MARIO SCHÖNBERG

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RELATIVISTIC COMMUTATION RULES IN THE QUANTUM THEORY OF FIELDS (II)

by MARIO SCHÖNBERG

Istituto Fisico della R. Università degli Studi. Roma.

Summary

The analysis of the commutation rules between field variables at different times, begun in the first part of this paper [1]), is extended to more general cases, in connection with the Heisenberg-Pauli quantum theory of wave fields [2]). This theory is discussed in connection with the wave equations of the classical field. The theory involves difficulties in the case of first order equations of motion. It is shown that the commutation rules between field quantities at different times can be determined by means of the commutation rules at the same time and the equations of motion, when these equations are linear .The case of second order wave equations is explicitely treated, the commutation rules for the same time being known from the Heisenberg-Pauli theory. The study of the functions involved in the relativistic rules shows that the knowledge of these functions is equivalent to the complete solution of the wave equations, any solution being calculable by integrations involving the initial values.

1. We consider a field described by a set of real coordinates and whose wave equations can be obtained from a variational principle:

$$\delta \int \mathbf{L} dt = 0 \tag{1}$$

Assuming the wave equations to be of the first or second order in the time derivatives of the coordinates Q and relativistically invariant, the lagrangian $\mathbf{L}$ will be the space integral of an invariant L, involving the coordinates Q_a and their first order derivatives $\partial Q_a / \partial x_i$ and $\dot{Q}_a$:

$$\mathbf{L} = \int L(Q_a, \frac{\partial Q_a}{\partial x_i}, \dot{Q}_a)\, dv\,. \tag{2}$$

The wave equations, resulting from equation (1), are:

$$\frac{\partial L}{\partial Q_a} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \left(\frac{\partial Q_a}{\partial x_i} \right)} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{Q}_a} = 0\,. \tag{3}$$

The equations of motion can be transformed in to the hamiltonian form by the introduction of the momenta P_a, conjugated to the coordinates Q_a, and of the hamiltonian $\mathbf{H}$:

$$P_a = \frac{\partial L}{\partial \dot{Q}_a} = \frac{\partial L}{\partial \dot{Q}_a} \tag{4}$$

$$\mathbf{H} = \int H \, dv \tag{5}$$

$$H = \sum_a P_a \dot{Q}_a - L \tag{6}$$

When the wave equation is of the second order the momenta P_a must involve the time derivatives of the coordinates $\dot{Q}_a$ and it is then possible to eliminate the Q_a from the hamiltonian and consider it as a function of the coordinates, momenta and their space derivatives. In the case of first order wave equations, the time derivatives of the coordinates can be eliminated from the hamiltonian, only taking into account the wave equations, the momenta P_a not involving the time derivatives.

The canonical equations of motion are:

$$\dot{Q}_a = \frac{\delta \mathbf{H}}{\delta P_a} \qquad \dot{P}_a = -\frac{\delta \mathbf{H}}{\delta Q_a}. \tag{7}$$

2. The quantization of the field theory can be carried out, assuming the coordinates and momenta to be q-numbers, with the following commutation rules: *)

$$\left.\begin{aligned} P_a(\boldsymbol{r}, t)\, Q_{a'}(\boldsymbol{r}', t) - Q_{a'}(\boldsymbol{r}, t)\, P_a(\boldsymbol{r}, t) &= \frac{\hbar}{i}\, \delta_{aa'}\, \delta(\boldsymbol{r} - \boldsymbol{r}') \\ P_a(\boldsymbol{r}, t)\, P_{a'}(\boldsymbol{r}', t) - P_{a'}(\boldsymbol{r}, t)\, P_a(\boldsymbol{r}, t) &= 0 \\ Q_a(\boldsymbol{r}, t)\, Q_{a'}(\boldsymbol{r}', t) - Q_{a'}(\boldsymbol{r}', t)\, Q_a(\boldsymbol{r}, t) &= 0 \end{aligned}\right\} \tag{8}$$

