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INVESTIGATION INTO THE HIGHER MOMENTS OF A NUCLEON CASCADE.

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§1. In a previous publication (1) by one of us (H. M.) the fluctuation of a nucleon cascade in homogeneous nuclear matter was considered. Analytical expressions for average numbers and the mean square deviation were given, as well as numerical results which showed that the mean square deviation in the case of a nucleon cascade in homogeneous nuclear matter was very similar to that found previously by Jánossy and Messel (2) for the electron-photon cascade. In the present paper we generalise the procedure so as to obtain moments of a cascade which develops not in a single nucleus but in several nuclei. The problem of the traversal through a single nucleus is important for the analysis of stars in emulsions. The problem of a finite absorber is of importance for the interpretation of counter or cloud chamber experiments and also for the theory of the development of large nucleon showers in the air.

The first moment, i.e., the average number of particles in a cascade developing in a finite absorber was investigated by Jánossy (3). The treatment of the higher moments is more complicated and will be dealt with in the present paper. The problem can be approached starting with a generalisation of the G–equation given by Jánossy (4) and used by Messel in (1).

§2. In order to obtain a G–equation for a nucleon cascade passing through a finite absorber we regard the traversals through nuclei as single events transforming an incident nucleon into a group of several nucleons.

Write Φ for the total cross-section of a nucleus. The probability of a collision along a path $d\,\theta$ is then

$$a\,d\theta = N\,\Phi\,d\theta \tag{1}$$

where N is the number of nuclei per gram of absorber and θ is the path in mass equivalent. We shall chose the unit of θ so as to make

$$a = 1 \tag{2}$$

We write $\phi_\nu (\epsilon_1, \epsilon_2 \ldots \epsilon_\nu)\, d\,\epsilon_1 \ldots d\,\epsilon_\nu$ for the probability that a primary of energy E_0 hitting a nucleus should give rise to ν nucleons coming out of the nucleus the latter having energies in the respective intervals

$$\epsilon_i\, E_0,\ (\epsilon_i + d\,\epsilon_i)\, E_0 \qquad i = 1, 2 \ldots\ldots \nu$$

$\phi_\nu (\epsilon_1 \ldots \epsilon_\nu)$ is assumed to be symmetric in its variables and

$$a_\nu = \frac{1}{\nu!} \int_0^1 \ldots \int_0^1 \phi_\nu (\epsilon_1 \ldots \epsilon_\nu)\, d\epsilon_1 \ldots d\epsilon_\nu \tag{3}$$

is the total probability for a collision leading to ν particles. The probabilities should be normalized thus

$$\sum_{\nu = 1} a_\nu = 1 \tag{4}$$

$a_0 = 0$ as we assume that collisions always lead to the splitting up of energy but never to the actual loss of a particle. We shall also use the definitions

$$\phi_{n,\nu} (\epsilon_1, \ldots \epsilon_n) = (n!/\nu!) \int_0^1 \ldots \int_0^1 \phi_{\nu + n} (\epsilon_1, \ldots \epsilon_n, \epsilon_{n+1}, \ldots \epsilon_{\nu + n})\, d\epsilon_{n+1} \ldots d\epsilon_{\nu + n}$$

in particular

$$\phi_{n,0} (\epsilon_1 \ldots \epsilon_n) = \phi_n (\epsilon_1 \ldots \epsilon_n) \tag{5}$$

and

$$a_n (\epsilon_1, \ldots \epsilon_n) = \sum_{\nu = 0}^{\infty} \phi_{n,\nu} (\epsilon_1 \ldots \epsilon_n) \tag{6}$$

Finally we introduce the Mellin transform of the $a_n (\epsilon_1, \ldots \epsilon_n)$ as follows

$$b_n (s_1, s_2 \ldots s_n) = \frac{1}{n!} \int_0^1 \ldots \int_0^1 \epsilon_1^{s_1} \ldots \epsilon_n^{s_n}\, a_n (\epsilon_1 \ldots \epsilon_n)\, d\epsilon_1 \ldots d\epsilon_n \tag{7}$$

We shall give explicit expressions for the quantities thus defined in §4 and §5.

The cascade in an absorber can be described by a function $\pi (n, \epsilon, \theta)$. This function gives the probability that at a depth θ there should be n particles with energies exceeding $\epsilon\, E_0$ apart from any number of particles with lower energies.

It is convenient to introduce a generating function

$$G(u, \epsilon, \theta) = \sum_{\nu=0}^{\infty} u^{\nu} \pi(\nu, \epsilon, \theta) \tag{8}$$

Because of normalization we have

$$G(1, \epsilon, \theta) = 1 \tag{9}$$

We shall be interested in the first and second moments of the distribution. Namely

$$N(\epsilon, \theta) = \left(\frac{\partial G}{\partial u}\right)_{u=1} = \tilde{n} \tag{10}$$

and

$$T(\epsilon, \theta) = \left(\frac{\partial^2 G}{\partial u^2}\right)_{u=1} = \overline{n(n-1)} \tag{11}$$

§3. *The G–equation.*

The probability that a primary travels a distance θ' without a collision is $e^{-\theta'}$ as we have normalized the collision probability suitably. The probability that the primary should suffer a collision in the interval $d\,\theta'$ leading to ν nucleons in intervals $d\epsilon_1 \ldots d\epsilon_\nu$ is thus

$$e^{-\theta'} d\theta' \, \phi_\nu (\epsilon_1 \ldots \epsilon_\nu) \, d\epsilon_1 \ldots d\epsilon_\nu.$$

