}(t, t')\,\mathrm{d}t\,. \tag{117}$$

Under quite general conditions $\overline{R}_{+}(\lambda, t')$ will exist when $Im\lambda > A_{+}$, and $\overline{R}_{-}(\lambda, t')$ will exist when $I_m\lambda < A_{-}$, A_{+} and A_{-} being real constants, because $V(t, t')$ is an unitary operator. We shall take as generalized resolvent $R(\lambda, t')$

$$R(\lambda, t') = \begin{cases} \exp[-i\lambda t']\overline{R}_{+}(\lambda, t') & Im\lambda > 0, A_{+}\,, \\ \exp[-i\lambda t']\overline{R}_{-}(\lambda, t') & Im\lambda < 0, A_{-}\,. \end{cases} \tag{118}$$

When the hamiltonian H is time independent $R(\lambda, t') = R(\lambda)$.

The inversion of the Laplace transformation (117) is given by

$$V_{\pm}(t, t') = \frac{1}{2\pi i} \int_{-\infty + iB_{\pm}}^{+\infty + iB_{\pm}} \exp[-i\lambda t] \overline{R}_{\pm}(\lambda, t')\,\mathrm{d}\lambda\,, \tag{119}$$

with

$$B_{+} > A_{+}\,, \qquad B_{-} < A_{-}\,. \tag{119a}$$

It results from (114) and (117) that the operators $\overline{R}_{\pm}(\lambda, t')$ are determined by the following equations

$$-i \frac{\mathrm{d}}{\mathrm{d}t'} \overline{R}_{\pm}(\lambda, t') = \overline{R}_{\pm}(\lambda, t') H(t') - \exp[i\lambda t']\,, \tag{120}$$

$$\overline{R}_{+}(\lambda, \infty) = 0\,, \qquad \overline{R}_{-}(\lambda, -\infty) = 0\,. \tag{121}$$

These operators are also determined by an integral equation which we shall now derive. Let us introduce the Fourier integral expansion of $H(t)$

$$H(t) = \int_{-\infty}^{+\infty} \exp[-i\omega t] L(\omega)\,\mathrm{d}\omega\,. \tag{122}$$

It is easily seen that

$$i \int_{t'}^{\infty} \exp[i\lambda t] \left[i \frac{\mathrm{d}}{\mathrm{d}t} V(t, t') \right] \mathrm{d}t = \exp[i\lambda t'] + \lambda \overline{R}_{+}(\lambda, t')\,, \quad (Im\lambda > A_{+})\,, \tag{123}$$

and

$$(124)\qquad i\int_{t'}^{\infty}\exp[i\lambda t]H(t)V(t,t')\,\mathrm{d}t =$$

$$=\frac{1}{2\pi}\int_{t'}^{\infty}\exp[i\lambda t]\,\mathrm{d}t\int_{-\infty}^{+\infty}\exp[-i\omega t]L(\omega)\mathrm{d}\omega\int_{-\infty}^{+\infty}\exp[-i\mu t]\overline{R}_{+}(\mu,t')\mathrm{d}\mu =$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}L(\omega)\,\mathrm{d}\omega\int_{-\infty}^{+\infty}\frac{\exp[i(\lambda-\omega-\mu)t']}{\mu-(\lambda-\omega)}\overline{R}_{+}(\mu,t')\,\mathrm{d}\mu \qquad (Im\lambda > A_{+})\,.$$

We are assuming that $\overline{R}_{+}(\mu, t')$ exists for real values of μ. We get from the Cauchy residue theorem for $Im\lambda > 0$

$$(125)\qquad \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\exp[i(\lambda-\omega-\mu)t']}{\mu-(\lambda-\omega)}\overline{R}_{+}(\mu,t')\mathrm{d}\mu = \overline{R}_{+}(\lambda-\omega,t')\,,$$

by closing the integration path with a half-circle at infinity in the upper part of the μ-plane. It results from (123), (124) and (125) that

$$(126a)\qquad \int_{-\infty}^{+\infty}L(\omega)\overline{R}_{+}(\lambda-\omega,t')\,\mathrm{d}\omega-\lambda\overline{R}_{+}(\lambda,t')=\exp[i\lambda t'], \qquad (Im\lambda>0)\,.$$

