ℒegu 28/11/51

Reprinted from THE PHYSICAL REVIEW, Vol. 83, No. 1, 21-25, July 1, 1951
Printed in U. S. A.

On the Nucleon Cascade with Ionization Loss

H. MESSEL
School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin, Ireland
(Received October 25, 1950)

The fundamental equations for a nucleon cascade including ionization loss are given. Solutions satisfying the exact boundary conditions are found in a manner similar to that used by Bhabha and Chakrabarty for the electron-photon cascade. However, in the present treatment, second-order equations are not resorted to and all four cases (protons producing protons and neutrons; neutrons producing protons and neutrons) are treated simultaneously. The results are given in tabular and graphic form and can be adapted easily to any specific problem. A number of examples showing the effect of ionization loss have been computed and the ratio of neutrons to protons given. The calculations are carried out using the cross section formulated by Heitler and Jánossy.

I. INTRODUCTION

IN a series of recent papers[1-5] the theory of the nucleon cascade has been developed and compared with experiment. In the cited papers adequate treatments have been given for the deduction of the expressions for average numbers and the fluctuations to be expected from the average, both for homogeneous nuclear matter and inhomogeneous matter. However, so far no detailed treatment of the problem has been given when ionization losses are taken into consideration.

In the present paper the cascade equations are given for protons and neutrons taking into account ionization losses, and exact solutions are found satisfying the correct boundary conditions. We retain the assumptions regarding the form of the cross section, made in the papers mentioned above. Since little is still known regarding the exact behavior of the nucleon cross sections at low energies, the solutions found will in all probability be out by a large factor in this region.

When the nucleon cascade is considered as a proton-neutron cascade with ionization loss, then the problem immediately becomes analogous to that for the electron-photon cascade. It is natural therefore to treat the problem of the nucleon cascade in a manner analogous to that which Bhabha and Chakrabarty[6,7] used for the electron-photon cascade. There have appeared a number of criticisms of the Bhabha-Chakrabarty solution;[8,9] however, we feel that these criticisms deal with minor points only and apply at most to the low energy region where no theory so far can claim success.

Though the solution of the electron-photon cascade with ionization loss obtained by Bhabha and Chakrabarty is elegant in many respects, we feel that the extreme simplicity underlying its derivation has been somewhat obscured. We give a treatment which appears to be straightforward, without resorting to second-order equations. Numerical results and several examples are given, showing the effects of taking ionization losses into account.

II. THE DIFFUSION EQUATIONS

Let $S^{(i,k)}(E,\theta)dE$ be the average number of protons or neutrons (depending on the suffix k) in the energy range E, $E+dE$ at a depth θ in inhomogeneous matter, due to a primary particle of energy E_0. The suffix $i=1,2$ refers to the primary particle and $k=1,2$ to the secondary. The number 1 always refers to a proton and 2 to a neutron. The ionization loss β is assumed to be independent of energy and is taken to be equal to 130 Mev per 65 g/cm² air throughout the problem.

We define $V(E/E')dE/E'$ as the probability that a nucleon of energy E' falling on a nucleus gives rise to one or more nucleons, and that one of the emerging nucleons has an energy in the interval E, $E+dE$. The function $V(E/E')$ has been determined by Jánossy and Messel[3] and we use their result. The fundamental equations can now be written as

$$\frac{\partial}{\partial\theta}S^{(i,k)}(E,\theta)+S^{(i,k)}(E,\theta)=\tfrac{1}{2}\int_0^\infty\{S^{(i,k)}(E',\theta)+S^{(i,3-k)}(E',\theta)\}\times V(E/E')\frac{dE'}{E'}+\delta_{1,k}\beta\frac{\partial}{\partial E}S^{(i,k)}(E,\theta).\quad(1)$$

We take the Mellin transform of (1) with respect to (E/E_0), and in order to keep the transform dimensionless we let

$$P^{(i,k)}(s,\theta)=\int_0^\infty(E/E_0)^{s-1}S^{(i,k)}(E,\theta)dE.\quad(2)$$

There then results

$$\frac{\partial}{\partial\theta}P^{(i,k)}(s,\theta)+A(s)P^{(i,k)}(s,\theta)+B(s)P^{(i,3-k)}(s,\theta)=-\delta_{1,k}(\beta/E_0)(s-1)P^{(i,k)}(s-1,\theta),\quad(2a)$$