The nucleons with energies $\epsilon_1 \ldots \epsilon_\nu$ can be regarded as primaries of new cascades and the probability that the secondary cascades should give rise in the end to a cascade containing n particles with energies $> \epsilon E_0$ is

$$\sum_{n_1 + n_2 + \ldots n_\nu = n} \epsilon^{-\theta'} d\theta' \, \pi(n_1, \frac{\epsilon}{\epsilon_1}, \theta - \theta') \, \pi(n_2, \frac{e}{\epsilon_2}, \theta - \theta') \ldots .$$

$$\ldots . \pi(n_\nu, \frac{\epsilon}{\epsilon_\nu}, \theta - \theta') \, \phi_\nu (\epsilon_1 \ldots \epsilon_\nu) \, d\epsilon_1 \ldots d\epsilon_\nu .$$

Multiplying the above equation by

$$u^n = u^{n_1 + n_2 \ldots . n_\nu}$$

and summing over $n = 0, 1 \ldots$ we obtain on the right-hand side a produce of ν G functions. Integrating over θ' from 0 to θ and also over all possible intermediate values of the ϵ we find eventually

$$G(u, \epsilon, \theta = \int_0^0 e^{-\theta'} d\theta' \int_0^1 \ldots \int_0^1 \sum_{\nu=1}^{\infty} \frac{1}{\nu!} G(u, \frac{\epsilon}{\epsilon_1}, \theta - \theta') \ldots G(u, \frac{\epsilon}{\epsilon_\nu}, \theta - \theta')$$

$$\phi_\nu \, (\epsilon_1 \ldots \epsilon_\nu) \, d\epsilon_1 \ldots d\epsilon_\nu .$$

The above equation can be transformed easily into a form similar to that given in (1), namely,

$$\frac{\partial G(\epsilon, \theta)}{\partial \theta} = \int_0^1 \dots \int_0^1 \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \left\{ G\left(\frac{\epsilon}{\epsilon_1}, \theta\right) \dots G\left(\frac{\epsilon}{\epsilon_\nu}, \theta\right) - G(\epsilon, \theta) \right\} \phi_\nu (\epsilon_1 \dots \epsilon_\nu) \, d\epsilon_1 \dots d\epsilon_\nu . \quad (12)$$

We have omitted the variable u for simplicity. Differentiating (12) with respect to u, we find for $u = 1$ with help of (6), (9) and (10)

$$\frac{\partial N(\epsilon, \theta)}{\partial \theta} + N(\epsilon, \theta) = \int_0^1 N\left(\frac{\epsilon}{\epsilon_1}, \theta\right) a_1(\epsilon_1) \, d\epsilon_1 \quad (13)$$

The solution of (13) is obtained in the usual way by means of a Mellin transformation (see also equ. (7)).

$$N(\epsilon, \theta) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \epsilon^{-s} e^{-\gamma(s)\theta} \frac{ds}{s} \quad (14)$$

$$\gamma(s) = 1 - b_1(s) .$$

Similarly differentiating (12) twice with respect to u we find for $u = 1$ with help of (6), (9) and (11)

$$\frac{\partial T(\epsilon, \theta)}{\partial \theta} + T(\epsilon, \theta) = \int_0^1 T\left(\frac{\epsilon}{\epsilon_1}, \theta\right) a_1(\epsilon_1) \, d\epsilon_1$$
$$+ \int_0^1 \int_0^1 N\left(\frac{\epsilon}{\epsilon_1}, \theta\right) N\left(\frac{\epsilon}{\epsilon_2}, \theta\right) a_2(\epsilon_1, \epsilon_2) \, d\epsilon_1 \, d\epsilon_2 . \quad (15)$$

The last term in (15) can be expressed with help of (14) as follows (see (7))

$$\int_0^1 \int_0^1 N\left(\frac{\epsilon}{\epsilon_1}, \theta\right) N\left(\frac{\epsilon}{\epsilon_2}, \theta\right) a_2(\epsilon_1, \epsilon_2) \, d\epsilon_1 \, d\epsilon_2 = \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \epsilon^{-r-t}$$
$$e^{-\theta(\gamma(r) + \gamma(t))} \, b_2(r, t) \, \frac{dr \, dt}{rt} . \quad (16)$$

Apart from notation, equ. (15) is analogous to that for the second moment in homogeneous nuclear matter. Thus the solution can be written out immediately as

$$T(\epsilon, \theta) = \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \frac{\epsilon^{-(r+t)}}{rt} \frac{e^{-\theta(\gamma(r)+\gamma(t))} - e^{-\theta(\gamma(r+t))}}{\gamma(r+t) - \gamma(r) - \gamma(t)} b_2(r, t)\, dr\, dt \qquad (18)$$

$T(\epsilon, \theta)$ can thus be determined numerically by a double saddle point integration.

§4. It remains to determine the $\phi_\nu (\epsilon_1 \dots \epsilon_\nu)$'s and the quantities derived thereof. For this purpose it is necessary to consider the processes in a single nucleus in more detail. We consider functions

$$\overline{\phi}_\nu (\epsilon_1, \dots \epsilon_\nu ; x)\, d\epsilon_1 \dots d\epsilon_\nu$$

representing the differential probability that a primary should give rise to exactly ν primaries in specified intervals after traversing an amount x of homogeneous nuclear matter. The connection with the ϕ_ν's is as follows:

$$\phi_\nu (\epsilon_1 \dots \epsilon_\nu) = \int_0^D \overline{\phi}_\nu (\epsilon_1 \dots \epsilon_\nu ; x) \frac{2x dx}{D^2}. \qquad (19)$$

D is the nuclear diameter; $\frac{2x dx}{D^2}$ is the probability that the primary travels a distance between x and $x + dx$ through the nucleus. In ref. (3) it was shown that we may write

$$\phi_\nu (\epsilon_1 \dots \epsilon_\nu ; x) = a_\nu (\epsilon_1 \dots \epsilon_\nu)\, e^{-x} (1 - e^{-x})^{\nu - 1}. \qquad (20)$$