It is easily seen that

$$(126b)\qquad \int_{-\infty}^{+\infty}L(\omega)\overline{R}_{-}(\lambda-\omega,t')\,\mathrm{d}\omega-\lambda\overline{R}_{-}(\lambda,t')=\exp[i\lambda t']\,, \qquad (Im\lambda<0)\,.$$

A very important particular case is that of a time independent unperturbed hamiltonian H_0 with a time dependent perturbation $H'(t)$

$$(127)\qquad H(t)=H_0+H'(t)\,,$$

$$(128)\qquad L(\omega)=H_0\delta(\omega)+L'(\omega)\,,$$

with

$$(129)\qquad L'(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[i\omega t]H'(t)\,\mathrm{d}t\,.$$

It follows from equations (126) that in this case

$$(130)\qquad \overline{R}_{\pm}(\lambda,t')=R_{\pm,0}(\lambda)\left[\exp[i\lambda t']-\int_{-\infty}^{+\infty}L'(\omega)\overline{R}_{\pm}(\lambda-\omega,t')\,\mathrm{d}\omega\right],$$

and thus

$$(131)\qquad \overline{R}_{\pm}(\lambda, t') = R_{\pm,0}(\lambda)\left[\exp[i\lambda t'] - \int\limits_{-\infty+\lambda}^{+\infty+\lambda} L'(\lambda-\lambda_1)\overline{R}_{\pm}(\lambda_1, t')\,\mathrm{d}\lambda_1\right].$$

The application of the method of successive approximations to (131) leads to the expansion

$$(132)\qquad \overline{R}_{\pm}(\lambda, t') = R_{\pm,0}(\lambda)\left[\exp[i\lambda t'] + \sum_{n=1}^{\infty}(-1)^n \int\limits_{-\infty+\lambda}^{+\infty+\lambda} L'(\lambda-\lambda_1)R_{\pm,0}(\lambda_1)\,\mathrm{d}\lambda_1 \times\right.$$

$$\left.\times \int\limits_{-\infty+\lambda_1}^{+\infty+\lambda_1} L'(\lambda_1-\lambda_2)R_{\pm,0}(\lambda_2)\,\mathrm{d}\lambda_2 \dots \int\limits_{-\infty+\lambda_{n-1}}^{+\infty+\lambda_{n-1}} L'(\lambda_{n-1}-\lambda_n)R_{\pm,0}(\lambda_n)\exp[i\lambda_n t']\,\mathrm{d}\lambda_n\right].$$

The expansion for $R(\lambda, t')$, obtained from (132) by multiplication with $\exp[-i\lambda t']$, goes over into (77) when H is time independent.

The Feynman operator for time dependent external fields.

8. – In the general case of a time dependent electromagnetic field, it is no more possible to define the operator V_F in terms of eigenstates of the Dirac equation. It is nevertheless possible to extend the definition (70), by introducing the Fourier integral expansion of $V(t, t')$

$$(133)\qquad V(t, t') = \int\limits_{-\infty}^{+\infty} \exp[-i\omega t]F(\omega, t')\,\mathrm{d}\omega\,,$$

and taking

$$(134)\qquad V_F(t, t') = \eta(t-t')\int\limits_{0}^{\infty} \exp[-i\omega t]F(\omega, t')\,\mathrm{d}\omega -$$

$$-\eta(t'-t)\int\limits_{-\infty}^{0} \exp[-i\omega t]F(\omega, t')\,\mathrm{d}\omega\,.$$

This generalization is the correct one, as it can be seen from the analysis given in the appendix of Feynman's paper, in which his procedure is related to the second quantization formulation of the hole theory. $V_F(t, t')$ is obtained from

$V_+(t, t')$ by subtraction of the negative frequency part of $V(t, t')$

(135) $$V_F(t, t') = V_+(t, t') - \int_{-\infty}^{0} \exp[-i\omega t] F(\omega, t')\, d\omega .$$

It is important to notice that $V_F(t, t')$ does not satisfy equation (116a), in general, because the negative frequency part of $V(t, t')$ is not a solution of the first equation (115), since

(136) $$H(t)F(\omega, t') \neq \omega F(\omega, t') ,$$

$V_F(t, t')$ is nevertheless a solution of (116b)

(137) $$-i \frac{d}{dt'} V_F(t, t') - V_F(t, t') H(t') = i\delta(t - t') ,$$

because

(138) $$-i \frac{d}{dt'} F(\omega, t') = F(\omega, t') H(t') .$$

The correct generalization of the Feynman integral equation is

(139) $$V_F(t, t') = V_F^{(0)}(t - t') - i \int_{-\infty}^{+\infty} V_F(t, \tau) H'(\tau) V_F^{(0)}(\tau - t')\, d\tau ,$$

because it leads to (137). This equation is not altogether satisfactory, as it can be seen by an analysis similar to that of sections 5 and 6.