[1] W. Heitler and L. Jánossy, Proc. Phys. Soc. (London) A62, 374 (1949).
[2] H. Messel, Proc. Roy. Irish Acad. A 54, 125 (1951).
[3] L. Jánossy and H. Messel, Proc. Roy. Irish Acad. A 54, 245 (1951).
[4] H. Messel and D. M. Ritson, Proc. Phys. Soc. (London) (1950) A 63, 1359 (1950).
[5] P. Budini and N. Dallaporta, Nuovo Cimento 7, 3 (1950).
[6] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. A181, 267 (1943).
[7] H. J. Bhabha and S. K. Chakrabarty, Phys. Rev. 74, 1352 (1948).
[8] I. E. Tamm and S. Belenky, Phys. Rev. 70, 660 (1946).
[9] H. S. Snyder, Phys. Rev. 76, 1563 (1949).

where

$$\left.\begin{aligned} V(s)&=\int_0^\infty u^{s-1}V(u)du, \\ V(s)&=2[1-(1+D_A\alpha_s)\exp(-D_A\alpha_s)]/(D_A\alpha_s)^2 \end{aligned}\right\} \quad (2b)$$

$$A(s)=1-\tfrac{1}{2}V(s), \quad B(s)=-\tfrac{1}{2}V(s), \quad (2c)$$

$$\alpha_s=\int_0^\infty\int_0^\infty(1-\epsilon_1^{s-1}-\epsilon_2^{s-1})w(\epsilon_1,\epsilon_2)d\epsilon_1 d\epsilon_2, \quad (2d)$$

$$w(\epsilon_1,\epsilon_2)=15\epsilon_2^2(1-\epsilon_1). \quad (2e)$$

[See the references given in Sec. I for details of the cross section given in (2d) and (2e).]

In (2b) D_A is the number of collisions suffered by a nucleon on traversal of a nucleus (of atomic weight A) along the nuclear diameter; D_A was taken equal to 2.41 for air in the previous publications mentioned above. The solution of (2a) is immediately given by

$$P^{(i,k)}(s,\theta)=P_0^{(i,k)}(s,\theta)-(\beta/E_0)(s-1)\times\int_0^\theta P_0^{(1,k)}(s,\theta-\theta')P^{(i,1)}(s-1,\theta')d\theta', \quad (3)$$

where $P_0^{(i,k)}(s,\theta)$ is the solution of the homogeneous part of (2a), i.e., of

$$\frac{\partial}{\partial\theta}P^{(i,k)}(s,\theta)+A(s)P^{(i,k)}(s,\theta)+B(s)P^{(i,3-k)}(s,\theta)=0. \quad (4)$$

From (2), (4), and the initial condition

$$S^{(i,k)}(E,0)=\begin{cases}\delta(E-E_0) & \text{for } i=k\\ 0 & \text{for } i\neq k,\end{cases}$$

it is easily shown that

$$P_0^{(i,k)}(s,\theta)=\tfrac{1}{2}\{\exp[-(1-V(s))\theta]\pm\exp(-\theta)\} \quad \begin{matrix}\text{plus sign for } i=k\\ \text{minus sign for } i\neq k.\end{matrix} \quad (5)$$

That (3) is a solution of (2a) is most readily seen by substituting (3) directly into (2a) and seeing that (2a) is satisfied.