The a_ν's can be determined from the following regression :—

$$a_1 (\epsilon) = \delta (1 - \epsilon)$$

$$(\nu - 1)\, a_\nu (\epsilon_1 \dots \epsilon_\nu) = \int_0^1 \sum_{k \neq l} a_{\nu-1} (\epsilon'_1 \dots \epsilon'_{\nu-2} \dots, \epsilon)\, w\left(\frac{\epsilon_k}{\epsilon}, \frac{\epsilon_l}{\epsilon}\right) \frac{d\epsilon}{\epsilon^2} \qquad (21)$$

$$\nu \geqslant 2$$

where $\epsilon' \dots \epsilon'_{\nu-2} = \epsilon_1, \dots \epsilon_\nu$ omitting ϵ_k and ϵ_l and $w(\epsilon_1, \epsilon_2)$ is the differential collision probability for a single nucleon – nucleon encounter.

The ν–fold Mellin transform of (21) can be written as follows. Put

$$\frac{1}{\nu!}\int_0^1 \ldots \int_0^1 \epsilon_1^{s1}\, \epsilon_2^{s2}\, \epsilon_\nu^{s_\nu}\, a_\nu\, (\epsilon_1 \ldots \epsilon_\nu)\, d\,\epsilon_1 \ldots d\,\epsilon_\nu = T\,(s_1, s_2 \ldots s_\nu)$$

$$\int_0^1\int_0^1 w\,(\epsilon_1, \epsilon_2)\, \epsilon_1^s\, \epsilon_2^t\, d\epsilon_1\, d\epsilon_2 = W\,(s, t) \tag{22}$$

$$W\,(0,\ 0) = 1$$

We have

$$T_1\,(s) = 1,$$

$$T_\nu\,(s_1, s_2 \ldots s_\nu) = \frac{1}{\nu\,(\nu-1)} \sum_{k \neq l} T_{\nu-1}(s'_1, s'_2 \ldots s'_{\nu-2}, s_k + s_l)\, W\,(s_k, s_l) \tag{23}$$

$$s'_1 \ldots s'_{\nu-2} = s_1 \ldots s_\nu \text{ omitting } s_k \text{ and } s_l,\ \nu \geqslant 2\,.$$

From the recursion (23) the T's can be determined step by step. The expressions thus obtained become increasingly complicated with increasing ν. Fortunately, for our actual problem we do not need the complete expressions but it is sufficient to determine T values which contain only one or two s values different from zero. In fact generalising the procedure of §3 it can be readily seen that the k–th moments of the distribution depend on T's containing k "s" values different from zero. We thus introduce

$$t_{n,\,\nu}(s_1 \ldots s_k) = \frac{(n+\nu)!}{\nu!}\, T_{n+\nu}\,(s_1 \ldots s_n, 0 \ldots 0) \tag{24}$$

in particular for $k = 0$

$$t_{0,\,\nu} = T\,(0, 0 \ldots 0) \quad \text{and} \quad t_{n,\,0}\,(s_1 \ldots s_k) = n!\; T_n\,(s_1 \ldots s_n) \tag{25}$$

From (23) and (25) we find

$$t_{0,\,\nu} = 1 \qquad \nu = 1, 2 \ldots \tag{26}$$

Introducing the $t_{n,\,\nu}$ into (23) we find

$$t_{n,\,\nu}\,(s_1 \ldots s_n) = t^*_{n,\,\nu}\,(s_1 \ldots s_n) + t_{n,\,\nu-1}\,(s_1 \ldots s_n)\, \frac{W^*_n\,(s_1 \ldots s_n) + \nu}{n + \nu - 1} \tag{27a}$$

where

$$t^*_{1,\,\nu}\,(s) = 0$$

$$t^*_{n,\,\nu}\,(s_1 \ldots s_n) = \frac{1}{n+\nu-1} \sum_{k \neq l} t_{n-1,\,\nu}\,(s'_1 \ldots s'_{n-2}, s_k + s_l)\; W\,(s_k, s_l)$$

$$n > 1 \tag{27b}$$

and

$$W^*_n(s_1, \ldots s_n) = 2 \sum_{k=1}^{n} W(s_k, 0) - 1 \tag{27c}$$

(26), (27a, 27b, 27c) give a full recursion for the $t_{n,\nu}$'s. For $n = 1$ we find

$$t_{1,\nu} = \frac{W^*_1(s) + \nu}{\nu} \cdot t_{1,\nu-1}.$$

Since $t_{1,0} = 1$ we have

$$t_{1,\nu} = \frac{(W^*_1(s) + \nu)!}{\nu!\, W^*_1(s)!} = \binom{-1 - W^*_1(s)}{\nu} (-)^\nu \ldots \tag{28}$$

To deal with $n > 2$ we find from (27) as the result of a simple manipulation

$$t_{n,\nu}(s_1 \ldots s_n) = \sum_{\mu=0}^{\nu} t^*_{n,\mu} \frac{(n + \mu - 1)!}{(W^*_n + \mu)!} \cdot \frac{(W^*_n + \nu)!}{(n + \nu - 1)!} \tag{29}$$

$$n \geqslant 2$$

(29) and (27b) provide a full recursion.
The first step gives

$$t^*_{2\nu}(s_1, s_2) = \frac{2\,W(s_1, s_2)}{(\nu + 1)!} - \frac{(W^*_1(s_1 + s_2) + \nu)!}{W^*_1(s_1 + s_2)!}$$

And thus

$$t_{2\nu}(s_1, s_2) = \frac{2W(s_1, s_2)\,(-)^{\nu+1}}{W_2^*(s_1, s_2) - W_1^*(s_1 + s_2) - 1} \left\{ \binom{-W^*_2(s_1, s_2)}{\nu + 1} - \binom{-W_1^*)s_1 + s_2) - 1}{\nu + 1} \right\} \tag{30}$$

