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Quantum Theory of the Point Electron. I*

MARIO SCHÖNBERG
University of São Paulo, São Paulo, Brazil
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A new treatment of the quantum theory of the electromagnetic field is discussed. The interactions between the particles and their interactions with the radiation field are treated according to the ideas of the author's classical theory. The radiation field is taken as a difference of two fields analogous to the field of the Heisenberg-Pauli electrodynamics. The commutation rules for the radiation field differ essentially from those of the Heisenberg-Pauli electrodynamics. In our formalism it is possible to avoid all the divergences by a suitable choice of the wave function of the system particles plus field. The physically relevant wave functions satisfy wave equations similar to those of systems with finite numbers of degrees of freedom, although they are also exact solutions of the Schrödinger equation of the system particles plus field which has an infinite number of degrees of freedom. There is also the possibility of getting finite non-vanishing self-energies.

I. INTRODUCTION

THE theory of point particles interacting with a field presents well-known difficulties, both in classical and quantum theories. We have shown[1–3] that it is possible to get a satisfactory classical theory of the point electron by modifying the usual ideas of the interactions between the point particles and their interactions with fields. We have also attempted a quantum generalization of our theory for the case of one electron.[4] in which there is only the interaction with the radiation field. The method we used in that paper has some basic points in common with the method we shall use in this paper (description of the radiation field by means of two fields of the Heisenberg-Pauli type) but it differs in some essential aspects, because in the present paper we shall not use anti-Hermitian operators in the description of the advanced waves.**

The essential point in our classical and quantum treatments lies in the modification of the field concepts of Faraday and Maxwell which underlie the quantum electrodynamics of Heisenberg and Pauli,[5,***] as well as more recent

* Revised form of a paper sent to The Physical Review in February, 1947.

[1] M. Schönberg, Phys. Rev. **69**, 211 (1946).

[2] M. Schönberg, Summa Brasiliensis Math. **1**, 41, 77 (1946).

[3] M. Schönberg, Summa Brasiliensis Phys. **1**, 51 (1947).

[4] M. Schönberg, Phys. Rev. **67**, 193 (1945).

** In that paper the advanced waves give negative contributions to the energy of the field in the classical formalism. The use of anti-Hermitian operators transforms the negative energy waves into positive energy photons so that the zero-point energy of the field is not canceled (see J. Leite Lopes, Anais Acad. Brasil. Ciencias **19**, 51 (1947).

[5] W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929), and **59**, 168 (1930).

*** A more detailed discussion of the methods of field quantization and of the Heisenberg-Pauli electrodynamics will appear soon in the Anais da Academia Brasileirs de Ciencias.

treatments.[6–9] In our theory we distinguish the direct intercation between two electrons from their interactions with the photons. The direct interactions between the electrons are treated as relativistic actions at a distance; they are assumed to be half-retarded, half-advanced actions of the type considered in classical theory by Tetrode[10] and Fokker;[11] no field concepts are used in the description of the direct interactions between the electrons. The photons are described by a radiation field which is a field of the Faraday-Maxwell type. In our classical theory the radiation field is one-half the difference between the retarded and advanced fields of the system of electrons; it satisfies homogeneous Maxwell equations.

The interaction between the particles and the radiation field gives rise to the emission and absorption of radiation. In order to ensure the conservation of energy and momentum in the emission and absorption of radiation, the energy and momentum of the radiation field must not be defined as in Maxwell's theory: *The energy of the radiation field is not necessarily positive, even in classical theory.* It follows from that circumstance that in the quantum formalism there are photons with positive and negative energies.

The fundamental difference between our quantum theory and the Heisenberg-Pauli quantum electrodynamics lies in the commutation rules for the field which interacts with the particles. *The potentials of our radiation field taken at any two-world points do commute, whereas they do not commute in the Heisenberg-Pauli electrodynamics when one of the world-points belongs to the light cone of the other one.* It follows from the irrestricted commutability of the radiation field potentials that the Fourier coefficients of these potentials do commute with their complex conjugates, so that it is possible to get states of the radiation field in which any set of Fourier coefficients and their complex conjugates have the value zero. We call the degress of freedom of the radiation field corresponding to the aforementioned Fourier coefficients the "frozen" degrees of freedom. *The possibility of the "freezing" allows us to get rid of the divergences arising from the infinite number of degrees of freedom of the field.*

In our quantum formalism there are exact solutions of the Schrödinger equation of the system particles plus radiation field describing states of motion of the system in which the particles interact only with a finite number of degrees of freedom of the field, provided we enclose the field in a box and impose periodic boundary conditions. Those solutions correspond to states in which the degrees of freedom which do not interact with the particles are "frozen."

In order to compute the transition probabilities and other physically interesting values we need a "freezing" rule and a physical interpretation rule. The "freezing" rule indicates which particular solution of the Schrödinger equation of the system ought to be used in the computation of the experimentally measurable quantities. The physical interpretation rule is necessary because of the existence of photons with positive and negative energies in our formalism. There are several acceptable interpretation rules. *The simplest interpretation rule can be considered as the quantum analog of the time boundary conditions of the classical theory.*

There is no difficulty in setting up a "freezing" criterion in the computation of the transition probabilities. We have not been able to find a reasonable "freezing" rule for the computation of self-energies, there is a large arbitrariness which seems to require the introduction of some elementary length, although there is no difficulty in eliminating the infinite self-energies.

