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CLASSICAL THEORY OF THE POINT ELECTRON

Part II

Stationary motions

1. The Lorentz-Dirac equations of motion do not describe satisfactorily all kinds of motion of charged particles, in particular they do not allow for radiationless accelerated motions. These motions are called stationary in quantum theory and we will conserve the same denomination in their classical theory. Some of the motions investigated by Dirac (1938) and Eliezer (1943), in which the Lorentz-Dirac equations lead to physically meaningless results, belong to the stationary type.

It may look absurd to assume the existence of radiationless accelerated motions of charged particles in classical theory, there is, however, a very simple consideration which indicates that the existence of classical stationary states results precisely from the existence of the quantum ones. Indeed, there will be quantum stationary states for any value we give to the Planck constant h, therefore it is certain that such radiationless states will exist when we take $h=0$, provided we assume that it is possible to go over continuously from classical to quantum theory, by making h tend to 0.

It was shown in the first part of this paper that the radiation losses and the acceleration momentum result both from the interaction between a particle and the radiation field. Moreover, if we assume that in certain states of motion there are no radiation losses there will be no acceleration energy either, since both can nct be separated, as it was shown in section 3 of Part I. Therefore, in stationary states of motion the partiticles do not generate radiated fields, so that *the total field of a particle in stationary motion coincides with the attached field*. But we have seen in section 13 of Part I that attached fields give rise to actions at a distance, hence:

The forces between the particles of a system in stationary motion are pure actions at a distance.

If we assume that there are no radiated fields, the generalized Tetrode-Fokker principle, equations (74)-(75) of Part I, applied to a system under no external influence becomes the original Tetrode-Fokker principle:

$$\Delta L^* = 0 \tag{1}$$

$$L^* = -\sum_j m_j c \int_{-\infty}^{+\infty} ds_j -$$

$$-\frac{1}{2}\sum_j \sum_{i \neq j} \frac{e_i e_j}{c} \iint_{-\infty}^{+\infty} \frac{dx_i^\mu}{ds_i} \frac{dx_{j,\mu}}{ds_j} \, \delta\left(\{x_i^\rho - x_j^\rho\}\{x_{i,\rho} - x_{j,\rho}\}\right) ds_i \, ds_j \tag{2}$$

Instead of the Lorentz-Dirac equations of motion we have now

$$m_j c \frac{d^2 x_j^\mu}{ds_j^2} = \frac{e_j}{c} \sum_{i \neq j} F_{i,at}^{\mu\rho} \frac{dx_{j,\rho}}{ds_j} \tag{3}$$

It is worthwhile to remark that the Maxwell equations are satisfied by the attached fields created in stationary motions, since the attached and retarded fields are solutions of the same Maxwell system.

In stationary motions there are no difficulties due to the acceleration energy because there is no radiated field and the acceleration energy, which is a potential energy of the particle in relation to its own radiated field, vanishes.

2. In the preceding section we derived the equations of motion of a system in stationary motion by using the field theory of the forces. But, since the forces between the particles of a system in stationary motion are actions at a distance, it is presumable that the theory of the stationary motions can be developped without using field ideas. We will see that it is indeed possible to get directly the original Tetrode-Fokker action principle by a suitable extension of the Hamilton principle of non relativistic dynamics based on the following assumptions:

1) In a stationary state of motion of a system each particle acts on the others but not on itself.

2) The force acting on each particle depends on its velocity but not on derivatives of order above unity of its coordinates.

3) The forces between the particles may be advanced or retarded, or of both types, but "propagate" themselves in both directions of time with the velocity of light.

4) The forces exerted on each particle by the others are independent and add up vectorially.

5) The force exerted on a particle by another may depend on different instants of the motion of this particle, in such a way that the parts due to different instants of the motion add simply up.

6) There is a hamiltonian action principle for each particle and a hamiltonian action principle for the whole system.

7) All the potentials appearing in the action integrals are to be considered as functions of the particles' variables and variated accordingly.

8) The potentials do not involve the masses of the particles.

From assumption (6) it results that there is a potential function Ψ_j for the force acting on the particle Q_j, such that the equations of motion are given by the hamiltonian action principle:

$$\delta \int_{-\infty} \left(m_j c^2 \sqrt{1 - \beta_j^2} + \Psi_j \right) dt = 0 \tag{4}$$

From assumption (2) it results that the Ψ_j are linear functions of the components of the velocity of Q_i:

$$\Psi_j = e_j^* \sqrt{1 - \beta_j^2} \frac{dx_j^\mu}{ds_j} \sum_{i \neq j} A_{i,\mu} (x_j) \tag{5}$$

The numbers e_j^* are characteristic coefficients of the Q_j. The summation in (5) takes account of assumptions (1) and (4). It will be assumed from now on, for simplicity's sake, that there are only two particles in the system. Since there is a hamiltonian action principle for the system, according to assumption (6), there must be an integral L^* such that the equations of motion of the particles of the system result from the variational equation:

$$\Delta L^* = 0 \tag{6}$$

L^* is a sum of three terms

$$L^* = L_1^* + L_2^* + L_{12}^* \tag{7}$$

L^* depending only on the variables of Q_j and L_{12}^* corresponding to the interaction between the two particles of the system. Taking in account (4) and (5) it follows that, by neglecting eventually divergence expressions and a constant factor, we may take:

$$L_j^* = - m_j c \int_{-\infty}^{+\infty} ds_j \tag{8}$$

$$L_{12}^* = - \frac{e_1^*}{c} \int_{-\infty}^{+\infty} \frac{dx_1^\mu}{ds_1} A_{2,\mu}\, ds_1 = - \frac{e_2^*}{c} \int_{-\infty}^{+\infty} \frac{dx_2^\mu}{ds_2} A_{1\mu}\, ds_2 \tag{9}$$

From assumption (5) it results that A_j^μ may be represented by an integral taken along the world line of Q_j. Taking in account (9) we get

$$A_j^{\mu}(x) = \int_{-\infty}^{+\infty} a_j^{\mu\nu}(x_j, x) \frac{dx_{j\nu}}{ds_j}\, ds_j \tag{10}$$

the $a_j^{\mu\nu}(x_j, x)$ being some functions of the x_j^{ρ} and x^{ρ}. But, since the forces "propagate" themselves with the velocity of light, $a_j^{\mu\nu}$ must be some δ — like function vanishing everywhere on the world line of Q_j, excepting at most the two zeros of $(x_j^{\mu} - x^{\mu})(x_{j,\mu} - x_{\mu})$. Therefore, $A_j^{\mu}(x)$ is a sum of two four vectors built respectively with the variables of Q_j taken at the points where $(x_j^{\mu} - x^{\mu})(x_{j,\mu} - x_{\mu})$ vanishes:

$$A_j^{\mu} = A_{j,ret}^{\mu} + A_{j,adv}^{\mu} \tag{11}$$

But, the only four vectors with the dimensions of a potential that we can build with the variables of Q_j, at a point of its world line, and the coordinates x^{μ} of a point P are of the form

$$\alpha_j \frac{e_j^* \dfrac{dx_j^{\mu}}{ds_j}}{\dfrac{dx_j^{\rho}}{ds_j}(x_{\rho} - x_{j,\rho})} \tag{12}$$