The $t_{k\nu}$ for $k = 3, 4 \ldots$ can be determined in a similar way. It can be shown by induction that $t_{k\nu}$ can be expressed as a linear combination of certain binomial coefficients

$$\binom{A}{\nu + k - 1}$$

§5. It remains to express the $b_n(s_1 \ldots s_n)$ defined by (5), (6) and (7) in terms of the $t_{n,\nu}(s_1 \ldots s_n)$. With help of (19) we have

$$b_n (s_1 \ldots s_n) = \int_0^D \frac{2x dx}{D^2} \sum \frac{1}{\nu!} \int_0^1 \cdots \int_0^1 \epsilon_1^{s_1} \ldots$$

$$\ldots \epsilon_n^{s_n} \int_0^1 \cdot \int_0^1 \bar{\phi}_{n+\nu} (\epsilon_1 \ldots \epsilon_{n+\nu}, x)\, d\epsilon_1 \ldots d\epsilon_n$$

expressing the integrals over the $\bar{\phi}_{n+\nu}$ in terms of the $t_{n,\nu}$'s we have further

$$b_n (s_1 \ldots s_n) = \int_0^D \frac{2x dx}{D^2} \sum_{\nu=0}^{\infty} e^{-x} (1 - e^{-x})^{\nu+n-1} t_{n,\nu} (s_1 \ldots s_n)$$

Thus for $n = 1$ we find with help of (28), and writing for short

$$W^*_1 (s) = -a_s$$

and

$$f(\lambda) = 2 \int_0^1 e^{-\lambda x} x\, dx \qquad (33)$$

then

$$b_1 (s) = f(-D\, W^*_1 (s)) = f(D\, a_s)$$

for $n = 2$,

$$b_2 (s_1, s_2) = 2W (s_1, s_2) \,.\, \frac{f(D(1 - W^*_2(s_1, s_2))) - f(D(-W^*_1(s_1 + s_2)))}{W_2^* (s_1, s_2) - W_1^* (s_1 + s_2) - 1}$$

$$= 2\, W (s_1, s_2) \frac{f\{D(a_{s_1} + a_{s_2})\} - f(D\, a_{s_1} + s_2)}{a_{s1+s2} - a_{s1} - a_{s2}} \qquad (34)$$

where

$$1 - W^*_2(s_1, s_2) = a_{s_1} + s_{s_2}$$

for

$$\lambda \to 0 \quad f(\lambda) \to 1 - {}^2/_3\, \lambda + \lambda^2/_4 \ldots .$$

thus for $D = a \to 0$ we have

$$\left. \begin{array}{l} b_1 (s) \to a_s \\ b_2 (s_1, s_2) \to 2\, a\, W_1 (s_1, s_2) \end{array} \right\} \qquad (34a)$$

Equ. (34a) shows that for small values of D the expressions (14) and (18) approach the expressions found for homogeneous nuclear matter in ref. (3). In order to facilitate computation we give in section 5 a resumé of formulae appearing in (1) and in the present paper.

§5. *Resumé of formulae for the development and fluctuation of a nucleon cascade in homogenous and inhomogeneous matter.*

(a) Homogeneous Nuclear Matter.

The average number of nucleons with energies greater than ϵE_0 at a depth x collisions due to a primary nucleon of energy E_0 is given by

$$N(\epsilon,\ x) = \frac{1}{2\ \pi i}\int_{s_0-i\infty}^{s_0+i\infty} \epsilon^{-s} e^{-a_s x}\,\frac{ds}{s} \tag{35}$$

where

$$a_s = \int_0^\infty\int_0^\infty (1 - \epsilon^s{}_1 - \epsilon^s{}_2)\ w\ (\epsilon_1,\ \epsilon_2)\ d\,\epsilon_1\ d\,\epsilon_2 \tag{36}$$

and $w\ (\epsilon_1,\ \epsilon_2)\ d\,\epsilon_1,\ d\,\epsilon_2$ is the cross section.

Considering a process in which a nucleon of energy E_0 loses energy and also gives rise to a recoil nucleon, it is assumed that the probability for a collision to occur is given by

$$w(E_0\ ;\ E_1,\ E_2)\ dE_1\ dE_2 = w\left(\frac{E_1}{E_0},\frac{E_2}{E_0}\right)\frac{dE_1 dE_2}{E_0{}^2} = \sigma\ \epsilon_1{}^\beta\ (1-\epsilon_1)^\gamma\ d\epsilon_1\ d\epsilon_2 \tag{37}$$

We chose $\beta = 2\quad \gamma = 1$ and $\sigma = 15$.

The expression for the difference between the first and second moments of a nucleon cascade was given in (1) as

$$\tau\ (\epsilon,\ x) = \overline{n^2} - \bar{n}^2 = B^*\,(r,\ r)\,\frac{1}{(2\ \pi i)^2}\int_{r_0-i\infty}^{r_0+i\infty}\int_{t_0-i\infty}^{t_0+i\infty} \epsilon^{-(r+t)}$$

$$\{\exp - (a_r + a_t)\,x + \exp - a_{r+t}\,x\}\,\frac{dr}{r}\,\frac{dt}{t} \tag{38}$$

and

$$\frac{\tau\ (\epsilon,\ x)}{\bar{n}^2 + n} = B^*\ (r,\ r) \tag{39}$$

with

$$B^*(r,\ t) = \left\{\frac{2\ W\,(r,\ t)}{a_{r+t} - a_r - a_t} - 1\right\}\frac{1 - \exp\,(a_r + a_t - a_{r+t})x}{1 + \exp\,(a_r + a_t - a_{r+t})x} \tag{40}$$

and

$$W\,(r,\ t) = \int_0^\infty\int_0^\infty \epsilon^r{}_1\ \epsilon^t{}_2\,w\,(\epsilon_1,\ \epsilon_2)\ d\epsilon_1\,d\epsilon_2\,. \tag{41}$$

(*b*) Inhomogeneous Matter.