Next let us write

$$P^{(i,k)}(s,\theta)=\sum_{n=0}^\infty P_n^{(i,k)}(s,\theta)(-\beta/E_0)^n. \quad (6)$$

We substitute (6) into (3), equate equal powers of (β/E_0) and find the recursion formula

$$P_n^{(i,k)}(s,\theta)=(s-1)\int_0^\theta P_0^{(1,k)}(s,\theta-\theta')P_{n-1}^{(i,1)}(s-1,\theta')d\theta'. \quad (7)$$

Keeping in mind definition (2), we invert the Mellin transform and find

$$S^{(i,k)}(E,\theta)=\frac{1}{2\pi i}\frac{1}{E_0}\int_C\left(\frac{E_0}{E}\right)^s\times\left\{\sum_{n=0}^\infty P_n^{(i,k)}(s,\theta)(-\beta/E_0)^n\right\}ds. \quad (8)$$

The $(n+1)$th term of (8) is given by

$$\frac{1}{2\pi i}\frac{1}{E_0}\int_C\left(\frac{E_0}{E}\right)^s P_n^{(i,k)}(s,\theta)(-\beta/E_0)^n ds=\frac{1}{2\pi i}\frac{1}{E_0}(-\beta)^n\int_C\frac{E_0^{s-n}}{E^s}P_n^{(i,k)}(s,\theta)ds. \quad (9)$$

Replacing $s-n$ by s and shifting back the path of integration, we find the $(n+1)$th term is given by the expression

$$\frac{1}{2\pi i}\frac{1}{E_0}(-\beta/E)^n\int_C\left(\frac{E_0}{E}\right)^s P_n^{(i,k)}(s+n,\theta)ds. \quad (10)$$

Hence (8) can be written as

$$S^{(i,k)}(E,\theta)=\frac{1}{2\pi i}\frac{1}{E_0}\int_C\left(\frac{E_0}{E}\right)^s\times\left\{\sum_{n=0}^\infty(-\beta/E)^n P_n^{(i,k)}(s+n,\theta)\right\}ds. \quad (11)$$

We omit here the detailed mathematical proof that (11) is absolutely and uniformly convergent for $E>\beta\theta$, since the proof is entirely ananlogous to that given both by Iyengar[10] and Bhabha and Chakrabarty.[7]

For convenience we now write (11) as

$$S^{(i,k)}(E,\theta)=\frac{1}{2\pi i}\frac{1}{E_0}\int_C\left(\frac{E_0}{E}\right)^s\times\left\{\sum_{n=0}(-\beta/E)^n\frac{\Gamma(s+n)}{\Gamma(s)}Q_n^{(i,k)}(s,\theta)\right\}ds, \quad (12)$$

where

$$[\Gamma(s+n)/\Gamma(s)]Q_n^{(i,k)}(s,\theta)=P_n^{(i,k)}(s+n,\theta). \quad (12a)$$

Now let us define

$$g^{(i,k)}(s,\theta)=Q_1^{(i,k)}(s,\theta)/Q_0^{(i,k)}(s,\theta) \quad (13)$$

and

$$f_n^{(i,k)}(s,\theta)=\left\{Q_0^{(i,k)}(s,\theta)\frac{[g^{(i,k)}(s,\theta)]^n}{n!}-Q_1^{(i,k)}(s,\theta)\frac{[g^{(i,k)}(s,\theta)]^{n-1}}{(n-1)!}+\cdots\right\}. \quad (14)$$

[10] K. S. K. Iyengar, Proc. Indian Acad. Sci. A15, 195 (1942).

We write the integrand of (12) as follows:

$$E_0^s\sum_{n=0}^{\infty}(-\beta)^n\frac{\Gamma(s+n)}{\Gamma(s)}\frac{Q_n^{(i,k)}(s,\theta)}{\{(E+\beta g^{(i,k)}(s,\theta))-\beta g^{(i,k)}(s,\theta)\}^{n+s}}$$

and expand

$$\{(E+\beta g^{(i,k)}(s,\theta))-\beta g^{(i,k)}(s,\theta)\}^{n+s}$$

in powers of

$$\beta g^{(i,k)}(s,\theta)/(E+\beta g^{(i,k)}(s,\theta)).$$

Rearranging the resultant double series (this can be done since the double series is absolutely convergent) yields

$$S^{(i,k)}(E,\theta)=\frac{1}{2\pi i}\frac{1}{E_0}\int_C\left(\frac{E_0}{E+\beta g^{(i,k)}(s,\theta)}\right)^s \times\left\{\sum_{n=0}^{\infty}\left(\frac{\beta}{E+\beta g^{(i,k)}(s,\theta)}\right)^n\frac{\Gamma(s+n)}{\Gamma(s)}f_n^{(i,k)}(s,\theta)\right\}ds. \quad (15)$$