*) The partial functional derivatives of a functional

$$M(u_\beta(P'); \frac{\partial}{\partial x_i} u_\beta(P'); \dot{u}_\beta(P'),$$

taken with respect to the variable $u_\alpha(P)$ is given by the formula:

$$\frac{\delta M}{\delta u_\alpha(P)} = \lim_{\Delta u \to 0} \frac{M(u_\beta(P') + \delta_{\alpha\beta}\delta(P-P')\Delta u; \frac{\partial}{\partial x_i}[u_\beta(P') + \delta_{\alpha\beta}\delta(P-P')\Delta u]; \dot{u}_\beta(P')) - M(u_\beta(P'); \frac{\partial}{\partial x_i} u_\beta(P'); \dot{u}_\beta(P')}{\Delta u}$$

$$\frac{\delta \mathbf{L}}{\delta Q_a} = \frac{\partial L}{\partial Q_a} - \Sigma \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \left(\frac{\partial Q_a}{\partial x_i}\right)} \qquad \frac{\delta \mathbf{L}}{\delta \dot{Q}_a} = \frac{\partial L}{\partial \dot{Q}_a}$$

The quantum equations of motion of the field quantity A are:

$$i\hbar\,\dot{A} = A\mathbf{H} - \mathbf{H}A \tag{9}$$

The equations of motion for the coordinates and momenta resulting from the general quantum equation of motion (9) must be identical with the canonical set (7). This can be seen to be the case when the hamiltonian involves only the coordinates, momenta and their space derivatives. From commutation rules (8) results:

$$[\mathbf{F}, Q_a] = \frac{\hbar}{i}\frac{\delta \mathbf{F}}{\delta P_a} \qquad [P_a, \mathbf{F}] = \frac{\hbar}{i}\frac{\delta \mathbf{F}}{\delta Q_a} \tag{10}$$

F being a functional of the coordinates, momenta and their space derivatives, of the form of a space integral of a function:

$$\mathbf{F} = \int F\left(Q_a, P_a, \frac{\partial Q_a}{\partial x_i}, \frac{\partial P_a}{\partial x_i}\right) dv\,. \tag{11}$$

Taking into account the formulae (10) we obtain:

$$i\hbar\,\dot{Q}_a = [Q_a, \mathbf{H}] = i\hbar\,\frac{\delta \mathbf{H}}{\delta P_a} \qquad i\hbar\,\dot{P}_a = [P_a, \mathbf{H}] = -i\hbar\,\frac{\delta \mathbf{H}}{\delta Q_a}\,. \tag{12}$$

In the case of first order wave equations the hamiltonian involves the time derivatives of the coordinates, and fromulae (10) can no longer be established, the commutation of the coordinates and momenta with the $\dot{Q}_a$ being unknown. It can also be seen that the commutation rules (8) are not consistent, in the present case. Indeed, the momenta, not involving the $\dot{Q}_a$, are functions of the Q_a and $\partial Q_a/\partial x_i$, and so commute with the Q_a, contrary to the first rule (8).

The inapplicability of the Heisenberg-Pauli theory to the case of first order wave equations can be easily understood. This theory is obtained by passing from the quantum theory of a system with a discrete set of variables q_k to the case of an infinite set of coordinates $Q_a(\boldsymbol{r})$. The classical theory of the discrete system belongs to classical mechanics, with second order equations of motion for the q_k. So, there is no reason to expect the theory in the case of continuity to be applicable, when the classical equations of motion are of the first order. The theory must be developed on other grounds. The possibility of treating the case of Dirac's equation with the Heisenberg-Pauli theory is due to an inconsistent application of the formalism, as can be seen by introducing the real and complex parts of the wave functions as independent variables.

3. The commutation rules (8) correspond to the case of the Bose-Einstein statistics. Another possibility, corresponding to the Fermi-Dirac statistics, is to change in the commutation rules (8) the sign — into the sign $+$:

$$\left.\begin{aligned} P_\alpha(\boldsymbol{r}, t)\, Q_\alpha(\boldsymbol{r}', t) + Q_{\alpha'}(\boldsymbol{r}', t)\, P_\alpha(\boldsymbol{r}, t) &= \frac{\hbar}{i}\,\delta_{\alpha\alpha'}\,\delta(\boldsymbol{r} - \boldsymbol{r}') \\ P_\alpha(\boldsymbol{r}, t)\, P_{\alpha'}(\boldsymbol{r}, t) + P_{\alpha'}(\boldsymbol{r}', t)\, P_\alpha(\boldsymbol{r}, t) &= 0 \\ Q_\alpha(\boldsymbol{r}, t)\, Q_{\alpha'}(\boldsymbol{r}', t) + Q_{\alpha'}(\boldsymbol{r}', t)\, Q_\alpha(\boldsymbol{r}, t) &= 0 \end{aligned}\right\} \qquad (13)$$

The case of the Fermi-Dirac statistics does not allow the development of a theory of the Heisenberg-Pauli type, owing to the non-existence of a similar theory in the case of a discrete system. In general, it will not be possible to pass from the quantum mechanical equation of motion (9) to the canonical equations (7), the formulae (10) not resulting from the commutation rules (13). So we see that the possibility of any kind of statistics depends on the equations of motion. This point will be discussed elsewhere.