II. THE CLASSICAL VARIATIONAL PRINCIPLE

Let us consider a system of n charged point particles. The coordinates of the jth particle will be denoted by x_j^μ ($\mu=0, 1, 2, 3$), its electric charge and rest mass by e_j and m_j, respectively. The retarded and advanced fields of the jth particle will be denoted by $F_{j,\,\mathrm{ret}}^{\mu\nu}$ and $F_{j,\,\mathrm{adv}}^{\mu\nu}$. We shall introduce the attached fields of the particles

$$F_{j,\,\mathrm{at}}^{\mu\nu}=\tfrac{1}{2}(F_{j,\,\mathrm{ret}}^{\mu\nu}+F_{j,\,\mathrm{adv}}^{\mu\nu}) \tag{1}$$

[6] P. A. M. Dirac, V. Fock, and B. Podolsky, Physik. Zeits. Sowjetunion 2, 468 (1932).
[7] P. A. M. Dirac, An. Inst. H. Poincaré 9, 13 (1939).
[8] P. A. M. Dirac, Proc. Roy. Soc. A180, 1 (1942).
[9] P. A. M. Dirac, Com. Dublin Inst. A, 3 (1946).
[10] H. Tetrode, Zeits. f. Physik. 10, 317 (1922).
[11] A. D. Fokker, Zeits. f. Physik 58, 386 (1929).

and the radiation field of the system

$$F_{\text{rad}}^{\mu\nu}=\tfrac{1}{2}\sum_{j=1}^{n}(F_{j,\text{ret}}^{\mu\nu}-F_{j,\text{adv}}^{\mu\nu}). \tag{2}$$

The equations of motion of the jth particle are

$$m_jc^2\frac{d^2x_j^\mu}{ds_j^2}=e_j\Big[F_{\text{rad}}^{\mu\nu}(x_j)+\sum_{k\neq j}F_{k,\text{at}}^{\mu\nu}(x_j)\Big]\frac{dx_{j,\nu}}{ds_j}, \tag{3}$$

with

$$ds_j=(g_{\mu\nu}dx_j^\mu dx_j^\nu)^{\frac{1}{2}} \tag{4}$$

and

$$g_{00}=1,\quad g_{11}=g_{22}=g_{33}=-1,\quad g_{\mu\nu}=0\ (\mu\neq\nu). \tag{5}$$

We shall denote the potentials of a field F by the letter A.

$$F^{\mu\nu}=\partial A^\nu/\partial x_\mu-\partial A^\mu/\partial x_\nu. \tag{6}$$

The attached potentials $A_{j,\text{at}}^\mu$ can be conveniently represented by integrals

$$A_{j,\text{at}}^\mu(x)=e_j\int_{-\infty}^{+\infty}\frac{dx_j^\mu}{ds_j}\delta(x_j,x)ds_j, \tag{7}$$

$$\delta(x_j,x)=\delta((x_j^\nu-x^\nu)(x_{j,\nu}-x_\nu)). \tag{8}$$

$\delta(u)$ denotes the Dirac symbolic function of the argument u.

The radiation potentials are solutions of the homogeneous d'Alembert equation

$$\Box A_{\text{rad}}^\mu=0,\quad\Big(\Box=g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\Big). \tag{9}$$

They satisfy also the Lorentz condition

$$\frac{\partial}{\partial x^\mu}A_{\text{rad}}^\mu=0. \tag{10}$$

The equations of motion of the particles and the field can be derived from the variational principle

$$\delta\mathfrak{L}=0, \tag{11}$$

$$\mathfrak{L}=\mathfrak{L}_p+\mathfrak{L}_f. \tag{12}$$

$\mathfrak{L}_p$ is the Lagrangian integral of the particles

$$\mathfrak{L}_p=-\sum_{j=1}^{n}m_jc\int_{-\infty}^{+\infty}\Big(\frac{dx_j^\mu}{ds_j}\frac{dx_{j,\mu}}{ds_j}\Big)^{\frac{1}{2}}ds_j-\sum_{j=1}^{n}\frac{e_j}{c}\int_{-\infty}^{+\infty}A_{\text{rad}}^\mu(x_j)\frac{dx_{j,\mu}}{ds_j}ds_j-\frac{1}{2c}\sum_{j=1}^{n}\sum_{k\neq j}e_je_k\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dx_j^\mu}{ds_j}\frac{dx_{k,\mu}}{ds_k}\times\delta(x_j,x_k)ds_jds_k. \tag{13}$$

$\mathfrak{L}_f$ is the Lagrangian integral of the field

$$\mathfrak{L}_f=-\frac{1}{32\pi c}\int_{-\infty}^{+\infty}[F^{(+)\mu\nu}F_{\mu\nu}^{(+)}-F^{(-)\mu\nu}F_{\mu\nu}^{(-)}]d_4x-\frac{1}{16\pi c}\int_{-\infty}^{+\infty}\Big[\Big(\frac{\partial A^{(+)\mu}}{\partial x^\mu}\Big)^2-\Big(\frac{\partial A^{(-)\mu}}{\partial x^\mu}\Big)^2\Big]d_4x. \tag{14}$$

In the variational Eq. (11) the potentials $A^{(+)\mu}$ and $A^{(-)\mu}$ are treated as independent field variables. Their equations of motion are

$$\Box A^{(\pm)\mu}=4\pi\sum_{j=1}^{n}J_j^\mu, \tag{15}$$

$$J_j^\mu(x)=e_j\int_{-\infty}^{+\infty}\frac{dx_j^\mu}{ds_j}\prod_{\lambda=0}^{3}\delta(x_j^\lambda-x^\lambda)ds_j. \tag{16}$$

We must assume that

$$\tfrac{1}{2}(A^{(+)\mu}-A^{(-)\mu})=A_{\text{rad}}^\mu, \tag{17}$$

so that we must have

$$\frac{\partial}{\partial x^\mu}(A^{(+)\mu}-A^{(-)\mu})=0. \tag{18}$$

In order to satisfy (18) we shall impose the Lorentz conditions

$$\frac{\partial}{\partial x^\mu}A^{(\pm)\mu}=0. \tag{19}$$

We could obviously take for $A^{(+)\mu}$ the potentials of the retarded field of the system of particles, and for $A^{(-)\mu}$ the corresponding advanced potentials.