α_j being an arbitrary dimensionless constant, because of assumption (8). Hence, we must have

$$A_j^{\mu}(x) = \alpha_j^{(1)} \left[\frac{e_j^* \dfrac{dx_j^{\mu}}{ds_j}}{\dfrac{dx_j^{\rho}}{ds_j}(x_{\rho} - x_{j,\rho})} \right]_{ret} + \alpha_j^{(2)} \left[\frac{e_j^* \dfrac{dx_j^{\mu}}{ds_j}}{\dfrac{dx_j^{\rho}}{ds_j}(x_{j,\rho} - x_{\rho})} \right] \tag{13}$$

or, in integral representation

$$A_j^{\mu}(x) = 2\, e_j^* \alpha_j^{(1)} \int_{-\infty}^{s_j'} \frac{dx_j^{\mu}}{ds_j}\, \delta\left(\{x_j^{\rho} - x^{\rho}\}\{x_{j,\rho} - x_{\rho}\}\right) ds_j +$$

$$+ 2 e_j^* \alpha_j^{(2)} \int_{s'}^{\infty} \frac{dx_j^\mu}{ds_j} \, \delta \left(\{x_j^\rho - x^\rho\} \{x_{j,\rho} - x_\rho\} \right) ds_j \tag{14}$$

In order to fulfill the symetry condition (9) we must take

$$\alpha_1^{(1)} = \alpha_1^{(2)} = \alpha_2^{(1)} = \alpha_2^{(2)} = \alpha \tag{15}$$

so that

$$\Psi_j = e_j \sqrt{1 - \beta_j^2} \, \frac{dx_{j,\mu}}{ds_j} A_{i,at}^\mu \tag{16}$$

$$e_j = 2 \, \alpha \, \overset{*}{e_j} \tag{16a}$$

The expression (16) of Ψ_j shows that the equations of motion are of the form (3). The action integral L^* can be put in the form given by equation (2).

3. The simplest case of stationary motion is that of an isolated particle, in which equations (3) become

$$\frac{d^2 x^\mu}{ds^2} = 0 \tag{17}$$

so that

$$\frac{dx^\mu}{ds} = const. \tag{18}$$

The preceding result shows that the principle of inertia for a charged particle can be taken as

The motion of an isolated point charge is stationary.

In this way we get rid of the self acceletated free motions allowed by the Lorentz-Dirac equations of motion.

The stationary Kepler problem

4. Let us consider a system of two charged particles, one of them having an infinite mass and the other a finite mass. The particle with infinite mass is in uniform rectilinear motion, so that it is possible to choose a Lorentz reference frame in which the heavy particle is at rest. In this reference frame the equations of motion of the light particle are:

$$\frac{d}{dt}\frac{m\mathbf{v}_1}{\sqrt{1-\beta_1^2}} = -\frac{e_1 e_2}{r_{12}^3}\,\mathbf{r}_{12} \tag{19}$$

e_1 and e_2 being respectively the charges of the light and heavy particles; $\mathbf{v}_1$ is the light particles velocity. Equations (19) are precisely the equations studied by Sommerfeld in the old quantum theory of the atom, hence:

There are classical stable orbits in the electromagnetic Kepler problem.

This result suggests that there may be a classical theory of non collapsible atoms. Of course this point can only be settled by a study of the stationary two body problem, taking in account the recoil of the heavy particle. It is, however, very probable that such non – collapsible atomic models do exist because there is conservation of energy in stationary motions, as it was already shown by Fokker (1929).

Conservation of energy and momentum

5. Fokker has shown that, in motions described by the action principle (1) – (2), there are no permanent losses of energy or momentum.

In order to get an intuitive image of the conservation laws in stationary motions, it is convenient to compare the mathematical formalism of their theory with the very similar one of a system of four dimensional interacting flexible strings. The double integrals which appear in L^* correspond to the interaction between the strings, and, as in the case of the strings, in order to get conservation of energy it is necessary to consider the entire lines, because the energy can be distributed and move along the strings or world lines.

Let us transcribe the equations of motion (3) expressing the $F_i^{\mu\nu}$ by means of the potentials $A^{\mu}_{i,at}$:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^2} = \frac{e_j}{c} \sum_{i \neq j} \frac{dx_j^\nu}{ds_j} \left(\frac{\partial A_{i,at;\nu}}{\partial x_j^\mu} - \frac{\partial A_{i,at;\mu}}{\partial x_j^\nu} \right) =$$

$$= - \frac{e_j}{c} \sum_{i \neq j} \left(\frac{dA_{i,at;\mu}}{ds_j} - \frac{dx_j^\nu}{ds_j} \frac{\partial A_{i,at;\nu}}{\partial x_j^\mu} \right) \tag{20}$$

Hence:

$$\frac{d}{ds_j} \left(m_j c \frac{dx_{j\mu}}{ds} + \frac{e_j}{c} \sum_{i \neq j} A_{i,at;\mu} \right) = \frac{e_j}{c} \sum_{i \neq j} \frac{dx_j^\nu}{ds_j} \frac{\partial A_{i,at;\nu}}{dx_j^\mu} \tag{21}$$

We do not have ordinary laws of conservation of energy and momentum because the expression in the right-hand side of (21) is no exact differential. In order to get the conservation laws, let us observe that

$$A^{\nu}_{i,at}(x_j) = e_i \int_{-\infty}^{+\infty} \frac{dx_i^\nu}{ds_i} \delta\left(\{x_i^\rho - x_j^\rho\} \{x_{i,\rho} - x_{j,\rho}\} \right) ds_i \tag{22}$$

and consider the integral E_{pot}

$$E_{pot} = \frac{1}{2} \sum_j \sum_{i \neq j} \frac{e_i e_j}{c} \int\int_{-\infty}^{+\infty} \delta\left(\{x_i^\rho - x_j^\rho\} \{x_{i,\rho} - x_{j,\rho}\} \right) \frac{dx_i^\mu}{ds_i} \frac{dx_{j,\mu}}{ds_j} ds_i \, ds_j \tag{23}$$

We can put equations (20) in the form:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^2} = \frac{\delta E_{pot}}{\delta x_j^{\mu}(s_j)} \tag{24}$$

Equations (24) put in evidence the conservative character of the stationary motions, they are a straightforward generalization of the equations of motion of a conservative system of non relativistic dynamics. Indeed, let us consider a system of points of non relativistic dynamics acting on each other by forces deriving from a potential Ψ, wich may depend on derivatives of any order of the coordinates. It can be seen easily that the equations of motion of such a system are

$$m_j \frac{d^2 x_{j,a}}{dt^2} = \frac{\delta}{\delta x_j^a} \int_{-\infty}^{+\infty} \Psi \, dt \quad (a=1,2,3) \tag{25}$$

because the components of the force acting on the j-th particle are

$$F_j^a = -\frac{\partial \Psi}{\partial x_j^a} + \frac{d}{dt}\left(\frac{\partial \Psi}{\partial \dot{x}_j^a}\right) - \frac{d^2}{dt^2}\left(\frac{\partial \Psi}{\partial \ddot{x}_j^a}\right) \cdots \tag{26}$$

E_{pot} is, therefore, analogous of the time integral of the potential Ψ, which is the potential energy in the case of purely positional forces. In the relativistic case there is no analogon of Ψ but only of Ψ's time integral. This is the reason why the conservation laws of the theory of stationary motions do not have the usual form but must be taken as average conservation laws.

It is worthwhile to remark that E_{pot} is a relativistic generalization of the magnetic potential energy of a system of linear electric currents.

6. The preceding considerations can be extended to the case of non stationary motions and to the Frenkel-Synge theory (Frenkel 1925, Synge 1940).