The solution (15) is now entirely analogous to that given by Bhabha and Chakrabarty for the electron-photon cascade; however, in the present development the complete four cases are covered (proton-proton, proton-neutron, neutron-neutron and neutron-proton). In order to find the expression for integral spectra we integrate (15) from E to E_0 and find

$$N^{(i,k)}(E,\theta)=\sum_{n=0}^{\infty}N_n^{(i,k)}(E,\theta) \quad (16)$$

with

$$N_n^{(i,k)}(E,\theta)=\frac{1}{2\pi i}\int_C\frac{1}{s+n-1}\left(\frac{E_0}{\beta}\right)^{s-1} \times\left(\frac{\beta}{E+\beta g^{(i,k)}(s,\theta)}\right)^{s+n-1}\frac{\Gamma(s+n)}{\Gamma(s)}f_n^{(i,k)}(s,\theta)ds. \quad (17)$$

The integration of (17) is carried out along a line parallel to the imaginary axis running from $s_0-i\infty$ to $s_0+i\infty$ with $s_0>1$.

The solution is now complete and appears in the form of a series. However, as was shown by Bhabha and Chakrabarty, it suffices for all practical purposes to take the first two terms of the series, and for many cases where we are considering particles above a given energy E, which is not too small, the first term suffices. It should be noted that at present we are not justified in considering particles with energies which are very low, because it is precisely in this region that we know little of the true behavior of a nucleon cross section.

III. DETERMINATION OF $g^{(i,k)}(s,\theta)$ AND $f_n^{(i,k)}(s,\theta)$

From the definitions (13) and (14) and the recurrence relation (7) explicit expressions for the $g^{(i,k)}(s,\theta)$ can easily be obtained. We give here only the results of the integration.

$$g^{(i,k)}(s,\theta)=\left\{\pm\theta\exp(-\theta)+\sum_{v=0}^{2}A_v^{(i,k)}(s)\exp[(u_v(s)\theta)]\right\} \times\{P_0^{(i,k)}(s,\theta)\}^{-1} \quad \begin{matrix}\text{plus sign for } i=k\\ \text{minus sign for } i\neq k,\end{matrix} \quad (18)$$

where

$$\left.\begin{aligned}u_0(s)&=-1,\\ u_1(s)&=-(1-V(s)),\\ u_2(s)&=-(1-V(s+1)),\end{aligned}\right\} \quad (18a)$$

$$\left.\begin{aligned}A_0^{(2,2)}(s)&=-A_0^{(1,1)}(s)=\frac{V(s)+V(s+1)}{4V(s)V(s+1)},\\ A_1^{(2,1)}(s)&=A_1^{(1,1)}(s)=\frac{2V(s)-V(s+1)}{4V(s)[V(s)-V(s+1)]},\\ A_2^{(1,2)}(s)&=A_2^{(1,1)}(s)\\ &=\frac{V(s)-2V(s+1)}{4V(s+1)[V(s)-V(s+1)]},\\ A_0^{(2,1)}(s)&=-A_0^{(1,2)}(s)=\frac{V(s)-V(s+1)}{4V(s)V(s+1)},\\ A_1^{(2,2)}(s)&=A_1^{(1,2)}(s)=\frac{V(s+1)}{4V(s)[V(s)-V(s+1)]},\\ A_2^{(2,2)}(s)&=A_2^{(2,1)}(s)\\ &=\frac{-V(s)}{4V(s+1)[V(s)-V(s+1)]}.\end{aligned}\right\} \quad (18b)$$

For real s greater than 0.1 the terms proportional to $\exp(-\theta)$ are negligible for all except very small θ; hence to a good approximation the first two terms of (18) can be dropped. Using this simplifying approximation we find

$$g^{(i,k)}(s,\theta)=2\{A_1^{(i,k)}(s)+A_2^{(i,k)}(s) \times\exp(-[V(s)-V(s+1)]\theta)\}. \quad (19)$$

The values of $f_n^{(i,k)}(s,\theta)$ can also be easily obtained by simply using the recursion relations. The work involved in finding $f_,^{(i,k)}(s,\theta)$ is not as great as would appear since one can readily utilize the numerical work completed in finding $Q_1^{(i,k)}(s,\theta)$. We do not carry out the evaluation for the second term of the series, since at present we do not feel that such a refinement is justified.