The average numbers with energies greater than E_0 due to a primary nucleon of energy E_0, in this case at a depth θ in ordinary matter is

$$N(\epsilon, \theta) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \epsilon^{-s} \exp - \theta g (D_A a_s) \frac{ds}{s} \tag{42}$$

with

$$g(t) \doteq (1 - f(t)) = 1 - 2 \frac{1 - (1 + t) e^{-t}}{t^2}. \tag{43}$$

(We write D_A for D appearing in (19)). D_A is the diameter of a nucleus in units as defined by (22), that is D_A is the average number of collisions made by a nucleon along the diameter of a nucleus. In order to determine D_A one may proceed as follows. Denote the range of nuclear forces by R_k. Let R_A be the radius of a nucleus whose atomic weight is A. When a nucleon traverses a nucleus along its diameter then it sweeps out a volume equal to $2\pi R^2_k R_A$. Hence

$$3/2\, A \frac{R^2_k}{R^2_A}$$

gives the number of collisions suffered by a nucleon along the nuclear diameter. The radius of a nucleus is

$$R_A = 1{\cdot}37 \times 10^{-13} A^{1/3} \text{ cm.}$$

Hence

$$D_A = 1{\cdot}5\, A^{1/3} \left\{ \frac{R_k}{1{\cdot}37 \times 10^{-13}} \right\}^2. \tag{44}$$

For air D_A was taken to be equal to 2·41. The depth θ is actually $\theta = \bar{\theta}\, n\, \Phi_A$, where $\bar{\theta}$ is the depth in g/cm² and $n\, \Phi_A$ is the reciprocal cross-section. The expression corresponding to (38) for the mean square deviation is given now as

$$\tau(\epsilon, \theta) = C^*(r, r) \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \epsilon^{-(r+t)} \Big\{ \exp - [g(D_A a_r) + g(D_A a_t)]\,\theta + \exp g(D_A a_{r+t})\,\theta \Big\} \frac{dr}{r} \frac{dt}{t} \tag{45}$$

and

$$\frac{\tau(\epsilon, \theta)}{\bar{n}^2 + \bar{n}} = C^*(r, r) \tag{46}$$

with

$$C^*(r, t) = \left\{ \frac{2\,W(r, t)\,D_A}{a_{r+t} - a_r - a_t} \cdot \frac{g(a_{r+t}) - g(a_r + a_t)}{g(a_{r+t}) - g(a_r) - g(a_t)} - 1 \right\} \frac{1 - \exp\{g(a_r) + g(a_t) - g(a_{r+t})\}\,\theta}{1 + \exp\{g(a_r) + g(a_t) - g(a_{r+t})\}\,\theta} \tag{47}$$

where we have written $a_r = D_A\, a_r$.

The integrals (42) and (46) were evaluated following the methods outlined in Part I. Table I gives some of the numerical results, figures 1 to 4 the graphs obtained from the calculation and from calculations carried out in (1) Part I.

TABLE I.

r	a_r	a'_r	$f(D_A\,a_r)$	$f'(D_A\,a_r)$
0·1	−0·7501744	2·2133847	−2·6263486	14·5837199
0·2	−0·5518336	1·7758575	−1·5412526	7·9775800
0·3	−0·3912770	1·4507220	−0·9198872	4·8031072
0·4	−0·2590895	1·2039510	−0·5323304	3·1115834
0·5	−0·1486292	1·0131478	−0·2741840	2·1344480
0·6	−0·0551043	0·8631647	−0·0931040	1·5325601
0·7	0·0250114	0·7435116	0·0394672	1·1313645
0·8	0·0943594	0·6467706	0·1394178	0·8771515
0·9	0·1549496	0·5675988	0·2173042	0·6910532
1·0	0·2083333	0·5020834	0·2793276	0·5562470
1·1	0·2557237	0·4473151	0·3297028	0·4560393
1·2	0·2980809	0·4011018	0·3713286	0·3798065
1·3	0·3361730	0·3617703	0·4062316	0·3206672
1·4	0·3706210	0·3280290	0·4358752	0·2739848
1·5	0·4019314	0·2988701	0·4613362	0·2365648
1·6	0·4305212	0·2734995	0·4834218	0·2061579
1·7	0·4567364	0·2512857	0·5027474	0·1811467
1·8	0·4808666	0·2317216	0·5197912	0·1603457
1·9	0·5031553	0·2143975	0·5349272	0·1428744
2·0	0·5238096	0·1989796	0·5484548	0·1280661
2·1	0·5430056	0·1851932	0·5606124	0·1154129
2·2	0·5608950	0·1728121	0·5715958	0·1045194
2·3	0·5776085	0·1616474	0·5815646	0·0950766
2·4	0·5932597	0·1515412	0·5906514	0·0868394
2·5	0·6079476	0·1423611	0·5989662	0·0796126
2·6	0·6217590	0·1339944	0·6066020	0·0732381
2·7	0·6347704	0·1263451	0·6136378	0·0675872
2·8	0·6470492	0·1193319	0·6201402	0·0625551
2·9	0·6586555	0·1128840	0·6261666	0·0580546
3·0	0·6696428	0·1069410	0·6317664	0·0540139
3·1	0·6800588	0·1014500	0·6369824	0·0503722
3·2	0·6899463	0·0963653	0·6418524	0·0470787
3·3	0·6993441	0·0916468	0·6464086	0·0440906
3·4	0·7082868	0·0872598	0·6506796	0·0413715
3·5	0·7168061	0·0831728	0·6546910	0·0388897
3·6	0·7249305	0·0793588	0·6584646	0·0366188
3·7	0·7326862	0·0757935	0·6620206	0·0345355
3·8	0·7400968	0·0724556	0·6653772	0·0326196
3·9	0·7471842	0·0693255	0·6685496	0·0308537
4·0	0·7539683	0·0663863	0·6715524	0·0292227