4. In this section we will examine the possibility of deriving the commutation rules for different times, knowing those for the same time. The two statistics can be treated on the same lines. The equations of motion will be supposed to be linear and homogeneous.

Let us consider first the case of first order equations of motion. The commutation rules for the same time will be:

$$Q_\alpha(\boldsymbol{r}, t)\, Q_{\alpha'}(\boldsymbol{r}', t) - Q_{\alpha'}(\boldsymbol{r}', t)\, Q_\alpha(\boldsymbol{r}, t) = (\boldsymbol{r}, \alpha\, |A|\, \boldsymbol{r}', \alpha') \qquad (14)$$

The problem of the relativistic commutation rules consists in the determination of $(\boldsymbol{r}, t, \alpha\, |B|\, \boldsymbol{r}', t', \alpha')$:

$$Q_\alpha(\boldsymbol{r}, t)\, Q_{\alpha'}(\boldsymbol{r}', t') - Q_{\alpha'}(\boldsymbol{r}', t')\, Q_\alpha(\boldsymbol{r}, t) = (\boldsymbol{r}, t, \alpha\, |B|\, \boldsymbol{r}', t', \alpha') \qquad (15)$$

From the comparison of (14) and (15) results:

$$(\boldsymbol{r}, t, \alpha\, |B|\, \boldsymbol{r}', t, \alpha') = (\boldsymbol{r}, \alpha\, |A|\, \boldsymbol{r}', \alpha') \qquad (16)$$

The equations of motion being linear and homogeneous, formula (14) shows that the $U_\alpha = (\boldsymbol{r}, t, \alpha\, |A|\, \boldsymbol{r}', t', \alpha')$ are a solution of the equations of motion. $(\boldsymbol{r}, t, \alpha\, |B|\, \boldsymbol{r}', t', \alpha')$ is completely determined as the solution of a first order differential equation, assuming for $t = t'$ the form $(\boldsymbol{r}, \alpha\, |A|\, \boldsymbol{r}', \alpha')$.

The case of second order equations of motion can be more thoroughly analysed, the form of the commutation rules between field variables at the same time being known explicitly. As before, the problem is to determine a function $(\boldsymbol{r}, t, \alpha\, |C|\, \boldsymbol{r}', t', \alpha')$ in such a way that:

$$Q_a(\boldsymbol{r}, t)\, Q_{a'}(\boldsymbol{r}', t') - Q_{a'}(\boldsymbol{r}', t')\, Q_a(\boldsymbol{r}, t) = (\boldsymbol{r}, t, \alpha\, |C|\, \boldsymbol{r}', t', \alpha') \quad (17)$$

For $t = t'$ commutation rule (17) is identical with the third rule (8) so that:

$$(\boldsymbol{r}, t, \alpha\, |C|\, \boldsymbol{r}', t, \alpha) = 0 \quad (18)$$

$(\boldsymbol{r}, t, \alpha)\, |C|\, \boldsymbol{r}', t', \alpha')$ is a solution of the equations of motion and for $t = t'$ takes the value 0, it will therefore be completely determined by finding the form of the time derivative for $t = t'$. From the derivative of both sides of formula (17) with respect to t, follows:

$$\dot{Q}_a(\boldsymbol{r}, t)\, Q_{a'}(\boldsymbol{r}', t) - Q_{a'}(\boldsymbol{r}', t)\, \dot{Q}_a(\boldsymbol{r}, t) = \frac{\partial}{\partial t}\, (\boldsymbol{r}, t, \alpha\, |C|\, \boldsymbol{r}', t', \alpha')_{t=t'} \quad (19)$$

The commutation rule between $\dot{Q}_a$ and $Q_{a'}$ can be obtained from rules (8) and therefore the time derivative of C will be known for $t = t'$. Indeed the momenta P_a are linear combinations of the coordinates Q_β and their derivatives $\partial Q_\beta/\partial x_i$ and $\dot{Q}_\beta$:

$$P_a = \Sigma\, c_{a\beta}\, Q_\beta + \sum_\beta \sum_i c_{a\beta i} \frac{\partial Q_\beta}{\partial x_i} + \sum_\beta d_\beta\, \dot{Q}_\beta \quad (20)$$