The application of the method of successive approximations to (139) leads to the expansion

(140) $$V_F(t, t') = V_F^{(0)}(t - t') + \sum_{n=1}^{\infty} (-i)^n \int_{-\infty}^{+\infty} V_F^{(0)}(t - \tau_1) H'(\tau_1) V_F^{(0)}(\tau_1 - \tau_2) H'(\tau_2) \ldots \times$$
$$\times H'(\tau_n) V_F^{(0)}(\tau_n - t') \prod_{j=1}^{n} d\tau_j = V_F^{(0)}(t - t') + \sum_{n=1}^{\infty} V_F^{(n)}(t, t') .$$

It is easily seen that such an expansion is not satisfactory, in general, because

(141) $$\left(i \frac{d}{dt} - H_0\right) V_F^{(n)}(t, t') = H'(t) V_F^{(n-1)}(t, t'),$$

and thus we should have

(142) $$i \frac{d}{dt} V_F(t, t') - H(t) V_F(t, t') = i\delta(t - t') .$$

We shall see later that it is possible to replace (140) by a more satisfactory expansion.

It results from (134) that

$$V_F(t,t') = \frac{1}{2\pi i}\int\limits_{C'} \frac{V(\lambda + t', t')}{t - t' - \lambda}\, \mathrm{d}\lambda\,, \tag{143}$$

C' being of the form represented in fig. 2, with the loops taken large enough to contain the points $\pm(t-t')$. In order to show the equivalence of (143) and (134), let us introduce into (143) the Fourier integral expansion of $V(t,t')$

$$\begin{aligned} V_F(t,t') = {} & \frac{1}{2\pi i}\int\limits_0^{\infty} \exp[-i\omega t']F(\omega,t')\,\mathrm{d}\omega \int\limits_{C'} \frac{\exp[-i\omega\lambda]}{t-t'-\lambda}\,\mathrm{d}\lambda \, + \\ & + \frac{1}{2\pi i}\int\limits_{-\infty}^{0} \exp[-i\omega t']F(\omega,t')\,\mathrm{d}\omega \int\limits_{C'} \frac{\exp[-i\omega\lambda]}{t-t'-\lambda}\,\mathrm{d}\lambda\,. \end{aligned} \tag{144}$$

We shall close the integration path C' with a half-circle at infinity lying in the lower part of the λ-plane in the first integral, and with a half-circle at infinity lying in the upper part of the λ-plane in the second integral. The integrals with respect to λ can be evaluated by residues and we get (134).

By deformation of C' into the real axis (143) becomes

$$V_F(t,t') = \frac{1}{2\pi i}\,\text{p.v.}\int\limits_{-\infty}^{+\infty} V(u,t')\,\frac{\mathrm{d}u}{t-u} + \frac{1}{2}\,\varepsilon(t-t')V(t,t')\,. \tag{145}$$

This equation is a generalization of (105).

$V_F(t,t')$ can be expressed in terms of the generalized resolvent $R(\lambda,t')$ by a formula which generalizes (72)

$$V_F(t,t') = \frac{1}{2\pi i}\int\limits_{C} \exp[-i\lambda(t-t')]R(\lambda,t')\,\mathrm{d}\lambda\,. \tag{146}$$