IV. EVALUATION AND NUMERICAL RESULTS

In Tables I and II we have tabulated the functions $g^{(1,1)}(s,\theta)$ and $g^{(1,2)}(s,\theta)$ for a wide range of s-values

TABLE I. The quantity $g^{(1,1)}(s,\theta)$ as a function of s for various constant values of the depth in air, θ. θ is measured in units of 65 g/cm².

$s \diagdown \theta$	1	2	4	6	8	10	15	20
				$g^{(1,1)}(s,\theta)$				
0.1	0.318	0.308	0.307	0.307	0.307	0.307	0.307	0.307
0.2	0.462	0.467	0.458	0.458	0.458	0.458	0.458	0.458
0.3	0.587	0.654	0.640	0.638	0.637	0.637	0.637	0.637
0.4	0.682	0.847	0.850	0.844	0.843	0.843	0.843	0.843
0.5	0.752	1.024	1.083	1.069	1.073	1.072	1.072	1.072
0.6	0.802	1.177	1.325	1.328	1.326	1.325	1.325	1.325
0.7	0.839	1.303	1.565	1.595	1.598	1.599	1.600	1.600
0.8	0.867	1.405	1.791	1.864	1.882	1.889	1.895	1.895
0.9	0.889	1.488	1.998	2.130	2.174	2.192	2.209	2.212
1.0	0.905	1.556	2.185	2.385	2.464	2.501	2.538	2.548
1.1	0.918	1.611	2.350	2.627	2.749	2.809	2.879	2.901
1.2	0.929	1.656	2.496	2.850	3.024	3.113	3.226	3.268
1.3	0.937	1.694	2.625	3.056	3.286	3.407	3.574	3.644
1.4	0.944	1.726	2.737	3.243	3.534	3.689	3.920	4.026
1.5	0.950	1.752	2.836	3.414	3.768	3.959	4.258	4.409
1.6	0.955	1.775	2.923	3.569	3.986	4.213	4.586	4.790
1.7	0.959	1.794	3.000	3.709	4.191	4.454	4.905	5.165
1.8	0.963	1.811	3.068	3.837	4.381	4.680	5.210	5.531
1.9	0.966	1.826	3.128	3.952	4.559	4.892	5.501	5.887
2.0	0.969	1.839	3.182	4.057	4.724	5.090	5.778	6.231
2.1	0.971	1.850	3.230	4.153	4.879	5.276	6.041	6.563
2.2	0.973	1.860	3.273	4.240	5.023	5.450	6.290	6.880
2.3	0.975	1.869	3.312	4.319	5.157	5.613	6.525	7.184
2.4	0.976	1.876	3.347	4.392	5.283	5.766	6.749	7.474
2.5	0.978	1.884	3.379	4.459	5.401	5.909	6.958	7.751
2.6	0.979	1.890	3.409	4.520	5.512	6.045	7.165	8.014
2.7	0.981	1.896	3.434	4.576	5.615	6.170	7.343	8.264
2.8	0.982	1.901	3.459	4.629	5.713	6.289	7.520	8.502
2.9	0.983	1.906	3.480	4.677	5.805	6.401	7.687	8.727
3.0	0.984	1.910	3.501	4.721	5.891	6.506	7.844	8.941

TABLE II. The quantity $g^{(1,2)}(s,\theta)$ as a function of s for various constant values of the depth in air, θ. θ is measured in units of 65 g/cm².