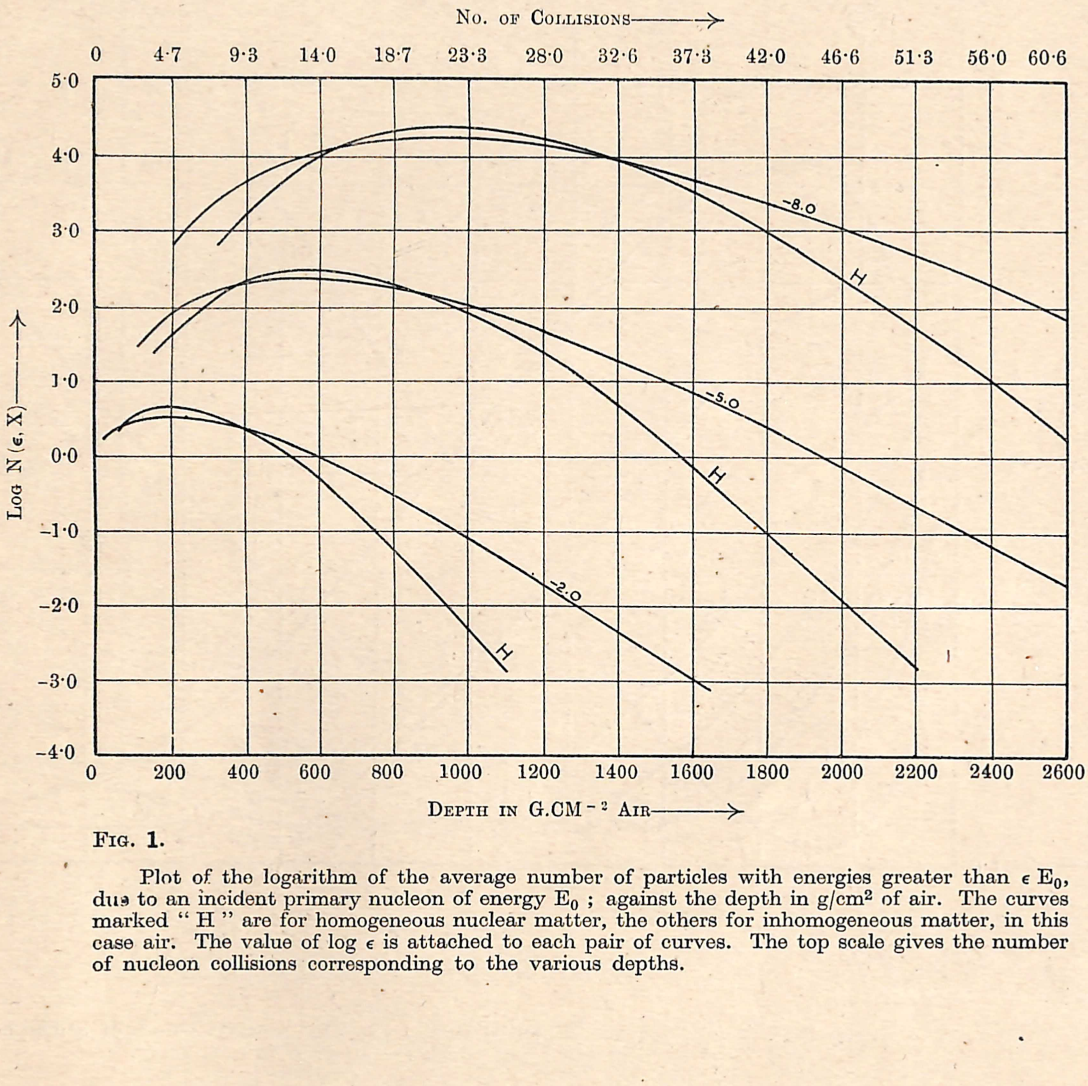

Fig. 1.

Plot of the logarithm of the average number of particles with energies greater than ϵE_0, due to an incident primary nucleon of energy E_0 ; against the depth in g/cm^2 of air. The curves marked " H " are for homogeneous nuclear matter, the others for inhomogeneous matter, in this case air. The value of log ϵ is attached to each pair of curves. The top scale gives the number of nucleon collisions corresponding to the various depths.