$c_{a\beta}$, $c_{a\beta i}$ and d_β are c numbers depending on $\boldsymbol{r}$ and t. Substituting the expression of the momenta in the first rule (8) the result is:

$$\dot{Q}_a(\boldsymbol{r}, t)\, Q_{a'}(\boldsymbol{r}', t) - Q_{\beta a'}(\boldsymbol{r}', t)\, \dot{Q}_a(\boldsymbol{r}, t) = \frac{\hbar}{i}\, \delta_{aa'},\, \delta(\boldsymbol{r} - \boldsymbol{r}') d_\beta^{-1} \quad (21)$$

The commutation rules can be directly deduced from formulae (8) and (21), by consideration of the properties of the function $V(\boldsymbol{r}, t, \alpha; \boldsymbol{r}', t', \alpha')$. V is the solution of the equations of motion, taking for $t = t'$ the value 0 and whose time derivative for $t = t'$ is $\delta_{a_{a'}}, \delta(\boldsymbol{r} - \boldsymbol{r}')$. V has the property expressed by the following equation:

$$f_a(\boldsymbol{r}, t) = \sum_{a'} \int f_{a'}(\boldsymbol{r}', t') \frac{\partial}{\partial t} V(\boldsymbol{r}, t, \alpha, \boldsymbol{r}', t', \alpha') dv' + $$
$$+ \sum_{a'} \int \frac{\partial}{\partial t'} f_a(\boldsymbol{r}', t')\, V(\boldsymbol{r}, t, \alpha, \boldsymbol{r}', t', \alpha') dv' \quad (22)$$

$f_a(\boldsymbol{r}, t)$ being an arbitrary solution of the equations of motion. Both sides of equation (22) are solutions of the equations of motion, identical and have identical time derivatives for $t = t'$, so that the equation is verified for any value of t.

Multiplying both sides of equalities (21) and the third rule in (8), respectively by

$$V(\boldsymbol{r}'', t'', \alpha''; \boldsymbol{r}', t', \alpha') \text{ and } \frac{\partial}{\partial t''} V(\boldsymbol{r}'', t'', \alpha''; \boldsymbol{r}', t', \alpha'),$$

integrating with respect to $\boldsymbol{r}'$ and summing with respect to α' we obtain rule (17).

The whole set of rules (8) can be obtained from rule (17). Taking into account the expression (20) for the momenta, this becomes evident for the first and third rules. To obtain the second rule it is sufficient to prove that

$$\frac{\partial^2}{\partial t \partial t'} (\boldsymbol{r}, t, \alpha \,|C|\, \boldsymbol{r}', t', \alpha')_{t=t'} = 0,$$

and this results from the application of formula (22) to

$$\frac{\partial}{\partial t'} (\boldsymbol{r}, t, \alpha \,|C|\, \boldsymbol{r}', t', \alpha').$$

The commutation rule (17) is consistent with the equations of motion, but is not sufficient to individuate completely them for their complete separate establishment. Indeed, $(\boldsymbol{r}, t, \alpha \,|C|\, \boldsymbol{r}', t', \alpha')$ being a solution of the equations of motion the operators $G_a(Q_\beta)$ must commute with the field variables and are therefore diagonal; the equations of motion must then be of the form $G_a(Q_\beta) = K_a$, but c number K_a remains arbitrary. This point will be examined in the following section.

The function $V(\boldsymbol{r}, t, \alpha; \boldsymbol{r}', t', \alpha')$ is closely related to the Green functions of the equations of motion. Let the equations of motion be:

$$G_a(Q_\beta) = \Sigma\, a_{a\beta}\, \ddot{Q} + \sum_{\beta i} b_{a\beta} \frac{\partial \dot{Q}_\beta}{\partial x_i} + \sum_{\beta} c_{a\beta}\, \dot{Q}_\beta + M_a = 0 \qquad (23)$$

The Green functions of the equations (23) are the solutions of the following system:

$$G_a(V_{\beta a'}) = \delta_{a_{a'}}\, \delta(\boldsymbol{r} - \boldsymbol{r}')\, \delta(t - t') \qquad (24)$$