$s \diagdown \theta$	1	2	4	6	8	10	15	20
				$g^{(1,2)}(s,\theta)$				
0.1	0.033	0.032	0.031	0.031	0.031	0.031	0.031	0.031
0.2	0.062	0.069	0.065	0.065	0.065	0.065	0.065	0.065
0.3	0.095	0.118	0.118	0.117	0.117	0.117	0.117	0.117
0.3	0.130	0.179	0.193	0.191	0.190	0.190	0.190	0.190
0.5	0.162	0.243	0.287	0.289	0.288	0.288	0.287	0.287
0.6	0.192	0.306	0.393	0.409	0.411	0.410	0.410	0.410
0.7	0.219	0.366	0.505	0.545	0.557	0.558	0.559	0.559
0.8	0.244	0.420	0.615	0.690	0.720	0.727	0.733	0.733
0.9	0.265	0.469	0.720	0.837	0.896	0.916	0.931	0.935
1.0	0.285	0.513	0.819	0.983	1.077	1.113	1.151	1.161
1.1	0.302	0.552	0.910	1.122	1.257	1.317	1.387	1.409
1.2	0.317	0.588	0.993	1.253	1.433	1.522	1.635	1.677
1.3	0.331	0.619	1.069	1.375	1.602	1.723	1.890	1.960
1.4	0.343	0.647	1.137	1.488	1.762	1.917	2.147	2.253
1.5	0.355	0.673	1.199	1.592	1.911	2.102	2.402	2.553
1.6	0.365	0.696	1.255	1.687	2.050	2.278	2.651	2.854
1.7	0.374	0.716	1.306	1.774	2.180	2.443	2.894	3.154
1.8	0.382	0.735	1.352	1.854	2.299	2.597	3.128	3.449
1.9	0.389	0.752	1.394	1.927	2.408	2.741	3.351	3.737
2.0	0.396	0.767	1.432	1.993	2.510	2.876	3.563	4.017
2.1	0.402	0.782	1.467	2.055	2.603	3.001	3.765	4.287
2.2	0.408	0.794	1.498	2.111	2.689	3.116	3.955	4.546
2.3	0.413	0.806	1.528	2.163	2.768	3.223	4.135	4.794
2.4	0.418	0.817	1.555	2.210	2.840	3.323	4.307	5.031
2.5	0.422	0.827	1.580	2.254	2.908	3.416	4.464	5.257
2.6	0.427	0.837	1.603	2.295	2.970	3.503	4.623	5.472
2.7	0.430	0.845	1.623	2.332	3.027	3.582	4.754	5.675
2.8	0.434	0.853	1.644	2.368	3.081	3.656	4.888	6.869
2.9	0.437	0.860	1.661	2.400	3.130	3.726	5.012	6.052
3.0	0.440	0.867	1.679	2.429	3.176	3.790	5.129	6.226

and eight different values of the depth θ. The evaluation for small depths was carried out using the exact expression (18); however, for $\theta \geq 10$ we used the simplified expression given by (19). It will be noted from (18), (18a) and (18b) that the value of $g^{(i,k)}(s,\theta)$ remains finite for all positive s and θ values.

In order to evaluate $N^{(i,k)}(E,\theta)$ we may use a single saddle integration on the $N_n^{(i,k)}(E,\theta)$, yielding:

$$N_n^{(i,k)}(E,\theta)=[-2\pi J''(s_n,\theta)]^{-\frac{1}{2}}\exp[-J(s_n,\theta)], \quad (20)$$

where

$$\begin{aligned}-J(s,\theta)=&(s-1)\ln(E_0/\beta)\\&+(s+n-1)\ln[\beta/E+\beta g^{(i,k)}(s,\theta)]\\&-\ln(s+n-1)+\ln\Gamma(s+n)\\&-\ln\Gamma(s)+\ln[f_n(i,k)(s,\theta)],\end{aligned} \quad (21)$$