The Lorentz-Dirac equations of motion are:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^2} = \frac{e_j}{e} \sum_{i \neq j} \frac{dx_j^\nu}{ds_j} \left(\frac{\partial A_{i,at;\nu}}{\partial x_j^\mu} - \frac{\partial A_{at;\mu}}{\partial x_j^\nu} \right)$$

$$+ \frac{e_j}{e} \sum_i \frac{dx_j^\nu}{ds_j} \left(\frac{\partial A_{i,rad;\nu}}{\partial x_j^\mu} - \frac{\partial A_{i,rad,\mu}}{\partial x_j^\nu} \right) \tag{27}$$

Therefore:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^2} = \frac{\delta E_{pot}}{\delta x_j^\mu (s_j)} + \frac{e_j}{c} \frac{dx_j^\nu}{ds_j} F_{rad,\mu\nu} \tag{28}$$

The dissipative character of the non stationary motion arises from the circumstance that the expressions

$$\frac{e_j}{c} \frac{dx_j^\nu}{ds_j} F_{rad,\mu\nu}$$

are not functional derivatives of a function of the world lines of the particles. This can be easily seen representing the A^μ_{rad} by means of integrals taken along the world lines of the particles and computing the functional derivatives of the second order.

In the Frenkel-Synge theory it is assumed that no part of a particle's field reacts on it and each particle acts on the others by means of its retarded field. The corresponding equations of motion are:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^2} = \frac{e_j}{c} \sum_{i \neq j} \frac{dx_j^\nu}{ds_j} (F_{i,at;\mu\nu} + F_{i,rad;\mu\nu}) \tag{29}$$

Hence:

$$m_j c \frac{d^2 x_{j,\mu}}{ds_j^{\,2}} = \frac{\delta E_{pot}}{\delta x_j^{\mu}(s_j)} + \frac{e_j}{c} \sum_{i \neq j} F_{i,rad;\mu\nu} \frac{dx_j^{\nu}}{ds_j} \tag{30}$$

It can be seen that, now, the system is not conservative, in the same way as in the case of the Lorentz-Dirac equations of motion. In the Frenkel-Synge theory the Larmor loss disappears, because it is due to the interaction between a particle and its own radiated field, but there is still a loss due to the interaction bwtween a particle and the radiated fields of the others.

7. In section 1 the theory of the stationary motions was developed within the frame of the field theory. Now we will complete the field theory of the stationary motions by introducing the stress tensor of the field. The stress tensor can be defined, as in the case of non stationary motions:

$$4\pi T^{\mu}_{\nu} = F^{\mu\rho} F_{\rho\nu} + \frac{1}{4} \delta^{\mu}_{\nu} (F^{\rho\sigma} F_{\rho\sigma}) -$$

$$- \sum_j F^{\mu\rho}_{j,at} F_{j,at;\rho\nu} - \frac{1}{4} \delta^{\mu}_{\nu} \left(\sum_j F^{\rho\sigma}_{j,at} F_{j,at;\rho\sigma} \right) \tag{31}$$

$$F^{\rho\nu} = \sum_j F^{\rho\nu}_{j.at} \tag{32}$$

We get, as in the case of non stationary motions, the conservation equations:

$$\frac{\partial T^{\mu}_{\nu}}{\partial x^{\mu}} = - \sum_i \sum_{j \neq i} F_{j,at;\nu\rho} \frac{dx_i^{\rho}}{ds_i} e_i \, \delta(\mathbf{r} - \mathbf{r}_i) \sqrt{1 - \beta_i^2} \tag{33}$$

There is a four vector of energy and momentum of the field G^{μ}_{f}

$$G^{\mu}_{f} = \int T^{o\mu} d\tau \tag{34}$$

which satisfies the equations:

$$\frac{dG_f^{\mu}}{dt} = - c \sum_i \sum_{j \neq i} F_{j,at}^{\mu\rho} \frac{dx_{i,\rho}}{ds_i} e_i \sqrt{1 - \beta_i^2} -$$

$$- \lim_{\Sigma \to \infty} \sum_{\rho=1}^{3} \int_{\Sigma} c \, T^{\mu\rho} u_\rho \, d\Sigma \tag{35}$$

u is the unitary vector on the outer normal to the surface Σ. The boundary integral in the right-hand side of equation (35) does not vanish, in general, it vanishes only in the average, as we shall see later, in the discussion of the boundary behaviour of the field. Thus the field theory of the stationary motions leads to the same conclusion regarding the conservation laws as the theory of actions at a distance: they must be considered as average laws. The relations between E_{pot} and G_f^0 are discussed in the appendix.

PART III

Behaviour of the fields at infinity

8. In order to examine the behaviour of the various kinds of fields at infinity, it will be assumed that the charges are distributed continuously and a four vector of charge and current density J^μ will be considered. This assumption does not introduce any restriction, since the case of discrete charges is the particular one corresponding to δ – like densities.

The retarded, advanced and attached potentials satisfy the same inhomogeneous d'Alembert equations:

$$\frac{\partial^2 A^\mu}{\partial x^\rho \partial x_\rho} = 4\pi J^\mu(x) \tag{36}$$

We must find the boundary conditions which individuate the three types of solutions:

$$A^\mu_{ret} = \int J^\mu \left(Q, t - \frac{r}{c}\right) \frac{d\tau}{r} \tag{37}$$

$$A^\mu_{adv} = \int J^\mu \left(Q, t + \frac{r}{c}\right) \frac{d\tau}{r} \tag{38}$$

$$A^\mu_{at} = \frac{1}{2} \int \left[J^\mu \left(Q, t - \frac{r}{c}\right) + J^\mu \left(Q, t + \frac{r}{c}\right) \right] \frac{d\tau}{r} \tag{39}$$

Equations (37) show that the contribution of each charge element to the retarded potential is an outgoing spherical wave and equations (38) show that the advanced field results from the superposition of incoming spherical waves, each charge element behaving as a sink. The attached field results from the superposition of both incoming and outgoing spherical waves and is symmetrical in relation to both. Thus we get an intuitive picture of the reason why charges radiate in non stationary motions, when they create retarded fields, and do not radiate in stationary motions, when they only create attached fields.

It is clear that the boundary conditions of a non stationary state are compatible with the existence of external waves, since these waves behave at the boundary as a superposition of outgoing spherical waves. But the boundary conditions of a stationary state must be symetrical in both outgoing and incoming spherical waves so that they do not allow for the existence of external waves.

9. In order to get the boundary conditions it is necessary to consider the general solution of the d'Alembert equations (36) given by the Kirchhoff-Lorenz formula (*):

$$A^{\mu} = \int_V \frac{J^{\mu}\left(Q,t-\frac{c}{r}\right)}{r}\,dt - \frac{1}{4\pi}\int_{\Sigma} \frac{1}{r}\,\frac{\partial A^{\mu}\left(Q,t-\frac{c}{r}\right)}{\partial n}\,d\Sigma \; +$$

$$+ \frac{1}{4\pi}\int_{\Sigma}\left[A^{\mu}\left(Q,t-\frac{r}{c}\right)\frac{\partial \frac{1}{r}}{dn} + \frac{1}{cr}\,\frac{\partial A^{\mu}\left(Q,t-\frac{r}{c}\right)}{\partial t}\right]d\Sigma \qquad (40)$$

There is a formula analogous to (40) in which appear advanced instead of retarded quantities:

$$A^{\mu} = \int_V \frac{J^{\mu}\left(Q,t+\frac{r}{c}\right)}{r}\,dt - \frac{1}{4\pi}\int_{\Sigma} \frac{1}{r}\,\frac{\partial A^{\mu}\left(Q,t+\frac{r}{c}\right)}{\partial n}\,d\Sigma \; +$$

(*) n is the inner normal to Σ.

$$+\frac{1}{4\pi}\int_{\Sigma}\left[A^{\mu}\left(Q,t+\frac{r}{c}\right)\frac{\partial\frac{1}{r}}{\partial n}-\frac{1}{cr}\frac{\partial A^{\mu}\left(Q,t+\frac{r}{c}\right)}{\partial n}\right]d\,\Sigma \qquad (41)$$

In formulae (40) and (41) V is the region of space enclosed by the surface Σ. From (40) and (41) we can get easily conditions which charcterize the advanced and retarded potentials:

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}+\frac{1}{c}\frac{\partial}{\partial t}\right)A^{\mu}_{ret}\left(\mathbf{r},\,t-\frac{r}{c}\right)=0 \qquad (42)$$

$$\lim_{r\to\infty} r\left(\frac{\partial}{\partial r}-\frac{1}{c}\frac{\partial}{\partial t}\right)A^{\mu}_{adv}\left(\mathbf{r},\,t+\frac{r}{c}\right)=0 \qquad (43)$$

From the definitions of the retarded and advanced potentials it follows that these conditions are also necessary, as we shall see. Conditions (42) and (43) generalize respectively Sommerfeld's "Ausstrahlungsbedingung" and "Einstrahlungsbedingung". (See Frank — v. Mises, Differentialgleichungen der Physik, vol. II, pg. 804, 1935).