III. CONJUGATED MOMENTA AND POISSON BRACKETS

We get at once the momenta conjugated to the particle and field variables from the Lagrangian integral $\mathfrak{L}$:

$$p_{j,\mu}=-\frac{\delta\mathfrak{L}}{\delta(dx_j^\mu/ds_j)}=m_jc\frac{dx_{j,\mu}}{ds_j}+\frac{e_j}{c}\Big[A_{\text{rad},\mu}(x_j)+\sum_{k\neq j}A_{k,\text{at},\mu}(x_j)\Big], \tag{20}$$

$$\Pi_0^{(\pm)}=-\frac{\delta\mathfrak{L}}{\delta(\partial A^{(\pm)0}/\partial x_0)}=\pm\frac{1}{8\pi c}\frac{\partial A^{(\pm)\mu}}{\partial x^\mu}, \tag{21}$$

$$\Pi_a^{(\pm)} = -\frac{\delta \mathfrak{L}}{\delta(\partial A^{(\pm)a}/\partial x_0)} = \mp\frac{1}{8\pi c}F_{a0}^{(\pm)} \quad (a=1, 2, 3). \quad (22)$$

The Poisson brackets are defined in the usual way. We get

$$\left(A^{(+)\mu}(\mathbf{x}, x_0), \frac{\partial}{\partial x_0}A^{(+)\nu}(\mathbf{x}', x_0)\right) = -8\pi c g^{\mu\nu}\delta(\mathbf{x}-\mathbf{x}'), \quad (23)$$

$$\left(A^{(-)\mu}(\mathbf{x}, x_0), \frac{\partial}{\partial x_0}A^{(-)\nu}(\mathbf{x}', x_0)\right) = 8\pi c g^{\mu\nu}\delta(\mathbf{x}-\mathbf{x}'), \quad (24)$$

whence

$$\left(A_{\text{rad}}^{\mu}(\mathbf{x}, x_0), \frac{\partial}{\partial x_0}A_{\text{rad}}^{\nu}(\mathbf{x}', x_0)\right) = 0. \quad (25)$$

IV. ENERGY AND MOMENTUM OF THE FIELD

The four vector of energy and momentum of the field can be taken as

$$P_f^{\mu} = \int_{-\infty}^{+\infty} T_f^{\mu 0}d_3x + \frac{1}{C}\sum_{j=1}^{n} e_j A_{\text{rad}}^{\mu}(x_j), \quad (26)$$

with

$$T_f^{\mu\nu} = \frac{1}{16\pi c}g^{\mu\nu}\left[\frac{\partial A^{(+)\rho}}{\partial x^{\sigma}}\frac{\partial A_{\rho}^{(+)}}{\partial x_{\sigma}} - \frac{\partial A^{(-)\rho}}{\partial x^{\sigma}}\frac{\partial A_{\rho}^{(-)}}{\partial x_{\sigma}}\right] - \frac{1}{8\pi c}\left[\frac{\partial A^{(+)\rho}}{\partial x_{\mu}}\frac{\partial A_{\rho}^{(+)}}{\partial x_{\nu}} - \frac{\partial A^{(-)\rho}}{\partial x_{\mu}}\frac{\partial A_{\rho}^{(-)}}{\partial x_{\nu}}\right]. \quad (27)$$

cP_f^0 includes the potential energy of the particles with respect to the radiation field plus the energy of the radiation field. The Hamiltonian of the system particles plus radiation field is H:

$$H = H_p + H_f, \quad (28)$$

$$H_p = \sum_j m_j c^2 \frac{dx_j^0}{ds} + \sum_j e_j\left[A_{\text{rad}}^0(x_j) + \sum_{k\neq j} A_{k,\,\text{at}}^0(x_j)\right], \quad (29)$$

$$H_f = c\int_{-\infty}^{+\infty} T_f^{00}d_3x. \quad (30)$$

It is easily seen that

$$H_f = \frac{1}{16\pi}\int_{-\infty}^{+\infty}\left[\sum_{a=1}^{3}\{F_{a0}^{(+)2} - F_{a0}^{(-)2}\} + \sum_{a=1}^{3}\sum_{b>a}\{F_{ab}^{(+)2} - F_{ab}^{(-)2}\}\right]d_3x - \frac{1}{16\pi}\int_{-\infty}^{+\infty}\left[\left(\frac{\partial A^{(+)\mu}}{\partial x^{\mu}}\right)^2 - \left(\frac{\partial A^{(-)\mu}}{\partial x^{\mu}}\right)^2\right]d_3x. \quad (31)$$

The energy density of the radiation waves may be negative even when the Lorentz conditions (19) *are taken into account.*

V. COMMUTATION RULES FOR THE QUANTUM POTENTIALS

Equations (23)–(25) are still valid in quantum theory. The quantum Poisson bracket of two quantities B and C is proportional to their commutator

$$(B, C) = -\frac{i}{\hbar}[B, C], \quad (32)$$

$$[B, C] = BC - CB, \quad (33)$$

The commutation rules corresponding to the values of the classical Poisson brackets and the equations of motion of the potentials allow us to write the relativistic commutation rules. Let us put

$$\Delta(x, x') = 2\text{sig n}(x_0 - x_0')\delta(x, x'). \quad (34)$$

The relativistic commutation rules are

$$[A^{(+)\mu}(x), A^{(+)\nu}(x')] = 2i\hbar c g^{\mu\nu}\Delta(x, x'), \quad (35)$$

$$[A^{(+)\mu}(x), A^{(-)\nu}(x')] = 0, \quad (36)$$

$$[A^{(-)\mu}(x), A^{(-)\nu}(x')] = -2i\hbar c\Delta(x, x'), \quad (37)$$

whence we get

$$[A_{\text{rad}}^{\mu}(x), A_{\text{rad}}^{\nu}(x')] = 0. \quad (38)$$

If the A_{rad}^{μ} would be potentials of the Heisenberg-Pauli type we would have instead of (38)

$$[A_{\text{rad}}^{\mu}(x), A_{\text{rad}}^{\nu}(x')] = i\hbar c g^{\mu\nu}\Delta(x, x').$$

It is well known that in quantum electrodynamics the potentials which come in the interaction with the particles are Wentzel potentials.[6] From now on we shall always assume that our potentials are

Wentzel potentials, which depend on two time variables: the time of the field x^0 and the time of the particles $x_p{}^0$. The commutation rules (35)–(37) are also valid for the Wentzel potentials.