and

$$J''(s_n,\theta)=\{\partial^2 J(s,\theta)/\partial s^2\}_{s=s_n}. \quad (21a)$$

The saddle point, s_n, is determined by the relation

$$\partial J(s,\theta)/\partial s=0 \quad (22)$$

i.e., from

$$\begin{aligned}&\ln(E_0/\beta)-\frac{1}{s+n-1}+\frac{\partial}{\partial s}[\ln\Gamma(s+n)-\ln\Gamma(s)]\\&+\frac{\partial}{\partial s}\ln[f_n^{(i,k)}(s,\theta)]+\Bigg\{\ln\frac{\beta}{E+\beta g^{(i,k)}(s,\theta)}\\&-(s+n-1)\frac{\beta}{E+\beta g^{(i,k)}(s,\theta)}\frac{\partial}{\partial s}g^{(i,k)}(s,\theta)\Bigg\}=0.\end{aligned} \quad (23)$$

In carrying out the evaluation of (23) it will be noted that for $E\gg\beta$ the second term in the last set of brackets in (23) can be neglected without appreciable error; however, for E equal to or of the order of β, it cannot be neglected. This difficulty is remedied by noting that the saddle point, s_n, varies only very slowly with E and thus

$$\{\beta/[E+\beta g^{(i,k)}(s,\theta)]\}^{s+n-1}$$

can be treated as a slowly varying factor (in comparison with $f_n^{(i,k)}(s,\theta)$). It can thus be removed from the expression for $J(s,\theta)$ and treated as a purely multiplicative factor, evaluated at the points $s=s_n$, where s_n

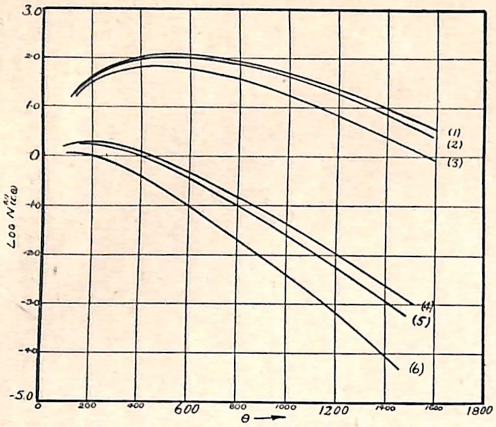

Fig. 1. The logarithm of the average number of protons above a given energy E at an atmospheric depth of θ g/cm² due to an incident primary proton of energy E_0. In curve (1), $E_0/E=10^5$, $\beta=0$; curve (2), $E_0/E=10^5$, $E_0=13\times10^{13}$ ev, $E=10\beta$; curve (3), $E_0/E=10^5$, $E_0=13\times10^{12}$ ev, $E=\beta$; curve (4), $E_0/E=10^2$, $\beta=0$; curve (5), $E_0/E=10^2$, $E_0=13\times10^{10}$ ev, $E=10\beta$; curve (6), $E_0/E=10^2$, $E_0=13\times10^9$ ev, $E=\beta$.

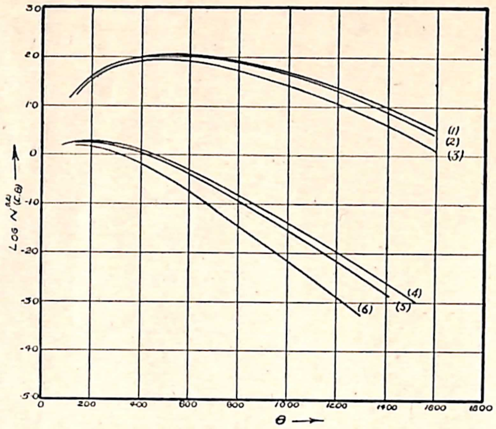

FIG. 2. The logarithm of the average number of neutrons above a given energy E at an atmospheric depth of θ g/cm² due to an incident primary proton of energy E_0. In curve (1), $E_0/E=10^5$, $\beta=0$; curve (2), $E_0/E=10^5$, $E_0=13\times10^{13}$ ev, $E=10\beta$; curve (3), $E_0/E=10^5$, $E_0=13\times10^{12}$ ev, $E=\beta$; curve (4), $E_0/E=10^2$, $\beta=0$; curve (5), $E_0/E=10^2$, $E_0=13\times10^{10}$ ev, $E=10\beta$; curve (6), $E_0/E=10^2$, $E_0=13\times10^9$ ev, $E=\beta$.

is now determined from (23) by neglecting the factor in the curly brackets.