10. In order to get a clearer understanding of the behaviour at infinity it is necessary to compute the Poynting vector $\mathbf{P}$ and its flux. For this purpose the fields may be reduced to their parts varying inversely with the distance, which are given by the formulae (*)

$$\mathbf{E}_{ret} = -\frac{1}{c}\dot{\mathbf{A}}^{tr}_{ret} \qquad \mathbf{H}_{ret} = [\mathbf{u}\times\mathbf{E}_{ret}] \qquad (44)$$

$$\mathbf{E}_{adv} = -\frac{1}{c}\dot{\mathbf{A}}^{tr}_{adv} \qquad \mathbf{H}_{adv} = [\mathbf{E}_{adv}\times\mathbf{u}] \qquad (45)$$

(*) See Pauli, Handbuch der Physik XXIV/1, 203, 1933.

$$\mathbf{A}^{tr}_{ret} = \frac{1}{R}\int \mathbf{J}^{tr}\left(Q, t - \frac{R}{c} + \frac{(\mathbf{u}.\mathbf{r}_Q)}{c}\right) d\tau \tag{46}$$

$$\mathbf{A}^{tr}_{adv} = \frac{1}{R}\int \mathbf{J}^{tr}\left(Q, t + \frac{R}{c} - \frac{(\mathbf{u}.\mathbf{r}_Q)}{c}\right) d\tau \tag{47}$$

$$\mathbf{J}^{tr} = \mathbf{J} - (\mathbf{J}.\mathbf{u})\,\mathbf{u} \tag{48}$$

$\mathbf{u}$ is the unit vector in the direction from a fixed point O, inside the charge distribution, to the point in the wave zone where the field is computed, R is the distance between O and this point; $\mathbf{r}_Q$ is the position vector $\mathbf{OQ}$ and $\mathbf{J}$ is the current density. divided by c.

In the wave zone we have expressions of the type (46) or (47) for the retarded and advanced potentials respectively, therefore conditions (42) and (43) are necessary as we said in section 9.

From (44) and (45) result the following values of the Poynting vectors $\mathbf{P}$ of the fields:

$$\left.\begin{aligned} \mathbf{P}_{ret} &= \frac{c}{4\pi} E^2_{ret}\,\mathbf{u} \\ \mathbf{P}_{adv} &= -\frac{c}{4\pi} E^2_{adv}\,\mathbf{u} \end{aligned}\right| \tag{49}$$

Equations (49) show that there is an outflow af energy in a retarded field and an inflow of energy in an advanced field.

The Poynting vector of a field $F^{\mu\nu}$ obtained by superposition of retarded and advanced fields

$$F^{\mu\nu} = a\, F^{\mu\nu}_{ret} + b\, F^{\mu\nu}_{adv} \tag{50}$$

is

$$\mathbf{P} = a^2\,\mathbf{P}_{ret} - b^2\,\mathbf{P}_{adv} \tag{51}$$

In the case of the attached field $a = b = \frac{1}{2}$ and so

$$\mathbf{P}_{at} = \frac{c}{16\pi} \left(E_{ret}^2 - E_{adv}^2 \right) \mathbf{u} \tag{52}$$

We will see in a moment that the time integrals of the square of the retarded and advanced eletric fields of the same charge distribution are equal. in the wave zone. Therefore:

$$\int_{-\infty}^{+\infty} \mathbf{P}_{at}\, dt = 0 \tag{53}$$

Equation (51) shows, by taking in account the aforementioned equality of the time integrals of the square of the retarded and advanced fields, that *the only linear combination of the advanced and retarded fields which does not lead to radiation losses and satisfies the Maxwell equations is the attached field.* Indeed, to get a vanishing time integral of the Poynting vector **P** we must take $a^2 = b^2$ and in order to satisfy the Maxwell equations we must have $a + b = 1$.

In order to compute the time integral of the Poynting vector it is convenient to represent **J** by a Fourier integral:

$$\mathbf{J} = \int_{-\infty}^{+\infty} \mathbf{I}(k)\, e^{-ickt}\, dk \tag{54}$$

The Fourier integrals of the advanced and retarded potentials and fields in the wave zone are:

$$\left.\begin{aligned} \mathbf{A}_{ret}^{tr} &= \frac{1}{R} \int\int_{-\infty}^{+\infty} \mathbf{I}^{tr}(k)\, e^{-ik[ct - R + (\mathbf{u}.\mathbf{r}_Q)]}\, d\tau\, dk \\ \mathbf{A}_{adv}^{tr} &= \frac{1}{R} \int\int_{-\infty}^{+\infty} \mathbf{I}^{tr}(k)\, e^{-ik[ct + R - (\mathbf{u}.\mathbf{r}_Q)]}\, d\tau\, dk \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} \mathbf{E}_{ret} &= \frac{i}{R}\int\!\!\int_{-\infty}^{+\infty} \mathbf{I}^{tr}(k)\, k e^{-ik[ct - R + (\mathbf{u}.\mathbf{r}_Q)]}\, d\tau\, dk \\ \mathbf{E}_{adv} &= \frac{i}{R}\int\!\!\int_{-\infty}^{+\infty} \mathbf{I}^{tr}(k)\, k e^{-ik[ct + R - (\mathbf{u}.\mathbf{r}_Q)]}\, d\tau\, dk \end{aligned}\right\} \quad (56)$$

It is well known that the time integral of the square of a quantity M represented by the Fourier integral

$$M(t) = \int_{-\infty}^{+\infty} e^{-ickt}\, m(k)\, dk \quad (57)$$

is given by the formula

$$\int_{-\infty}^{+\infty} M^2\, dt = \frac{2\pi}{c}\int_{-\infty}^{+\infty} m(k)\, m(-k)\, dk \quad (58)$$

Taking in account formulae (57) and (58) we see that, in the wave zone, the time integrals of the squares of the retarded and advanced fields are equal

$$\int_{-\infty}^{+\infty} \mathbf{E}_{ret}^2\, dt = \int_{-\infty}^{+\infty} \mathbf{E}_{adv}^2\, dt \quad (59)$$

From (51) and (59) it results that the time integral of Poynting vector of the radiated field $\mathbf{P}_{rad}$ vanishes:

$$\int_{-\infty}^{+\infty} \mathbf{P}_{rad}\, dt = \frac{c}{4\pi}\int_{-\infty}^{+\infty}\left[\mathbf{E}_{rad} \times \mathbf{H}_{rad}\right] dt = 0 \quad (60)$$

Conservation of the energy of the field and particles

11. Now we can complete the discussion of the conservation laws. We have seen in sections 6, 7 and 9 of Part I that in the non stationary states of motion there is no exact compensation between the energy lost by the field and gained by the particles, on account of certain fluxes at infinity, formulae (36), (42), (48) and (49) of Part I.