We shall now expand the Wentzel potentials in Fourier series by enclosing the field in a cubic box of volume V and imposing periodic boundary conditions.

$$A^{(\pm)\mu}(x)=\left(\frac{4\pi\hbar c}{V}\right)^{\frac{1}{2}}\sum_k k_0{}^{-\frac{1}{2}}$$

$$\times\{Q^{(\pm)\mu}(k)\exp[-ik^\rho x_\rho]+\bar{Q}^{(\pm)\mu}(k)\exp[ik^\rho x_\rho]\}, \quad (39)$$

$$k_0=|\mathbf{k}|. \quad (39a)$$

The commutation rules for the Fourier coefficients are

$$[Q^{(+)\mu}(k),\,\bar{Q}^{(+)\nu}(k')]=-g^{\mu\nu}\delta_{kk'}, \quad (40)$$

$$[Q^{(-)\mu}(k),\,\bar{Q}^{(-)\nu}(k')]=g^{\mu\nu}\delta_{kk'}. \quad (41)$$

The Fourier coefficients of the radiation potentials are the $Q_{\text{rad}}{}^\mu(k)$:

$$Q_{\text{rad}}{}^\mu(k)=\tfrac{1}{2}[Q^{(+)\mu}-Q^{(-)\mu}]. \quad (42)$$

It results from (40) and (41) that

$$[Q_{\text{rad}}{}^\mu(k),\,\bar{Q}_{\text{rad}}{}^\nu(k')]=0. \quad (43)$$

Let us consider now the operators $N_\mu{}^{(\pm)}(k)$:

$$N_a{}^{(+)}=\bar{Q}_a{}^{(+)}Q_a{}^{(+)},\quad N_0{}^{(+)}=Q_0{}^{(+)}\bar{Q}_0{}^{(+)}, \quad (44)$$

$$N_a{}^{(-)}=Q_a{}^{(-)}\bar{Q}_a{}^{(-)},\quad N_0{}^{(-)}=\bar{Q}_0{}^{(-)}Q_0{}^{(-)}. \quad (45)$$

The operators N have the eigenvalues $0, 1, 2, 3\cdots$. It is easily seen that the Hamiltonian of the radiation waves H_f has the following expression

$$H_f=\sum_k\left[\left\{\sum_{a=1}^{3}N_a{}^{(+)}(k)-N_0{}^{(+)}(k)\right\}-\left\{\sum_{a=1}^{3}N_a{}^{(-)}(k)-N_0{}^{(-)}(k)\right\}\right]\hbar k_0. \quad (46)$$

VI. THE QUANTUM EQUATIONS OF MOTION

(A.) In our quantum formalism we shall introduce a single time variable for all the particles because of the actions at a distance. Therefore, we will have two equations of motion for any quantity B which does not depend explicitly on the time variables

$$i\hbar c\frac{\partial B}{\partial x_p{}^0}=[B,\,H_p], \quad (47)$$

$$i\hbar c\frac{\partial B}{\partial x^0}=[B,\,H_f]. \quad (48)$$

(It is more satisfactory to consider (47) and (48) as supplementary conditions.) We shall assume that our particles have spin $\frac{1}{2}$. Therefore the Hamiltonian of the particles will be

$$H_p=\sum_j H_j+H_{\text{int}}, \quad (49)$$

$$H_j=(c\boldsymbol{\alpha}_j\cdot\{\mathbf{p}_j-(e_j/c)\mathbf{A}_{\text{rad}}(x_j)\})+\beta_j m_j c^2+e_j A_{\text{rad}}{}^0(x_j), \quad (50)$$

$$H_{\text{int}}=\tfrac{1}{2}\sum_j e_j\alpha_j{}^\mu\sum_{k\neq j}[A_{k,\,\text{at},\,\mu}(x_j)]_{(x_j{}^0=x_p)}. \quad (51)$$

The attached potentials can be defined in quantum theory by integrals analogous to the classical ones (Eq. (7)):

$$A_{j,\,\text{at}}{}^\mu(x)=e_j\int_{-\infty}^{+\infty}\alpha_j{}^\mu(\tau)[\delta(x,\,x_j(\tau))]_{(x_j{}^0=\tau)}d\tau. \quad (52)$$

The α and β are the well-known Dirac operators. The δ-function which appears under the integral is defined as a symbolic limit

$$\delta(x,\,x_j(\tau))=\frac{1}{(\pi)^{\frac{1}{2}}}\lim_{\epsilon\to0}\epsilon^{-1}\exp[\epsilon^{-2}\{(x^\rho-x_j{}^\rho(\tau))\times(x_\rho-x_{j,\,\rho}(\tau))\}^2]. \quad (53)$$

Hence

$$A_{j,\,\text{at}}{}^\mu(x)=\frac{1}{(\pi)^{\frac{1}{2}}}e_j\lim_{\epsilon\to0}\epsilon^{-1}\int_{-\infty}^{+\infty}\alpha_j{}^\mu(\tau)\times\exp[\epsilon^{-2}\{(x_0-\tau)^2-|\mathbf{x}-\mathbf{x}_j(\tau)|^2\}^2]d\tau, \quad (54)$$

with

$$|\mathbf{x}-\mathbf{x}_j(\tau)|^2=\sum_{a=1}^{3}[x^a-x_j{}^a(\tau)]^2. \quad (54a)$$

It results from (47) that

$$\frac{\partial}{\partial x_p{}^0}A^{(\pm)\mu}(x)=\sum_j\alpha_j{}^\mu(x_p{}^0)[\Delta(x,\,x_j(x_p{}^0))]_{(x_j{}^0=x_p{}^0)}, \quad (55)$$

whence

$$\frac{\partial}{\partial x_p{}^0} A_{\rm rad}{}^\mu(x)=0. \qquad (56)$$

This equation is very important because it shows that the radiation potentials are pure field quantities since they do not depend on the time of the particles.