In order to derive (146) we shall replace $(t-t'-\lambda)^{-1}$ in (143) by the following expressions

$$(t-t'-\lambda)^{-1} = i\int\limits_0^{\infty} \exp[-i(t-t'-\lambda)\alpha]\,\mathrm{d}\alpha\,, \qquad (Im\lambda > 0) \tag{147a}$$

$$(t-t'-\lambda)^{-1} = -i\int\limits_{-\infty}^{0} \exp[-i(t-t'-\lambda)\alpha]\,\mathrm{d}\alpha, \qquad (Im\lambda < 0) \tag{147b}$$

$$(148)\quad V_F(t,t') = \frac{1}{2\pi}\int_{C'_+} d\lambda \int_0^{\infty} V(\lambda+t',t') \exp[-i(t-t'-\lambda)\alpha]\, d\alpha -$$
$$-\frac{1}{2\pi}\int_{C'_-} d\lambda \int_{-\infty}^{0} V(\lambda+t',t') \exp[-i(t-t'-\lambda)\alpha]\, d\alpha =$$
$$=\frac{1}{2\pi}\int_{C_+} \exp[-i(t-t')\alpha]\, d\alpha \int_0^{\infty} V(\lambda+t',t') \exp[i\alpha\lambda]\, d\lambda -$$
$$-\frac{1}{2\pi}\int_{C_-} \exp[-i(t-t')\alpha]\, d\alpha \int_{-\infty}^{0} V(\lambda+t',t) \exp[i\alpha\lambda]\, d\lambda =$$
$$=\frac{1}{2\pi i}\int_{C} \exp[-i(t-t')\alpha] R(\alpha,t')\, d\alpha\,,$$

C'_+ and C'_- are the parts of C' lying above and below the real λ-axis, respectively; C_+ and C_- are the corresponding parts of C.

We can derive an expansion of $V_F(t,t')$ from (146) and (132)

$$(149)\quad V_F(t,t') = V_F^{(0)}(t-t') + \frac{1}{2\pi i}\int_C \exp[-i\lambda(t-t')] \sum_{n=1}^{\infty} R_n(\lambda,t')\, d\lambda\,,$$

with

$$(150)\quad \begin{cases} R_n(\lambda,t') = -R_0(\lambda)\displaystyle\int_{-\infty}^{+\infty} \exp[-i\mu t'] L'(\mu) R_{n-1}(\lambda-\mu,t')\, d\mu, \\ R_0(\lambda,t') = R_0(\lambda)\,. \end{cases}$$

In order to compare this expansion with Feynman's one, we shall use the recurrence relation for the terms of (140)

$$(151)\quad V_F^{(n)}(t,t') = -i\int_{-\infty}^{+\infty} V_F^{(0)}(t-\tau) H'(\tau) V_F^{(n-1)}(\tau,t')\, d\tau\,,$$

and introduce the Fourier integral expansion

$$(152)\quad V_F^{(n)}(t,t') = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \exp[-i\lambda(t-t')] \mathcal{R}_n(\lambda,t')\, d\lambda\,.$$

Thus we get for the $\mathcal{R}_n(\lambda,t')$ the same recurrence equations (150)

$$(153)\quad \mathcal{R}_n(\lambda,t') = -\mathcal{R}_0(\lambda)\int_{-\infty}^{+\infty} \exp[-i\mu t'] L'(\mu) \mathcal{R}_{n-1}(\lambda-\mu,t')\, d\mu \quad (\mathcal{R}_0 = R_0)\,.$$

Therefore, we would have Feynman's equation, if it would be possible to deform C into the real axis in (149), and then to integrate the series term by term.

The invariant resolvent in quantum field theory.

9. – We shall consider now a dynamical system formed by two quantized interacting fields. The unperturbed hamiltonian H_0 will be taken as the hamiltonian of the non interacting fields and H' will denote the interaction. We shall assume that H' is the space integral of a scalar $\mathcal{H}(\boldsymbol{x})$, as it happens in the case of the system electromagnetic field plus electron field in interaction

$$H' = \int \mathcal{H}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} . \tag{154}$$

In order to get a covariant formalism we shall introduce the operator $U(\sigma, \sigma')$ depending on two space-like surfaces in space-time and satisfying the Tomonaga-Schwinger equations

$$i\,\frac{\delta U(\sigma, \sigma')}{\delta\sigma(x)} = \mathcal{H}(x)U(\sigma, \sigma') , \tag{155}$$

with

$$\mathcal{H}(x) = \exp\,[iH_0x_0]\mathcal{H}(\boldsymbol{x})\exp\,[-iH_0x_0] . \tag{156}$$