Formulae of the same form do exist in the case of stationary motions, for instance equation (35). Therefore *it cannot be be said that the enerqy and momentum of the system of the field and the charged particles is rigorously constant in any case*. Let us see wether thery fluctuate or there are permanent losses.

The surface integral that appears in equation (36) of Part I, and the analogous one of the case of stationary motions are:

$$U = \frac{c}{4\pi} \int_{\Sigma} \left([\mathbf{E} \times \mathbf{H}] - \sum_{i} [\mathbf{E}_{i,at} \times \mathbf{H}_{i,at}], \mathbf{n} \right) d\,\Sigma \tag{61}$$

Taking in account that the time integral of the Poynting vector of an attached field vanishes we get:

$$\int_{-\infty}^{+\infty} U\,dt = \frac{c}{4\pi} \int_{\Sigma} d\,\Sigma \int_{-\infty}^{+\infty} \left([\mathbf{E} \times \mathbf{H}], \mathbf{n} \right) dt \tag{62}$$

In the case of a stationary motion the total field is a sum of attached fields, therefore

$$\int_{-\infty}^{+\infty} U_{st}\,dt = 0 \tag{63}$$

Hence the energy oscillates. This result agrees with Fokker's and explains why there are no radiative losses in stationary motions.

The case of non stationary motions is more complicated. The time integral of the flux at infinity does not vanish necessarily because the Poynting vector points always outwards, unless the field at infinity falls off faster than r^{-1}. Let us consider a particular case in which this happens: a particle is in a state of uniform retilinear motion up to the time t_o, then gets accelereated and finally resumes a state of uniform motion at the time t_1. In such a case it is possible at any instant to find a closed surface dividing the space in two parts: inside the region which contains the particle — the field has a part varying as r^{-1}; outside the surface the field falls off as r^{-2}. The existence of this surface results from the finite velocity of propagation of the field disturbances and from the limited duration of the accelerated state of motion. Actually the preceding example is not a possible non — stationary state of motion; Dirac (1938) has shown that the particle will be already acceler a ted before it receives the external actions, so that the accelerated part of the motion will have an infinite duration; nevertheles the flux at infinity vanishes, because the corresponding retarded quantities vanish since the acceleration tends to zero for $t \to -\infty$.

The considerations developped in the case of a single particle can, of course, be extended to any system of particles in non stationary motion and show that:

There will be no difficulties arising from the conservation laws if it is assumed that the accelerations of the particles of systems in non stationary motions vanish for $t = -\infty$.

12. The analysis of the conservation laws led us to the introduction of a time boundary condition for $t = -\infty$. Dirac (1938) introduced a time boundary condition for $t = +\infty$, in order to determine completely the solution of the third order equations of motion, which are not yet completely determined by the knowledge of the initial positions and velocities. Dirac did not formulate his time boundary condition in the most general case, he considered only the case of a single particle which is under the action of external forces ceasing to act at a time t_o and imposed the condition that after t_o the acceleration of the particle vanishes.

We shall generalize Dirac's boundary condition in the following way

The accelerations of the particles of systems in non stationary motions which are under the action of external forces, tending to zero when t tends to $+\infty$, *tend also to zero for* $t = +\infty$.

This time boundary condition limits the decrease of the acceleration energy not allowing it to decrease indefinitely. If we apply this boundary condition to the motion of a free particle the self accelerating solutions will be discarded.

It may seem that it is not possible to introduce boundary conditions for both time boundaries $t = -\infty$ and $t = +\infty$. However, Dirac's, treatment of the motion of an electron which receives a light pulse shows that it is possible to do it in some cases which are precisely the only ones that have physical signification. It is possible to satisfy the time boundary conditions in motions of "hyperbolic" type in which the particles are infinitely separated for $t = -\infty$ and fly apart indefinitely when $t = \infty$.

Part IV

The "anti-particles"

13. Until now we have considered two kinds of motions of a charged particle: the stationary and non stationary motions, characterized by the kind of field which the particles create. In both the stationary and non stationary motions the part of a particle's field which reacts on it is the difference between the total field created by the particle and its attached field; the reacting part vanishes in stationary motions because in such states the total field coincides with the attached part. The stationary and non stationary motions we have considered, present a common feature: their kinetic energies are positive. By, reasons of symmetry we are led to look for motions in which the total field created by a particle is the advanced field and for motions with negative kinetic energy.

We will assume that in the states in which a particle generates an advanced field the part of the field which reacts on it is the difference between its total field and its attached field. This reacting part coincides, therefore, with minus the radiated field. of a non stationary state. The preceding assumption can be justified by an extension of the Hamilton principle analogous to that considered in section 14 of Part. I, by substituting the hypothesis that the total field created by a particle is the retarded field by the alternative hypothesis that it is the advanced field. We will call the part of a particle's field which reacts on it radiated part.

Equations of motion

14. When a particle generates an advanced field the reaction of the radiated field creates an acceleration momentum equal to minus the corresponding ones of the non stationary motion and a "Larmor gain", because the radiated field is now:

$$F^{\mu\nu}_{rad} = \frac{1}{2}\left(F^{\mu\nu}_{adv} - F^{\mu\nu}_{ret} \right) \tag{64}$$

The fact that there is now a gain, instead of a Larmor loss, is not surprising because the Poynting vector of an advanced field points inwards, in the wave zone, as it was shown in section 10. Such an inflow of energy is not contradictory with the conservation of energy, if we assume that *the acceleration of the particle vanishes for* $t=\infty$, in analogy with the corresponding assumption for the case of non stationary motions. Hence, if there are states in which a particle creates an advanced field its acceleration must vanish for $t=\infty$, while it is always gaining energy at the Larmor rate. If its kinetic energy would be positive such a gain would naturally tend to accelerate it (excluding physically insatisfactory increases of the acceleration energy). Therefore it is natural to assume that:

"*The total field created by a particle is the advanced field, when it is in a radiating state of motion with negative kinetic energy*".

We will call such motions non — stationary motions with negative kinetic energy and stationary motions with negative kinetic energy the radiationless ones.

The preceding assumption, together with (64), leads to the equations of motion by the same arguments of section 4, Part I:

$$\left.\begin{aligned}
&-\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right) + \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left\{\frac{\mathbf{v}\,\dot{\mathbf{v}}}{(1-\beta^2)^2}\right\} - \\
&\qquad -\frac{2}{3}\frac{e^2}{c^3}\frac{\dot{\mathbf{v}}^2 - \left[\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right]^2}{(1-\beta^2)^3} = (\mathbf{F}_{ext}\cdot\mathbf{v}) \\
&-\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\beta^2}}\right) + \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left\{\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-\beta^2}}\right)\right\} - \\
&\qquad -\frac{2}{3}\frac{e^2\mathbf{v}}{c^5}\frac{\dot{\mathbf{v}}^2 - \left[\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right]}{(1-\beta^2)^3} = \mathbf{F}_{ext}
\end{aligned}\right\} \tag{65}$$

$$\mathbf{F}_{ext} = e\left\{\mathbf{E}_{ext} + \left[\frac{\mathbf{v}}{c} \times \mathbf{H}_{ext}\right]\right\} \tag{66}$$

A particle with the same velocity and position but opposite charge would be under the action of the force $\mathbf{F}'_{ext}$:

$$\mathbf{F}'_{ext} = -\mathbf{F}_{ext} \tag{67}$$

The equations of motion (167) can be written as follows:

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\beta^2}}\right) - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left\{\frac{v\dot{v}}{(1-\beta^2)^2}\right\} + \frac{2}{3}\frac{e^2}{c^3}\frac{\dot{\mathbf{v}}^2 - \left[\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right]^2}{(1-\beta^2)^3} = (\mathbf{F}'_{ext}\cdot\mathbf{v}) \\ &\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\beta^2}}\right) - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left\{\frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt}\left(\frac{\mathbf{v}}{\sqrt{1-\beta^2}}\right)\right\} + \\ &+ \frac{2}{3}\frac{e^2\mathbf{v}}{c^5}\frac{\dot{\mathbf{v}}^2 - \left[\frac{\mathbf{v}}{c}\times\dot{\mathbf{v}}\right]^2}{(1-\beta^2)^3} = \mathbf{F}'_{ext} \end{aligned}\right| \tag{68}$$

Therefore:

"A particle in a non stationary state with negative kinetic energy moves in the same way as a particle with opposite sign and positive kinetic energy".