We can go over from the preceding formalism to the formalism with a single time variable by putting

$$x_p{}^0=x^0=x_s{}^0. \qquad (57)$$

$x_s{}^0/c$ is the time of the system particles plus radiation field. We have, obviously,

$$i\hbar c\frac{\partial B}{\partial x_s{}^0}=i\hbar c\left(\frac{\partial B}{\partial x_p{}^0}+\frac{\partial B}{\partial x^0}\right)_{(x_p{}^0=x^0=x_s{}^0)}=[B, H_p+H_f]. \qquad (58)$$

(B.) In the preceding section we used the Heisenberg picture in which the state of the system does not vary with time. We shall now go over to the Schrödinger picture in which the state of the system varies with time. In this picture we have the Schrödinger equation for the wave function Ψ:

$$i\hbar\frac{\partial\Psi}{\partial t}=\mathcal{H}\Psi, \quad (ct=x_s{}^0). \qquad (59)$$

$\mathcal{H}$ is the Hamiltonian in the Schrödinger picture; it cannot be taken as the Hamiltonian of the Heisenberg picture corresponding to a fixed value of t, $t=0$, for instance, because the attached potentials do depend explicitly on the time. $\mathcal{H}$ is given in terms of H by the formula

$$\mathcal{H}=U(t,0)HU^{-1}(t,0), \qquad (60)$$

which is a particular case of the general formula (79). The state Ψ must satisfy the Dirac-Fock-Podolsky supplementary conditions

$$R^{(\pm)}(x)\Psi=0, \qquad (61)$$

$$R^{(\pm)}(x)=\frac{\partial}{\partial x^\mu}A^{(\pm)\mu}(x)+\textstyle\sum_j e_j\Delta(x,x_j). \qquad (62)$$

It is remarkable that the supplementary condition for the radiation potentials is the Lorentz condition, as it results at once from (61).

It is more convenient to express the supplementary conditions in terms of the Fourier coefficients of the potentials. We have

$$\left\{k_\mu Q^{(\pm)\mu}(k)-\left(\frac{\pi}{hcV}\right)^{\frac{1}{2}}\textstyle\sum_j e_jk_0{}^{-\frac{1}{2}}\times\exp[ik_\mu x_j{}^\mu]_{(x_j{}^0=ct)}\right\}\Psi=0, \qquad (63)$$

$$\left\{k_\mu \bar{Q}^{(\pm)\mu}(k)-\left(\frac{\pi}{hcV}\right)^{\frac{1}{2}}\textstyle\sum_j e_jk_0{}^{-\frac{1}{2}}\times\exp[-ik_\mu x_j{}^\mu]_{(x_j{}^0=ct)}\right\}\Psi=0. \qquad (64)$$

VII. ELIMINATION OF THE LONGITUDINAL WAVES

Let us put

$$Q_T{}^{(\pm)a}(k)=Q^{(\pm)a}(k)+\frac{k^a}{k_0{}^2}[Q^{(\pm)b}(k)k_b]. \qquad (65)$$

The $Q_T{}^a(k)$ are components of transversal vectors $\mathbf{Q}_T(k)$ which can be conveniently decomposed along the directions of two unit vectors $\mathbf{u}_s(k)$ $(s=1,2)$ mutually orthogonal and orthogonal to $\mathbf{k}$.

$$[\mathbf{Q}_T{}^{(\pm)}(k)]_{(t=0)}=\sum_{s=1}^{2}Q_s{}^{(\pm)}(k)\mathbf{u}_s(k). \qquad (66)$$

By taking into account the supplementary conditions (63) and (64), we get

$$\mathcal{H}_f\Psi=\sum_k hck_0\left[\sum_{s=1}^{2}\{N_s{}^{(+)}(k)-N_s{}^{(-)}(k)\}\right]\Psi-\sum_{j=1}^{n}e_j[A_{\rm rad}{}^0(x_j)]_{(x_j{}^0=0)}\Psi. \qquad (67)$$

$\mathcal{H}_f$ is the Hamiltonian of the photons in the Schrödinger picture

$$\mathcal{H}_f=U(t,0)H_fU^{-1}(t,0). \qquad (68)$$

The N are the numbers of positive and negative energy photons

$$N_s{}^{(+)}=\bar{Q}_s{}^{(+)}Q_s{}^{(+)}, \quad N_s{}^{(-)}=Q_s{}^{(-)}\bar{Q}_s{}^{(-)}. \qquad (69)$$

It results from (67) that

$$\mathcal{H}\Psi=\sum_i\left\{-c\alpha_{j,a}\left(p_j{}^a-\frac{e_j}{c}A_{\text{rad}}{}^a(x_j)\right)+m_jc^2\beta_j\right.$$
$$\left.+\frac{e_j}{2}\sum_{l\neq j}\alpha_{j,\mu}A_{l,\text{at}}{}^\mu(x_j)\right\}_{\text{Schröd}}\Psi$$
$$+\left\{\sum_k\hbar ck_0\left[\sum_{s=1}^2(N_s{}^{(+)}(k)-N_s{}^{(-)}(k))\right]\right\}\Psi. \quad (70)$$