We shall assume the compatibility conditions

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 , \qquad (x^\mu - x'^\mu)(x_\mu - x'_\mu) < 0 . \tag{156a}$$

In order to introduce the invariant resolvent we shall consider a family of space-like surfaces σ_τ depending on an invariant parameter τ varying from $-\infty$ to $+\infty$. The surfaces σ_τ are such that any point of space-time belongs to one and only one of them. Furthermore, we shall assume that the points in which any time-like line intersects successive surfaces are ordered in time and that $\tau = -\infty$ and $\tau = +\infty$ correspond to surfaces infinitely remote in the past and future, respectively. All these conditions are fulfilled when the σ_τ are planes normal to an arbitrary time-like four-vector and τ denotes the oriented distance between the planes σ_τ and σ_0, measured in the positive sense of time flux.

The resolvent $\mathcal{R}(\lambda, \sigma_{\tau'})$ is defined by the following equations

$$\mathcal{R}(\lambda, \sigma_{\tau'}) = \begin{cases} \exp\,[-i\lambda\tau']\mathcal{R}_+(\lambda, \sigma_{\tau'}) , & Im\lambda > A_+ \\ \exp\,[-i\lambda\tau']\mathcal{R}_-(\lambda, \sigma_{\tau'}) , & Im\lambda < A_- \end{cases} \tag{157}$$

with

$$(158a)\qquad \overline{\mathscr{R}}_{+}(\lambda, \sigma_{\tau'}) = i\int\limits_{\tau'}^{\infty} \exp[i\lambda\tau]\,U(\sigma_\tau, \sigma_{\tau'})\,\mathrm{d}\tau\,,$$

$$(158b)\qquad \overline{\mathscr{R}}_{-}(\lambda, \sigma_{\tau'}) = -\,i\int\limits_{-\infty}^{\tau'} \exp[i\lambda\tau]\,U(\sigma_\tau, \sigma_{\tau'})\,\mathrm{d}\tau\,,$$

A_+ and A_- being respectively the lower and upper limits of the imaginary part of λ for the existence of $\overline{\mathscr{R}}_{+}(\lambda, \sigma_{\tau'})$ and $\overline{\mathscr{R}}_{-}(\lambda, \sigma_{\tau'})$. The resolvent $\mathscr{R}$ is obviously invariant.

We get from (155)

$$(159)\qquad i\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,U(\sigma_\tau, \sigma_{\tau'}) = \mathscr{K}'(\tau)\,U(\sigma_\tau, \sigma_{\tau'})\,,$$

with

$$(160)\qquad \mathscr{K}'(\tau) = \int\limits_{\sigma_\tau} \mathscr{H}(x)\left(\frac{\partial\tau}{\partial x^\mu}\,\frac{\partial\tau}{\partial x_\mu}\right)^{-1/2}\mathrm{d}\sigma\,.$$

Introducing the Fourier integral expansion of $\mathscr{K}'(\tau)$

$$(161)\qquad \mathscr{K}'(\tau) = \int\limits_{-\infty}^{+\infty} \exp[-i\omega\tau]\,\mathscr{L}'(\omega)\,\mathrm{d}\omega\,,$$

we get an integral equation for the $\overline{\mathscr{R}}_{\pm}(\lambda, \sigma_{\tau'})$ which corresponds to (126)

$$(162)\qquad \int\limits_{-\infty}^{+\infty} \mathscr{L}'(\omega)\,\overline{\mathscr{R}}_{\pm}(\lambda-\omega, \sigma_{\tau'})\,\mathrm{d}\omega - \lambda\overline{\mathscr{R}}_{\pm}(\lambda, \sigma_{\tau'}) = \exp[i\lambda\tau']\,.$$

This equation is not entirely analogous to (126) because, by going over to the interaction representation, we did already take into account the influence of the unperturbed hamiltonian H_0.