It should not be concluded from the preceding result that a particle with negative kinetic energy behaves altogether as a particle with positive kinetic energy and opposite sign, because the field it generates is not the same which a charge with the opposite sign would create. From now on we will call "anti-particle" a particle in a state of negative kinetic energy creating an advanced field.

The preceding analysis can be considred as a classical refinement of Dirac's reasonement which shows that a negative kinetic energy solution of the wave equation can be interpreted as describing a motion of a particle with opposite charge and positive kinetic energy (See Dirac, Quantum Mechanics, pg 270, 1935). Dirac's analysis does not take into account the reaction of radiation as ours does. We see that *Dirac's conclusions are still classically valid when the reaction of radiation is considered, if it is assumed that the anti-particle generates an advanced field. The advanced field, which results from a superposition of incoming waves, corresponds to the "hole" character of the positron.*

The flow of time for anti-particles

15. The theory of the anti-particles suggests a modification of some usual ideas concerning the flow of time. The time appears in the relativistic theories in two different forms:

1) As one of the variables describing the position of the events in the four dimensional universe.
2) As the proper time of a particle, measured by the length of the arc on its world line.

The real significance of time is connected with the length described on the world line. This can be seen very clearly in general relativity where it is not possible, in general, to choose x^o in such a way that its differentials coincide with the elements of proper time of the particles. If we consider the flow of time to be connected with the existence of a preferred direction on the world line of a particle, there is no reason to admit that it is always possible to choose x^o in such a way that

$$\frac{dx^o}{ds} > 0 \tag{69}$$

for all the particles in a flat space time. We may assume that there are states of motion of a particle in which (69) is satisfied and states in which

$$\frac{dx^o}{ds} < 0 \tag{70}$$

If the four vector of kinetic energy and momentum is defined by

$$G^{\mu} = m \frac{dx^{\mu}}{ds} \tag{71}$$

the kinetic energy will have the sign of $\frac{dx^o}{ds}$.

Assuming the preceding theory of the flow of time, the equations of motion of a particle will be always of the same form, both in non stationary states of positive and negative energy:

$$mc \frac{d^2x^{\mu}}{ds^2} = \frac{e}{c} (F^{\mu\nu}_{ext} + F^{\mu\nu}_{rad}) \tag{72}$$

These equations show clearly the equivalence between a change of orientation of the world line and a change of sign of the charge: in the factor that multiplies the fields, in the right hand side, the change of the signs of the charge and of ds have the same effect; the external field is not affected by the change of sign of ds or e but the change of either e or the orientation of the world line leads to a change of sign of the radiated field.

Now it is intuitive why a change of sign of the kinetic energy should be associated with a substitution of the retarded field by the advanced one:

The advanced field created by an anti-particle appears as a retarded field to an observer whose time flows in the same direction as the anti-particle's time.

Indeed, the advanced potential can be expressed by the integral

$$A^{\mu}_{adv}(z) = 2\,e \int_{t'}^{\infty} \frac{dx^{\mu}}{dt}\, \delta\left(\{x^{\rho}-z^{\rho}\}\, \{x_{\rho}-z_{\rho}\} \right) dt \qquad (73)$$

t being the proper time of the observer who sees the field created by the particle as an advanced field, t' is a value of t between the two zeros of the argument of the δ — function. An observer whose proper time τ flows in the opposite direction

$$\frac{d\tau}{dt} < 0 \qquad (74)$$

will obtain for the potentials of the same field the values:

$$A^{\mu}_{adv}(z) = -\,2\,e \int_{-\infty}^{\tau'} \frac{dx^{\mu}}{d\tau}\, \delta\left(\{x^{\rho}-z^{\rho}\}\, \{x_{\rho}-x_{\rho}\} \right) d\tau \qquad (75)$$

The expressions in the right hand side of (75) are precisely the retarded potentials for the second observer; the minus sign which apparently contradicts formula (84) of Part I arises from the circumstance that the argument of the δ function increases when τ varies from $-\infty$ to τ'.

The interpretation of negative kinetic energies as effects of a flow of time in two alternative directions leads naturally to the consideration of stationary motions with negative kinetic energy, in which the particles generate half-advanced, half-retarded fields. For an observer whose time flows in the same direction, the motion of such a system would be a stationary motion of the same kind of those studied in Part II, but for an observer with a proper time flowing in the opposite direction the motion will appear as a *stationary motion of a system of particles with opposite signs, because the attached fields are not changed by a change of the orientations of the world-lines of the respective particles.*

The conservation principles

16. Let us consider now the general case of a field containing n_I particles and n_{II} anti-particles. The stress tensor of the field can be defined as in section 7 of Part I:

$$4\pi T^{\mu}_{\nu} = F^{\mu\rho} F_{\rho\nu} + \frac{1}{4} \delta^{\mu}_{\nu} (F^{\rho\sigma} F_{\rho\sigma})$$

$$- \sum_{i=1}^{n_I + n_{II}} F^{\mu\rho}_{i,at} F_{i,at;\rho\nu} - \frac{1}{4} \delta^{\mu}_{\nu} \sum_{i=1}^{n_I + n_{II}} (F^{\rho\sigma}_{i,at} F_{i,at;\rho\sigma}) \qquad (76)$$

$$F^{\mu\nu} = \sum_{i=1}^{n_I} F^{\mu\nu}_{i,ret} + \sum_{i=n_I+1}^{n_I + n_{II}} F^{\mu\nu}_{i,adv} \qquad (77)$$

$F^{\mu\nu}$ is the tensor of the total field, sum of the retarded fields of the particles and advanced fields of the anti-particles; the particles correspond to the values of i from 1 to n_I and the anti-particles to the values of i from n_I+1 to n_I+n_{II}. As in section 7 of Part I we get:

$$\frac{\partial T^{\mu}_{\nu}}{\partial x^{\mu}} = \sum_i (F_{\rho\nu} - F_{i,at;\rho\nu})\, e_i\, \delta(\mathbf{r} - \mathbf{r}_i) \frac{dx^{\rho}_i}{ds_i} \sqrt{1-\beta_i^2} \qquad (78)$$

Integrating both sides of (78) over a volume V containing all the charges we get, by using the same notations of section 7 of Part I:

$$\frac{dG^{\nu}_f(V)}{dt} = \sum_i e_i (F^{\rho\nu} - F^{\rho\nu}_{i,at}) \frac{dx_{i,\rho}}{ds_i} \sqrt{1-\beta_i^2}$$

$$- \int_{\Sigma} \sum_{\mu=1}^{3} T^{\mu\nu} u_{\mu} \, d\Sigma \qquad (79)$$

$G_f^{\nu}(V)$ is the four vector of the energy and momentum of the volume V of the field:

$$G_f^{\nu}(V) = \int_v T^{0\nu} \, d\tau \qquad (80)$$

From (79) it follows that the rate of variation of the total energy of the field is (*):

$$\frac{dW_f}{dt} = -\sum_i e_i (\mathbf{F}_i \cdot \mathbf{v}_i) - \lim_{\Sigma \to \infty} \frac{c}{4\pi} \int_{\Sigma} \Big(\Big\{ [\mathbf{E} \times \mathbf{H}] -$$

$$- \sum_i [\mathbf{E}_{i,at} \times \mathbf{H}_{i,at}] \Big\} \cdot \mathbf{n} \Big) \, d\Sigma \qquad (81)$$

17. In order to get conservation of energy the contribution of the first term under the surface integral in (81) must vanish, at least in average. But now the total field is neither entirely retarded nor entirely advanced, it will nevertheless tend to O as r^{-2}, at infinity, if we assume that the accelerations of the particles vanish for $t = -\infty$ and the accelerations of the anti-particles for $t = \infty$, as we did respectively in sections 11 and 14. Indeed, the contributions of the particles to the field at infinity arise from their motions at $t = -\infty$ and the contributions of the anti-particles from their motions at $t = \infty$, since they create advanced fields.