All the operators in the right-hand side of (70) are taken in the Schrödinger picture. Let us introduce now the gauge invariant momenta $\mathbf{p}_j{}^\dagger$:

$$\mathbf{p}_j{}^\dagger=\mathbf{p}_j-\frac{e_j}{c}\mathbf{A}_{\text{rad},\,L}(x_j). \quad (71)$$

$\mathbf{A}_{\text{rad},\,L}$ is the longitudinal part of the vector potential $\mathbf{A}_{\text{rad}}$. By taking into account that the curl of $\mathbf{A}_{\text{rad},\,L}$ vanishes, we see that the components of the gauge invariant momenta do commute. The commutation rules between the gauge invariant momenta and the coordinates of the particles are of the same type as those for the ordinary momenta $\mathbf{p}$. The $\mathbf{p}^\dagger$ are gauge invariant because

$$[p_j{}^{\dagger a},\,R^{(\pm)}(x)]=0. \quad (72)$$

By introducing the gauge invariant momenta we can eliminate completely the longitudinal photons from the expression of $\mathcal{H}\Psi$:

$$\mathcal{H}\Psi=\sum_i\left\{-c\alpha_{j,a}\left(p_j{}^{\dagger a}-\frac{e_j}{c}A_{\text{rad},\,T}{}^a(x_j)\right)+m_jc^2\beta_j\right.$$
$$\left.+\frac{e_j}{2}\sum_{l\neq j}\alpha_{j,\mu}A_{l,\text{at}}{}^\mu(x_j)\right\}_{\text{Schröd}}\Psi$$
$$+\left\{\sum_k\hbar ck_0\left[\sum_{s=1}^2(N_s{}^{(+)}(k)-N_s{}^{(-)}(k))\right]\right\}\Psi. \quad (73)$$

$\mathbf{A}_{\text{rad},\,T}$ is the transverse part of $\mathbf{A}_{\text{rad}}$

$$\mathbf{A}_{\text{rad},\,T}(x)=\left(\frac{4\pi\hbar c}{V}\right)^{\frac{1}{2}}\sum_k k_0{}^{-\frac{1}{2}}$$
$$\times\{\mathbf{Q}_{\text{rad},\,T}(k)\exp[-ik^\rho x_\rho]$$
$$+\bar{\mathbf{Q}}_{\text{rad},\,T}(k)\exp[ik^\rho x_\rho]\}, \quad (74)$$

$$\mathbf{Q}_{\text{rad},\,T}=\tfrac{1}{2}[\mathbf{Q}_T{}^{(+)}-\mathbf{Q}_T{}^{(-)}]. \quad (75)$$

VIII. THE MÖLLER INTERACTION

We shall now establish first approximation formulas for the attached potentials which show that the direct interaction between particles coincides, in a first approximation, with the Möller interaction.

In the Heisenberg picture the time variation of a physical quantity B which does not depend explicitly on the time is given by a contact transformation

$$B(t)=U^{-1}(t,t')B(t')U(t,t'). \quad (76)$$

When the Hamiltonian does not depend explicitly on the time, we have

$$U(t,t')=\exp\left[-\frac{i}{\hbar}H(t-t')\right]. \quad (77)$$

We have in the Heisenberg picture, because of (52) and (76),

$$A_{j,\text{at}}{}^\mu(x)=ce_j\int_{-\infty}^{+\infty}U^{-1}(t',0)$$
$$\times[\alpha_j{}^\mu(0)\delta(x,x_j(0))]U(t',0)dt'. \quad (78)$$

We go over from the Heisenberg to the Schrödinger picture by a contact transformation

$$B_{\text{Schröd}}(t)=U(t,0)B(t)U^{-1}(t,0), \quad (79)$$

whence

$$[A_{j,\text{at}}{}^\mu(x)]_{\text{Schröd}}=ce_j\int_{-\infty}^{+\infty}U(t,t')$$
$$\times[\alpha_j{}^\mu\delta(x,x_j)]_{\text{Schröd}}U^{-1}(t,t')dt'. \quad (80)$$

In order to determine the first approximation expressions of the attached potentials we shall use the operator U corresponding to an unperturbed motion in which the interactions between the particles and their interactions with the radiation field are neglected. We shall denote the U of the unperturbed motion by U_0:

$$U_0(t,t')=\exp[-(i/\hbar)(\textstyle\sum_j H_j{}^0+H_f)(t-t')], \quad (81)$$

$$H_j{}^0=c(\boldsymbol{\alpha}_j\cdot\mathbf{p}_j)+\beta_jm_jc^2. \quad (82)$$

In the approximation we are using

$$[A_{j,\text{at}}{}^\mu(x)]_{\text{Schröd}}=ce_j\int_{-\infty}^{+\infty}\exp[-(i/\hbar)H_j{}^0(t-t')]$$
$$\times[\alpha_j{}^\mu\delta(x,x_j)]_{\text{Schröd}}$$
$$\times\exp[(i/\hbar)H_j{}^0(t-t')]dt'. \quad (83)$$

Let us denote by H_d the direct interaction between the particles in the Schrödinger picture

$$H_d=\tfrac{1}{2}\sum_j\sum_{l\neq j} e_j[\alpha_{j,\mu}A_{l,\mathrm{at}}{}^{\mu}(x_j)]_{\mathrm{Schröd}},\ (x_j{}^0=ct). \quad (84)$$