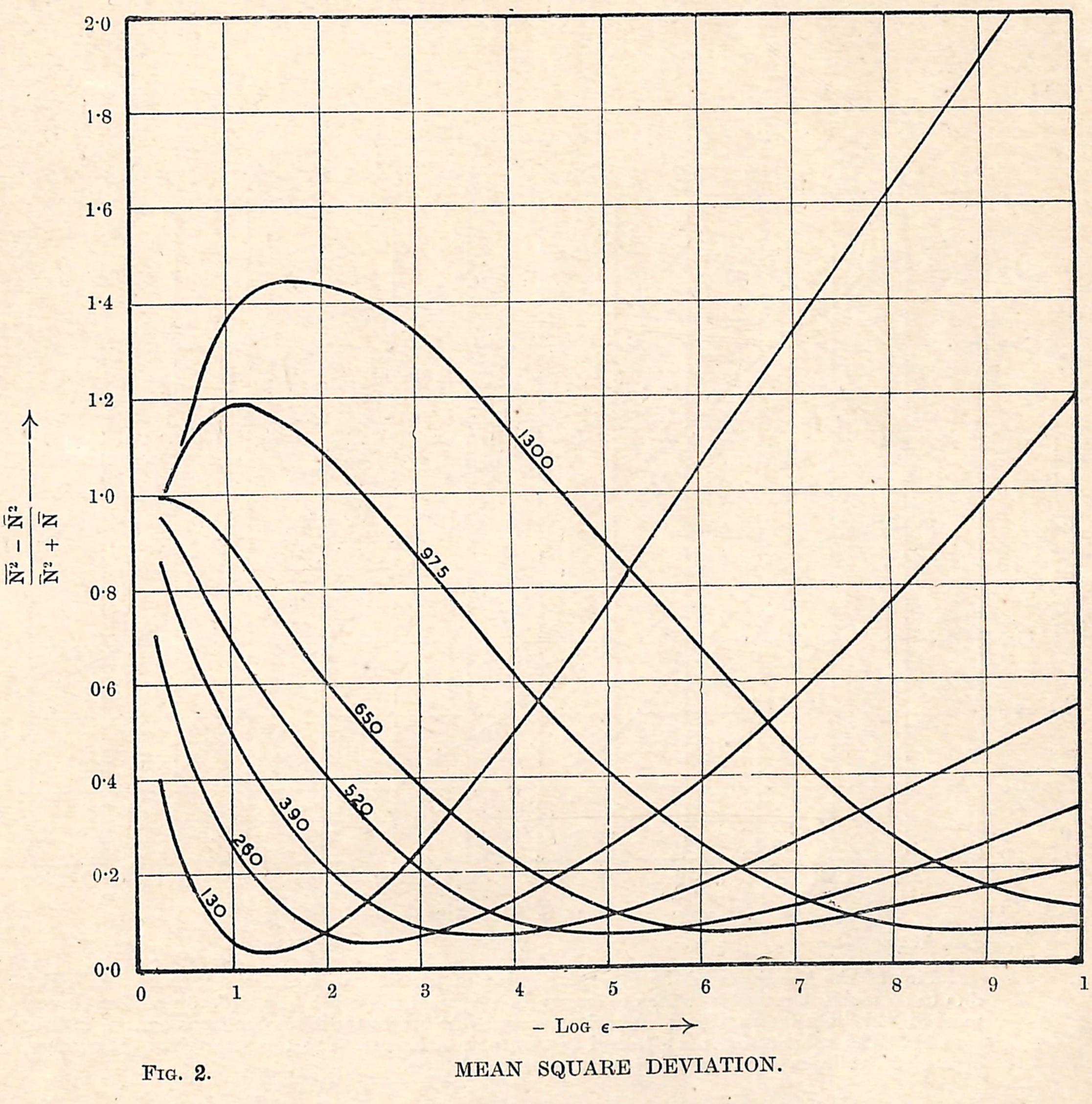

Fig. 2. MEAN SQUARE DEVIATION.

The "deviation" $\dfrac{\overline{N^2} - \bar{N}^2}{\bar{N}^2 + \bar{N}}$ in air is plotted against log ε for constant values of atmospheric depth in g/cm² which are attached to each curve.