The time boundary condition for the anti-particles at $t = \infty$ is of the same type as the boundary condition for the particles at $t = -\infty$ and is even identical to it for an observer for whom the direction of time-flow coincides with that of the anti-particles. In the same way as for the particles we must introduce another time boundary condition for the anti-particles, analogous to the generalized Dirac condition of section 12:

The non stationary motions of an isolated system of anti-particles go over into non accelerated ones either in a finite time or asymptotically for $t = -\infty$.

(*) $\mathbf{F}_i$ is the force acting on the i-th particle.

Part V

General principles of the theory

18. In the four preceding parts of this paper we considered three different kinds of fields created by charged particles: retarded fields, advanced fields and attached fields. To each type of field corresponds a different type of motion: non stationary motions with positive kinetic energy, stationary motions and non stationary motions of anti-particles. All those fields satisfy the Mawxell equations but they are not the only relativistic solutions: any linear combination $F^{\mu\nu}$ of the retarded and advanced fields of the particles

$$F^{\mu\nu} = \sum (a_i\, F^{\mu\nu}_{i,ret} + b_i\, F^{\mu\nu}_{i,adv}) \tag{82}$$

will satisfy the field equations, provided:

$$a_i + b_i = 1 \tag{83}$$

Why do the particles not create other linear combinations of the retarded and advanced fields different from the retarded, advanced and attached fields ? Our theory does not give any answer to this question. To any choice of the total field created by a particle corresponds the stress tensor of the field

$$4\pi\, T^{\mu}_{\nu} = F^{\mu\rho} F_{\rho\nu} + \frac{1}{4} \delta^{\mu}_{\nu} (F^{\rho\sigma} F_{\rho\sigma})$$

$$- \sum_i F^{\mu\rho}_{i,at} F_{i,at;\rho\nu} - \frac{1}{4} \delta^{\mu}_{\nu} \left(\sum_i F^{\rho\sigma}_{i,at} F_{i,at;\rho\sigma} \right) \tag{84}$$

in which $F^{\mu\nu}$ is the total field defined by (82). From (84) we get:

$$\frac{dW_f}{dt} = - \sum_i (\mathbf{F}_i \, . \, \mathbf{v}_i) -$$

$$- \lim_{\Sigma \to \infty} \frac{c}{4\pi} \int_{\Sigma} \left(\mathbf{n} . \left\{ [\mathbf{E} \times \mathbf{H}] - \sum_i [\mathbf{E}_{i,at} \times \mathbf{H}_{i,at}] \right\} \right) d\Sigma \tag{85}$$

$\mathbf{F}_i$ being the force acting on the i-th particle of the system. As in the cases in which the charges create retarded, advanced or attached fields, in general there are no difficulties arising from the conservation of energy because the surface integral in (85) can be made nihil, in average or exactly, by imposing time boundary conditions, or even no condition at all, as in the case of stationary motions.

19. In the discussion of the three types of motions of charged particles many assumptions were made, now we will present a general formulation of the principles which puts more clearly in evidence the relations between all the particular hypothesis.

In order to formulate the general principles of the theory of point charges it is convenient to combine the ideas of action at a distance and action through the field, as it was already done in section 13 of Part I. We will assume the following principles:

I) The equations of motion of the fields and particles are given by the action principle:

$$\Delta L = 0 \tag{86}$$

$$L = - \sum_i m_i c \int_{-\infty}^{+\infty} ds_i -$$

$$- \frac{1}{2} \sum_j \sum_{i \neq j} \frac{e_i e_j}{c} \int\int_{-\infty}^{+\infty} \frac{dx_i^{\mu}}{ds_i} \frac{dx_{j,\mu}}{ds_j} \, \delta \left(\{ x_i^{\nu} - x_j^{\nu} \} \{ x_{i,\nu} - x_{j,\nu} \} \right) ds_i \, ds_j$$

$$-\sum_i \frac{e_i}{c} \int_{-\infty}^{+\infty} \frac{dx_i^\mu}{ds_i} A_{rad,\mu}\, ds_i - \frac{1}{16\pi} \int_{-\infty}^{+\infty} [F^{\rho\sigma} F_{\rho\sigma} - \sum_{i,j} F_{i,at}^{\rho\sigma} F_{i,at;\rho\sigma}]\, d\tau\, dt \quad (87)$$

in which $F^{\mu\nu}$ is

$$F^{\mu\nu} = \sum_i F_{i,at}^{\mu\nu} + \sum_i F_{i,rad}^{\mu\nu} + F_{wav}^{\mu\nu} \quad (88)$$

and the potentials satisfy the supplementary conditions:

$$\frac{\partial A_{i,at}^\mu}{\partial x^\mu} = 0 \qquad \frac{\partial A_{i,rad}^\mu}{\partial x^\mu} = 0 \qquad \frac{\partial A_{wav}^\mu}{\partial x^\mu} = 0 \quad (89)$$

II) The element of world line of a particle ds_i can be positive or negative:

$$ds_i = \pm \sqrt{dx_i^\mu\, dx_{i,\mu}} \quad (90)$$

III) The attached field of a point charge e_i is defined by the potentials $A_{i,at}^\mu$

$$A_{i,at}^\mu (x) = e_i \int \frac{dx_i^\mu}{ds_i}\, \delta\Big(\{x^\rho - x_i^\rho\}\{x_\rho - x_{i,\rho}\}\Big)\, ds_i \quad (91)$$

IV) The radiated field of a particle is either nihil or defined by the potentials $A_{i,rad}^\mu$:

$$A_{i,rad}^\mu (x) = \left(\int_{-\infty}^{s'} - \int_{s'}^{+\infty}\right) e_i \frac{dx_i^\mu}{ds_i}\, \delta(\{x^\nu - x_i^\nu\}\{x_\nu - x_{i,\nu}\})\, ds_i. \quad (92)$$

V) In states in which the radiated fields vanish they vanish for all the particles and there is no field of incoming waves $F_{wav}^{\mu\nu}$.

VI) In states of motion with a non vanishing radiation field the accelerations of the particles vanish for $s_i = -\infty$ and their radiation fields tend to zero when $s_i = +\infty$.

The action principle (86)-(87) generalizes (72) of Part I by allowing for states of negative kinetic energy and taking the broader definition of the radiated field contained in the IV principle. This definition of the radiated field includes the two particular ones of the non-stationary motions of particles and anti-particles and the vanishing radiated field of the stationary motions.

Therefore the action principle (86)-(87) is a generalization of the principle (72) of Part I, including all kinds of motions considered in the preceding parts of this paper:

a) The non stationary motions of particles and anti-particles correspond to the two possible signs of the ds_i allowed by principle II.

b) The stationary motions correspond to the vanishing radiated fields of principle IV.