In order to derive Möller's formula for the matrix elements of the interaction between the particles, we shall consider two states of the system particles plus field $\Psi^{(r)}$ $(r=1, 2)$, whose wave functions are of the form

$$\Psi^{(r)}=\textstyle\prod_j \psi_j{}^{(r)}\psi_f. \quad (85)$$

The $\psi_j{}^{(r)}$ are solutions of the Dirac equations

$$H_j{}^0\psi_j{}^{(r)}=E_j{}^{(r)}\psi_j{}^{(r)}, \quad (86)$$

representing free particles with momenta $\mathbf{p}_j{}^{(r)}$. ψ_f is the wave function of the radiation field; we assume that ψ_f does not depend on the variables of the particles. By taking into account that

$$\exp[-(i/h)H_j{}^0(t-t')]\psi_j{}^{(r)} =\exp[-(i/h)E_j{}^{(r)}(t-t')]\psi_j{}^{(r)}, \quad (87)$$

we get the first approximation value of the matrix element of H_d:

$$(1\,H_d|2)=-\tfrac{1}{2}h^{3n}(h\pi)^{-1}\sum_j\sum_{l\neq j} e_je_l(1|\alpha_j{}^{\mu}\alpha_{l,\mu}|2)_{\mathrm{Schröd}} \times\delta(\mathbf{p}_j{}^{(1)}-\mathbf{p}_j{}^{(2)}+\mathbf{p}_l{}^{(1)}-\mathbf{p}_l{}^{(2)})\prod_{k\neq l,j}\delta(\mathbf{p}_k{}^{(1)}-\mathbf{p}_k{}^{(2)}) \times\left[\frac{1}{c^2}(E_j{}^{(1)}-E_j{}^{(2)})^2-|\mathbf{p}_j{}^{(1)}-\mathbf{p}_j{}^{(2)}|^2\right]^{-1}, \quad (88)$$

corresponding to a transition between the two states (Möller's formula).

IX. THE "FREEZING" OF THE DEGREES OF FREEDOM OF THE FIELD

In our formalism the number of degrees of freedom of the field corresponding to each wave number vector $\mathbf{k}$ is larger than in the Heisenbergs Pauli electrodynamics, because we have photon with negative energies. In this point our formalism is analogous to those introduced by Dirac.[8,9] There are, however, essential differences between Dirac's treatments and ours. Dirac uses the Wentzel potentials of the Heisenberg-Pauli field and introduces redundant variables by splitting the Heisenberg-Pauli field into two parts, whereas we use the potentials of the radiation field. In Dirac's formalism the interaction between the particles results from their interactions with the Heisenberg-Pauli field, whereas in our theory the main part of the interaction between the particles cannot be derived from their interactions with the radiation field.

Let us introduce the variables $p_s{}^{\pm}(k)$ and $q_s{}^{\pm}(k)$:

$$\left.\begin{aligned} Q_s{}^{(+)}(k)+Q_s{}^{(-)}(k)&=p_s{}^+(k)+ip_s{}^-(k)\\ \bar{Q}_s{}^{(+)}(k)+\bar{Q}_s{}^{(-)}(k)&=p_s{}^+(k)-ip_s{}^-(k)\end{aligned}\right\}(s=1,2); \quad (89)$$

$$\left.\begin{aligned} Q_s{}^{(+)}(k)-Q_s{}^{(-)}(k)&=q_s{}^-(k)-iq_s{}^+(k)\\ &=2Q_{\mathrm{rad},s}(k)\\ \bar{Q}_s{}^{(+)}(k)-\bar{Q}_s{}^{(-)}(k)&=q_s{}^-(k)+iq_s{}^+(k)\\ &=2\bar{Q}_{\mathrm{rad},s}(k)\end{aligned}\right\}. \quad (90)$$

The commutation rules for the new variables are

$$[q_s{}^+(k),\,p_{s'}{}^+(k')]=[q_s{}^-(k),\,p_{s'}{}^-(k')] =i\delta_{ss'}\delta_{kk'}, \quad (91)$$

$$[q_s{}^{\pm}(k),\,q_{s'}{}^{\pm}(k')]=[p_s{}^{\pm}(k),\,p_{s'}{}^{\pm}(k')]=0, \quad (92)$$

$$[q_s{}^+(k),\,p_{s'}{}^-(k')]=[q_s{}^-(k),\,p_{s'}{}^+(k')]=0. \quad (93)$$

It results from Eqs. (90) *that the interaction between a particle and the radiation field does not involve the variables* $p_s{}^{\pm}$.

It is easily seen that

$$N_s{}^{(+)}(k)-N_s{}^{(-)}(k) =p_s{}^+(k)q_s{}^-(k)-p_s{}^-(k)q_s{}^+(k). \quad (94)$$

We can take as variables of the radiation field in the wave function Ψ the $p_s{}^{\pm}(k)$. If the wave function Ψ does not depend on the $p_s{}^{\pm}(k)$ corresponding to a definite value of $\mathbf{k}$, we will have

$$[N_s{}^{(+)}(k)-N_s{}^{(-)}(k)]\Psi=0, \quad (95)$$

$$[\mathbf{Q}_{\mathrm{rad},T}(k)\exp(-ik^{\rho}x_{\rho}) +\bar{\mathbf{Q}}_{\mathrm{rad},T}(k)\exp(ik^{\rho}x_{\rho})]\Psi=0. \quad (96)$$

Therefore, in the state described by such a wave function the degrees of freedom of the field corresponding to the value of $\mathbf{k}$ *do not give any contribution to the energy of the field and to the interaction with the particles.* We shall call "frozen" degrees of freedom of the field the degrees of freedom corresponding to values of $\mathbf{k}$ and s whose

$p_s^{\pm}(k)$ are not involved in the wave function Ψ of the system field plus particles.

The numbers of positive and negative energy photons corresponding to a "frozen" set of values of $\mathbf{k}$ and s are not determined but they are equal, as it results from (95).