As was mentioned in Sec. II, for all practical purposes it suffices to take the first two terms of the series for $N^{(i,k)}(E,\theta)$ and if we do not consider energies less than the ionization loss β, then a fair approximation to $N^{(i,k)}(E,\theta)$ is provided by $N_0^{(i,k)}(E,\theta)$.

Thus for $E\approx\beta$, $E_0\gg\beta$,

$$N^{(i,k)}(E,\theta)\approx\left\{\frac{\beta}{E+\beta g^{(i,k)}(s,\theta)}\right\}^{s-1}\frac{1}{2\pi i}\times\int_C\left(\frac{E_0}{\beta}\right)^{s-1}f_0^{(i,k)}(s,\theta)\frac{ds}{s-1}. \quad (24)$$

(Note that on the right we now simply have a correction factor multiplying the solution neglecting ionization loss.)

In Figs. 1, 2, and 3 we have given the results of a calculation using (24).[11] Several different primary energies were chosen and the values of E considered were $E=\beta$ and $E=10\beta$. We have considered the cases $i=1$ and $k=1$ and 2.

V. DISCUSSION OF RESULTS

In Fig. 1 we have plotted the average numbers of protons above a given energy E in a cascade developing in a finite absorber, due to a primary proton of energy E_0, both when ionization losses are neglected and when they are taken into account. Various values of the primary energy E_0 and of the energy E were chosen, in order to show how the effect of ionization loss varies according to the energy range in which we are interested. Similarly, in Fig. 2 the average numbers of neutrons due to a primary proton for the same energy ranges are plotted. Finally, in Fig. 3 the ratio of neutrons to protons is given.

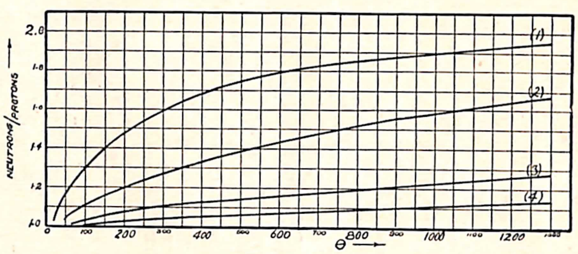

FIG. 3. The ratio of the number of neutrons to protons above a given energy E, at an atmospheric depth of θ g/cm² due to an incident primary proton of energy E_0. In curve (1), $E_0/E=10^2$, $E_0=13\times10^9$ ev, $E=\beta$; curve (2), $E_0/E=10^5$, $E_0=13\times10^{12}$ ev, $E=\beta$; curve (3), $E_0/E=10^2$, $E_0=13\times10^{10}$ ev, $E=10\beta$; curve (4), $E_0/E=10^5$, $E_0=13\times10^{13}$ ev, $E=10\beta$.

It is seen from Figs. 1, 2, and 3 that the effect of ionization is by no means negligible when $E=\beta$; however, with increasing energy E, the effect readily falls off and already for $E=10\beta$ is practically negligible.

Comparison of the theory (with ionization loss) and experimental data is being made in a subsequent paper.

My thanks are due to Professor E. Schrödinger for encouragement to carry on with the above research and to Professor L. Jánossy for many stimulating discussions. I would like to express my thanks to the National Research Council of Canada for provision of a Special Scholarship and to the School of Theoretical Physics at the Dublin Institute for Advanced Studies for the provision of a Research Scholarship.

[11] At small depths the calculation is out by a small factor due to the neglect in the numerical work of the second exponential occurring in $f_0^{(i,k)}(s,\theta)$. Due to this fact $N^{(1,1)}(E,\theta)$ is slightly greater and $N^{(1,2)}(E,\theta)$ slightly smaller than given by the curves for depths $\theta=0$ to 200 g/cm². As a consequence the ratio of neutrons to protons in the same depth range is exaggerated.