The function $W_{\beta a'}$ is a solution of the system:

$$G_a(W_{\beta a'}) = \delta_{aa'}\, \delta(\boldsymbol{r} - \boldsymbol{r}')\, \delta(t - t')\, a^{-1}_{\overline{a\beta}}\, (\boldsymbol{r}, t) \qquad (25)$$

$$W_{\alpha\alpha'}(\mathbf{r}, t; \mathbf{r},' t') = \tfrac{1}{2}\Phi(t - t')\, V(\mathbf{r}, t, \alpha; \mathbf{r}', t', \alpha') \tag{26}$$

$$\Phi(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i\alpha x}}{\alpha}\, d\alpha \tag{27}$$

$$\Phi'(x) = \delta(x) \tag{28}$$

This results immediately from the substitution of $W_{\beta\alpha'}$ in equations (25), taking into account the properties of the functions V and δ:

$$G_\alpha(W_{\beta\alpha'}) = \tfrac{1}{2}\Phi(t - t')\, G_\alpha\{V(\mathbf{r}, t, \beta; \mathbf{r}', t', \alpha')\} +$$

$$+ \delta(t - t')\sum_\beta a_{\alpha\beta}\frac{\partial}{\partial t} V(\mathbf{r}, t, \beta; \mathbf{r}', t', \alpha') + \tfrac{1}{2}\delta'(t - t')\Sigma\, a_{\alpha\beta}V(\mathbf{r}, t, \beta; \mathbf{r}', t', \alpha') +$$

$$+ \tfrac{1}{2}\delta(t - t')\sum_{\beta i} b_{\alpha\beta}\frac{\partial}{\partial x_i} V(\mathbf{r}, t, \beta; \mathbf{r}', t', \alpha') +$$

$$+ \tfrac{1}{2}\delta(t - t')\sum_\beta c_{\alpha\beta}V(\mathbf{r}, t, \beta; \mathbf{r}', t', \alpha') =$$

$$= \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t')\delta_{\alpha\alpha'}\, a_{\alpha\alpha'}(\mathbf{r}, t) \tag{29}$$

5. Now we will consider the case of linear, non-homogeneous equations of motion. The following theorem can be demonstrated: the relativistic commutation rules are the same in the case of homogeneous and non-homogeneous equations of motion. The theorem will be discussed in the case of second order equations of motion, the case of first order equations can be treated on the same lines if the commutation rules between the field variables at the same time do not depend on the inhomogeneity of the equations of motion. Let the relativistic commutation rule be:

$$Q_\alpha(\mathbf{r}, t)\, Q_{\alpha'}(\mathbf{r}', t') - Q_{\alpha'}(\mathbf{r}', t')\, Q_\alpha(\mathbf{r}, t) = (\mathbf{r}, t, \alpha\, |D|\, \mathbf{r}', t', \alpha') \tag{30}$$

The non-homogeneous equations of motion are of the form:

$$G_\alpha(Q_\beta) = K_\alpha \tag{31}$$

the K_α being known functions. Introducing $(\mathbf{r}, t, \alpha\, |D|\, \mathbf{r}', t', \alpha')$ in the equations $G_\alpha = 0$ and taking into account the commutation rule (40) and the equations of motion (31), there results:

$$G_\alpha((\mathbf{r}, t, \beta\, |D|\, \mathbf{r}', t', \alpha')) = 0 \tag{32}$$

$(\mathbf{r}, t, \alpha\, |D|\, \mathbf{r}', t', \alpha')$ is therefore a solution of the homogeneous equations. In the case of second order equations of motion the commutation rules between the field variables at the same time are given by set (8) and are evidently the same for the homogeneous and the inhomogeneous equations. The two matrices C and D must therefore satisfy the same conditions and are for that reason identical:

$$C = D \tag{33}$$

The equations of motion of the electromagnetic potentials in vacuo are the d'Alembert equations:

$$\square A_{\alpha} = 0 \tag{34}$$

$$(\alpha = 0, 1, 2, 3)$$

In the general case of a field with charges, the equations of motion become:

$$\square A_{\alpha} = \gamma_{\alpha} \tag{35}$$

γ being the four vector of the current.

Applying in the present case our theorem we see that the relativistic commutation rules for an electromagnetic field are the same as those for a field in vacuo. The Jordan-Pauli [3]) commutation rules are therefore applicable to any electro-magnetic field.

It can now tbe easily understood why the relativistic commutation rules to not determine completely the equations of motion of a field; there always remains namely the possibility of introducing an arbitrary known function into the equations of motion.

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