The application of the method of successive approximations to the solution of (162) gives an expansion of the $\overline{\mathscr{R}}_{\pm}(\lambda, \sigma_{\tau'})$

$$(163)\qquad \overline{\mathscr{R}}_{\pm}(\lambda, \sigma_{\tau'}) = -\,\frac{\exp[i\lambda\tau']}{\lambda}\left[1 + \sum_{n=1}^{\infty}\int\limits_{-\infty}^{+\infty}\mathscr{L}'(\omega_1)\,\frac{\mathrm{d}\omega_1}{\lambda-\omega_1}\times\right.$$

$$\left.\times\int\limits_{-\infty}^{+\infty}\mathscr{L}'(\omega_2)\,\frac{\mathrm{d}\omega_2}{\lambda-\omega_1-\omega_2}\cdots\int\limits_{-\infty}^{+\infty}\mathscr{L}'(\omega_n)\,\frac{\exp\left[-i\sum\limits_{j=1}^{n}\omega_j\tau'\right]}{\lambda-\sum\limits_{j=1}^{n}\omega_j}\,\mathrm{d}\omega_n\right].$$

We get from (163) by simple transformations

$$(164a)\qquad \overline{\mathcal{R}}_{+}(\lambda, \sigma_{\tau'}) = -\frac{1}{\lambda}\Bigg[\exp[i\lambda\tau'] + \sum_{n=1}^{\infty}(-i)^n \int_{\tau'}^{\infty}\mathrm{d}\tau_1 \int_{\tau_1}^{\infty}\mathrm{d}\tau_2 \ldots \int_{\tau_{n-1}}^{\infty}\mathrm{d}\tau_n\, \mathcal{K}'(\tau_n)\mathcal{K}'(\tau_{n-1}) \ldots \mathcal{K}'(\tau_1)\exp[i\lambda\tau_n]\Bigg],$$

$$(Im\lambda > 0)\;;$$

$$(164b)\qquad \overline{\mathcal{R}}_{-}(\lambda, \sigma_{\tau'}) = -\frac{1}{\lambda}\Bigg[\exp[i\lambda\tau'] + \sum_{n=1}^{\infty} i^n \int_{-\infty}^{\tau'}\mathrm{d}\tau_1 \int_{-\infty}^{\tau_1}\mathrm{d}\tau_2 \ldots \int_{-\infty}^{\tau_{n-1}}\mathrm{d}\tau_n\, \mathcal{K}'(\tau_n)\mathcal{K}'(\tau_{n-1}) \ldots \mathcal{K}'(\tau_1)\exp[i\lambda\tau_n]\Bigg].$$

$$(Im\lambda < 0)\,.$$

The expansions (164) show that the $\overline{\mathcal{R}}_{\pm}(\lambda, \sigma_{\tau'})$ are the solutions of the differential equation

$$(165)\qquad -i\frac{\mathrm{d}}{\mathrm{d}\tau'}\overline{\mathcal{R}}_{\pm}(\lambda, \sigma_{\tau'}) = -\exp[i\lambda\tau'] + \overline{\mathcal{R}}_{\pm}(\lambda, \sigma_{\tau'})\mathcal{K}'(\tau')\,,$$

defined by the initial conditions

$$(166)\qquad \overline{\mathcal{R}}_{+}(\lambda, \sigma_{\infty}) = 0\,, \qquad \overline{\mathcal{R}}_{-}(\lambda, \sigma_{-\infty}) = 0\,.$$

The differential equation (165) and the conditions (166) can be derived directly from (158), by taking into account that

$$(167)\qquad -i\frac{\delta}{\delta\sigma'(x')}U(\sigma, \sigma') = U(\sigma, \sigma')\mathcal{H}(x')\,.$$

The solution of (165) by the method of successive approximations leads again to the expansions (164).

RIASSUNTO (*)

Gli operatori risolventi della teoria delle equazioni funzionali lineari si applicano in genere al formalismo quantico e più specialmente alla formulazione della teoria dei positroni data da FEYNMAN. Si dà una generalizzazione degli operatori risol-

(*) *Traduzione a cura della Redazione.*

venti per trattare problemi con hamiltoniani dipendenti dal tempo. Si dimostra che la formulazione di Feynman equivale a considerare nel nucleo di propagazione onde divergenti per le energie cinetiche positive e onde convergenti per le energie cinetiche negative. Dai risolventi si derivano sviluppi dei nuclei di propagazione senza impiegare l'equazione integrale di Feynman che presenta difficoltà. Si definisce un risolvente relativisticamente invariante nella teoria dei campi interagenti quantizzati. Un operatore derivato dal risolvente serve a descrivere un nuovo tipo di collisione che può essere usato nella teoria dello stato fondamentale dei nuclei atomici.