It is obviously possible to impose on the solution of the Schrödinger equation of the system field plus particles the condition of not involving the $p_s^{\pm}(k)$ corresponding to as many values of $\mathbf{k}$ as we want, because the freezing conditions

$$Q_{\text{rad.}\,s}(k)\Psi=0, \quad \bar{Q}_{\text{rad.}\,s}(k)\Psi=0 \tag{97}$$

will be satisfied at any time if they are fullfiled for $t=0$. *We could, for instance, "freeze" all the degrees of freedom of the radiation field and thus obtain solutions of the Schrödinger equation representing states of motion in which the particles do not interact with the radiation field. The existence of such solutions is of fundamental importance because it shows that the existence of negative energy photons in our formalism does not exclude the existence of stationary states of motion of the particles.*

The "freezing" of degrees of freedom opens new possibilities in the quantum theory of fields. Thus, for instance, if we consider the one-electron problem, we will get an infinite self-energy if we define the vacuum as the state of the field in which there are no positive or negative energy photons. But, if we define the vacuum as the state of the field in which all its degrees of freedom are "frozen," we will get a vanishing self-energy. *It is also possible to get a finite self-energy by "freezing" only the degrees of freedom corresponding to frequencies above a certain limit (in the rest system of the electron which is supposed to have a definite momentum).*

X. THE PHYSICAL INTERPRETATION RULE

In the computation of the probability of a transition in which are involved r photons (the photons which appear in the initial and final states) with momenta $\hbar\mathbf{k}_r$ ($r=1, 2, \cdots l$), we shall use a wave function of the system particles plus field involving only the field variables corresponding to $\mathbf{k}$'s in the relevant ranges ($\mathbf{k}_r-\frac{1}{2}\Delta\mathbf{k}$, $\mathbf{k}_r+\frac{1}{2}\Delta\mathbf{k}$), $\Delta\mathbf{k}$ being a small vector with a length of the order of the line breadths. The degrees of freedom corresponding to values of $\mathbf{k}$ outside the relevant ranges will be "frozen." The "frozen" wave function will satisfy the Schrödinger equation of the system but it will also satisfy the reduced Schrödinger equation, which is obtained by dropping in the Hamiltonian all the terms corresponding to "frozen" degrees of freedom. We shall assume that: *The reduced Schrödinger equation must be treated as the wave equation of a system with a finite number of degrees of freedom. Its solutions will be normalized as if the "frozen" degrees of the field would not exist.* It would not be possible to normalize the "frozen" wave function by treating it as the wave function of a system with an infinite number of degrees of freedom, but it can be normalized as the wave function of a system with a finite number of degrees of freedom.

We shall now take as field variables in the "frozen" wave function the $N_s^{(\pm)}(k)$ corresponding to the "unfrozen" degrees of freedom. Let the initial and final states of the field in the physical world be characterized by the occupation numbers $n_i(k, s)$ and $n_f(k, s)$. We shall associate to that transition in the physical world the transition of the formalism between the states characterized by the occupation numbers $[N_s^{(\pm)}(k)]_i$ and $[N_s^{(\pm)}(k)]_f$.

$$[N_s^{(+)}(k)]_i=n_i(k, s), \quad [N_s^{(-)}(k)]_i=0, \tag{98}$$

$$[N_s^{(+)}(k)]_f=n_f(k, s), \quad [N_s^{(-)}(k)]_f=0. \tag{99}$$

We shall denote by N the total number of emitted and absorbed photons in the physical world. The physical interpretation rule is the following one:

The probability of the transition $n_i \rightarrow n_f$, in the physical world, is equal to the probability of the transition $[N]_i \rightarrow [N]_f$ in the formalism—computed according to the usual quantum rules—multiplied by 2^N.

The preceding interpretation rule does not use the negative energy photons. It is possible to give an interpretation rule in which no use is made of the positive energy photons, by interpreting the emission of a negative energy photon as an absorption in the physical world, and conversely the absorption of a negative energy photon as an emission in the physical world. It is also possible to give an interpretation rule in which the absorptions and emissions of photons in the physical world do correspond to emissions of negative energy and positive energy photons, respectively, in the formalism. The arbitrariness in the physical interpretation rule is analogous to that existing in Dirac's form of the Heisenberg-Pauli electrodynamics (see W. Pauli[12]). The interpretation rule we have formulated is the simplest one and can be justified by a correspondence argument as we shall see.

The existence of positive and negative energy photons in our quantum formalism gives rise to the multiplicity of the possible interpretation rules and seems to be an inconvenience of the theory. The negative energy photons are not introduced by a device of quantization, their existence is a consequence of the fact that the classical expression of the energy of the radiation field is not necessarily positive. The multiplicity of the possible physical interpretations will presumably disappear when the problem of the negative energy states for both electrons and photons will be better understood. We have shown that in classical theory it is necessary to assume that a particle with negative kinetic energy creates an advanced field,[1,2] in order that the reaction of radiation be a damping force, because the advanced field gives rise to a Larmor "gain." In quantum theory such a Larmor "gain" corresponds to an emission of negative energy photons, therefore, it may be unavoidable to introduce negative energy photons, because of the existence of negative energy states for the electrons.

Our physical interpretation rule was chosen by a criterion of simplicity, but it corresponds to the classical time boundary conditions. In the classical theory of the point electron the time boundary conditions are introduced in order to get rid of the motions with indefinite self-acceleration of the particles, as it was shown by Dirac[13] and Schönberg.[1,2] The indefinite self-acceleration is due to an indefinite decrease of the potential energy of a particle with respect to the radiation field, which is included in the energy of the radiation field. The classical time boundary conditions select the motions in which the potential energy of a particle with respect to the radiation field (acceleration energy) remains constant on the average, so that on the average there is an increase of the energy of the radiation field in the physically possible motions, because of the Larmor loss of the particles. *The emission of negative energy photons is the quantum equivalent of the classical self-acceleration and, thus, our physical interpretation which does not take into account the emission and absorption of negative energy photons corresponds to the classical exclusion of the self-accelerations on the average by means of the time boundary conditions.*

[12] W. Pauli, Rev. Mod. Phys. **15**, 175 (1943).

[13] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

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