Principle VI contains the two time boundary conditions which determine the solutions of the equations of motion and eliminate the diverging acceleration energies and energy fluxes at infinity in non-stationary motions.

Since we are adopting the conception of actions at a distance the actual field is the total radiation field. The $F^{\mu,\nu}_{i,at}$ are to be considered as functions of the particles world-lines, not as quantities to be determined by the differential equations resulting from the action principle (87). When we use the expression attached field we consider it only an abreviated way to say that the direct actions between the particles have the form of Lorentz forces arising from the attached fields.

To the six principles we may add a seventh one:

VII) The stress tensors of the total radiation field and the matter are respectively

$$4\pi\ T_{rad,\nu}^{\mu} = F^{\mu\rho} F_{\rho\nu} + \frac{1}{4}\ \delta_{\nu}^{\mu}\ F^{\rho\sigma} F_{\rho\sigma}$$

$$- \sum_{i,j} F_{i,at}^{\mu\rho}\ F_{j,at;\rho\nu} - \frac{1}{4}\ \delta_{\nu}^{\mu} \left(\sum_{i,j} F_{i,at}^{\rho\sigma}\ F_{j,at;\rho\sigma} \right) \tag{93}$$

$$4\pi\ .T_{part,\nu}^{\mu} = 4\pi \sum_{i} m_i c \frac{dx_i^{\mu}}{ds_i} \frac{dx_{i,\nu}}{ds_i}\ \delta\,(\mathbf{r} - \mathbf{r}_i)\ \frac{ds_i}{dt}$$

$$+ \sum_{i} \sum_{i \neq j} \left[F_{i,at}^{\mu\rho}\ F_{j,at;\rho\nu} + \frac{1}{4}\ \delta\ (F_{i,at}^{\rho\sigma}\ F_{j,at;\rho\sigma}) \right] \tag{94}$$

Formula (93) is a generalization of (67) of Part I including all types of motions of particles and anti-particles, the stress tensor (93) vanishes in the case of stationary motions; in (94) the stress tensor of the attached fields is added to the kinetic stress tensor of the particles. Adding the two stress tensors we get the total stress tensor:

$$4\pi\ T_{\nu}^{\mu} = 4\pi \sum_{i} m_i c \frac{dx_i^{\mu}}{ds_i} \frac{dx_{i,\nu}}{ds_i}\ \delta\ (\mathbf{r} - \mathbf{r}_i)\ \frac{ds_i}{dt}$$

$$+ F^{\mu\rho} F_{\rho\nu} + \frac{1}{4}\ \delta_{\nu}^{\mu}\ (F^{\rho\sigma}\ F_{\rho\sigma})$$

$$- \sum_{i} F_{i,at}^{\mu\rho}\ F_{i,at;\rho\nu} - \frac{1}{4}\ \delta_{\nu}^{\mu} \left(\sum_{i} F_{i,at}^{\rho\sigma}\ E_{i,at;\rho\sigma} \right) \tag{95}$$

20. The action principle (86) is of the Schwarzschild type as (72) of Part I, we can take instead a principle of the same kind as (55) of Part I, in which both the field components and potentials are taken as independent variables, an action principle leading to both sets of Maxwell equations. By making some partial integrations and neglecting boundary integrals we can go over from (86) to:

$$\Delta L^* = 0 \tag{96}$$

$$L^* = -\sum_i m_i c \int_{-\infty}^{+\infty} ds_i -$$

$$-\frac{1}{2}\sum_{i \neq j}\sum_j e_i e_j \int_{-\infty}^{+\infty}\int \frac{dx_i^{\mu}}{ds_i}\frac{dx_{j,\mu}}{ds_j}\,\delta\left(\{x_i^{\nu} - x_j^{\nu}\}\{x_{i,\nu} - x_{j,\nu}\}\right) ds_i\, ds_j$$

$$-\sum_i \frac{e_i}{c}\int_{-\infty}^{+\infty} \frac{dx_i^{\mu}}{ds_i} A_{rad,\mu} ds_i + \frac{1}{16\pi}\int_{-\infty}^{+\infty}\left[F^{\rho\sigma}F_{\rho\sigma} - \sum_{i,j} F_{i,at}^{\rho\sigma} F_{j,at\rho\sigma}\right] d\tau\, dt$$

$$-\frac{1}{4\pi}\int_{-\infty}^{+\infty}\left[A_\rho \frac{\partial F^{\rho\sigma}}{\partial x^\sigma} - \sum_{i,j} A_{i,at;\rho}\frac{\partial F_{j,at}^{\rho\sigma}}{dx^\sigma}\right] d\tau\, dt \tag{97}$$

The action principle (86) is analogous to the form of the Hamilton principle of non-relativistic dynamics which leads to the equations of Lagrange by variating the coordinates and not considering the velocities as independent quantities: the action principle (96)-(97) is analogous to the form of the Hamilton principle in which independent variations are given to both the coordinates and the conjugated momenta, a form of the action principle which leads directly to the first order Hamilton equations.

It is worthwhile to remark that the action principle (96) leads to both sets of Maxwell equations and there is no need of supplementary conditions of the Lorentz type.

APENDIX

In sections 5 and 7 we discussed the conservation of energy in stationary motions from two different point of views, introducing the functional E_{pot} and the energy G_f^o. The comparison with non relativistic dynamics shows that E_{pot} is analogous to the time integral of the potential energy of the system of particles; equation (35) shows that G_f^o can be considered as analogous to a potential energy, hence E_{pot} must be analogous to the time integral of $\bar{G}_f^o$, at least in the average. Indeed, we get from equation (24)

$$\frac{d}{dt}\sum_i \frac{m_i}{\sqrt{1-\beta_i^2}} = \sum_i \sqrt{1-\beta_i^2}\,\frac{\Delta E_{pot}}{\Delta x_i(s_i)} = \sum_i \frac{1}{c}\frac{\Delta E_{pot}}{\Delta x_i^o(t)} \tag{98}$$

From (35) and (98) it results that

$$\overline{\frac{dG_f^o}{dt}} = -\sum_i c\,\frac{\Delta E_{pot}}{\Delta x_i^o(t)} \tag{99}$$

the bars representing time mean values, because the time integral of the surface integral in the right-hand side of (35) vanishes according to (53). If we consider E_{pot} as a functional of the time t we may write:

$$\frac{\Delta E_{pot}}{\Delta t} = \sum_i \frac{\Delta E_{pot}}{\Delta x_i^o(t)} \tag{100}$$

But we have also

$$-\frac{dG_j^o}{dt} = \frac{\Delta}{\Delta t}\int_{-\infty}^{+\infty} G_j^o \, dt \qquad (101)$$

considering $\int G_j^o$ dt a functional of the fields and of t, so that

$$\overline{\frac{\Delta}{\Delta t}\int_{-\infty}^{+\infty} G_j^o \, dt} = \overline{\frac{\Delta E_{pot}}{\Delta t}} \qquad (102)$$

This equation puts clearly in evidence in which sense E_{pot} may be considered as an analogon of the time integral of G_j^o.

Resumo

Este trabalho é uma continuação da análise dos movimentos de um sistema de cargas puntiformes, iniciada no fascículo precedente desta revista; discutem-se os estados de movimento sem irradiação e os com energia cinética negativa. Mostra-se que é necessário admitir que, em movimentos não estacionários com energia cinética negativa, as partículas geram campos avançados. Toda a teoria dos movimentos com energia cinética negativa pode ser apresentada de uma maneira satisfatória, admitindo-se que em tais movimentos o tempo próprio do corpusculo está orientado em sentido oposto ao do observador.

Investiga-se o comportamento ao contorno dos campos retardado, avançado e semi-retardado — semi-avançado e dá-se uma formulação geral e unificada dos princípios da teoria.

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