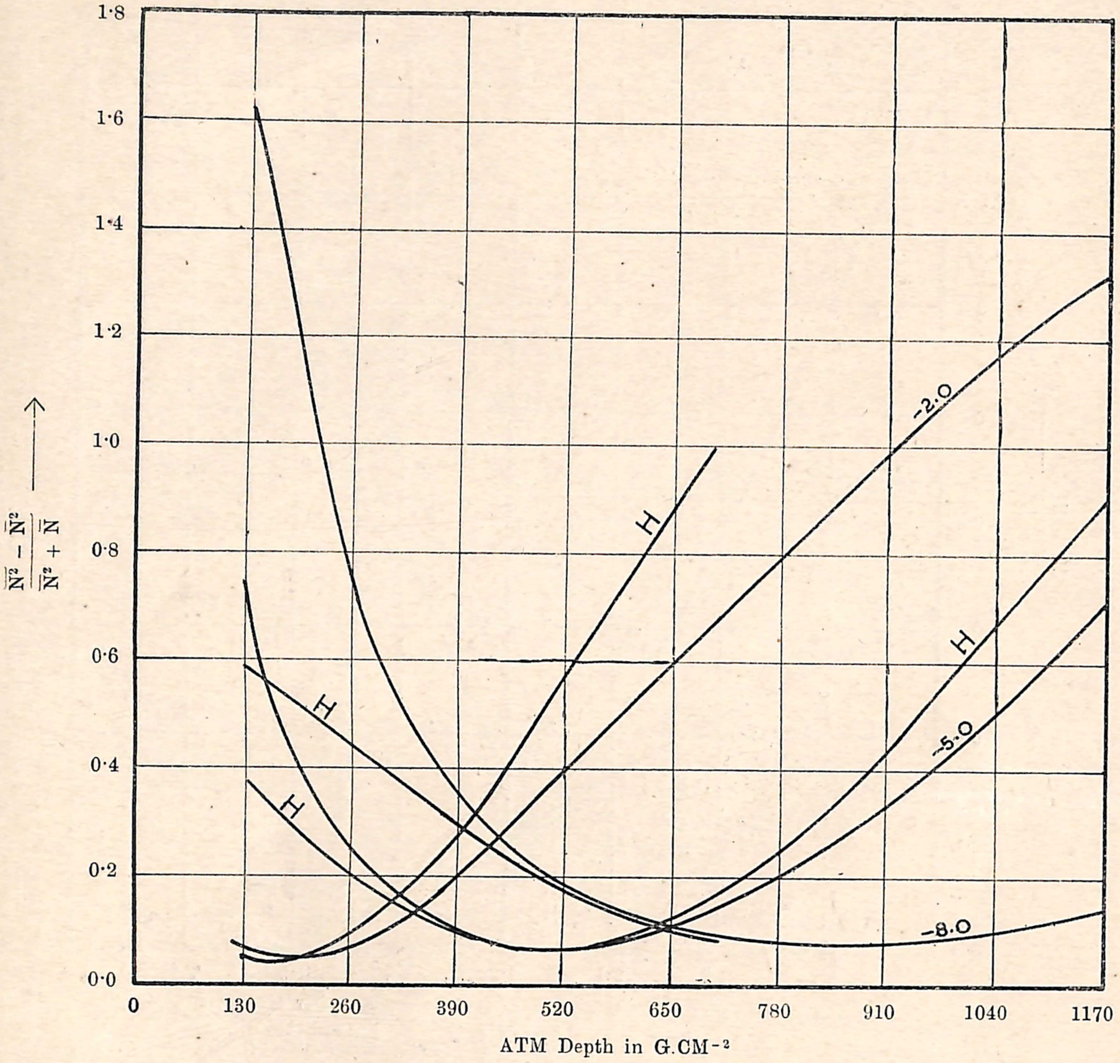

FIG. 3. MEAN SQUARE DEVIATION.

The " deviation " $\frac{\overline{N^2} - \bar{N}^2}{\bar{N}^2 + \bar{N}}$ for homogeneous nuclear matter (curves marked " H "), and for inhomogeneous matter, in this case air, plotted for constant values of log ϵ, against the atmospheric depth in g/cm^2. The value of log ϵ is attached to each pair of curves.

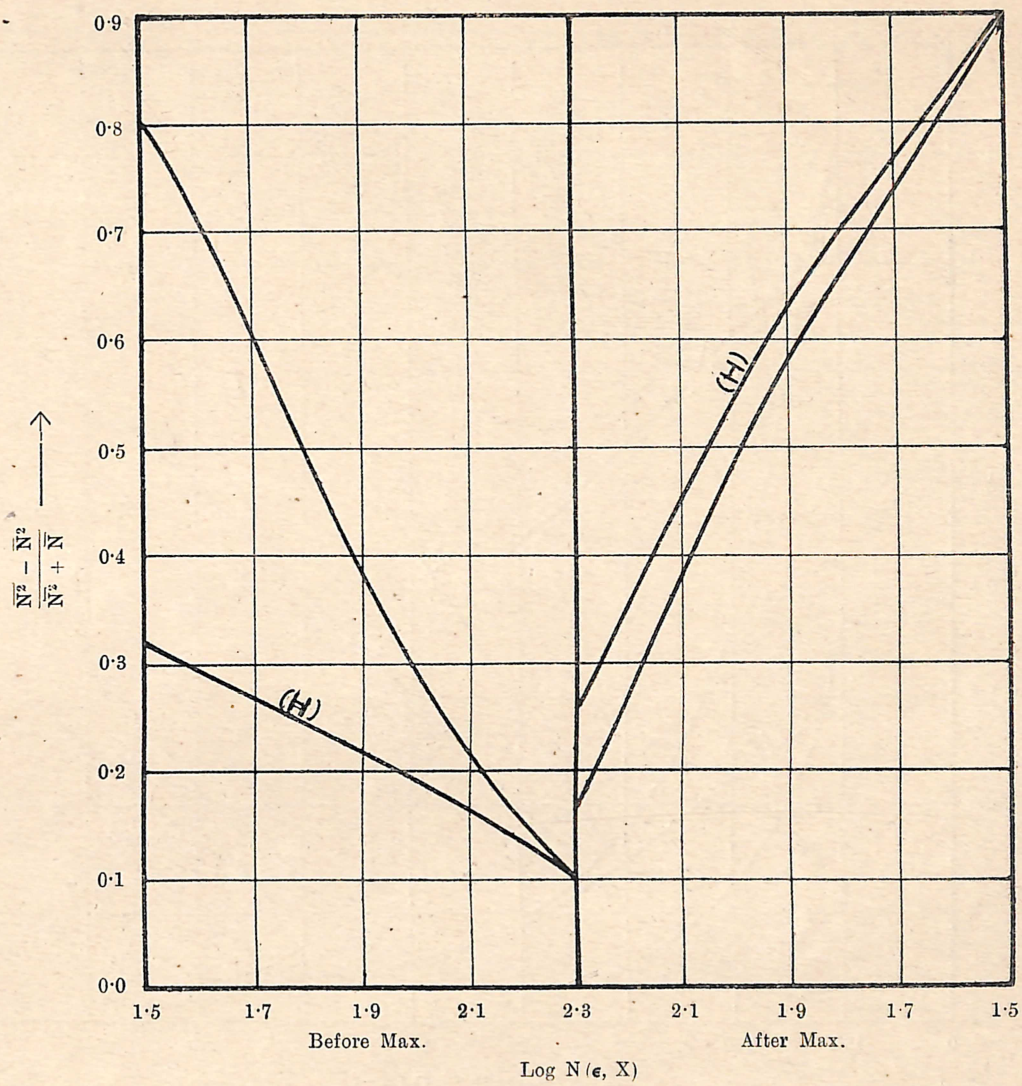

Fig. 4. MEAN SQUARE DEVIATION.

The "deviation" $\frac{\overline{N^2} - \overline{N}^2}{\overline{N}^2 + \overline{N}}$ for homogeneous nuclear matter (curves marked "H"), and for air plotted against the logarithm of the average numbers for log $\epsilon = -5{\cdot}0$. The "bold" centre line corresponds to the maximum

6. *Discussion of the results.*

In fig. 1 we have plotted the average numbers of particles of cascades developing in a finite absorber, i.e. in inhomogeneous nuclear matter. For the sake of comparison the curves for an equivalent amount of homogeneous nuclear matter are also given.

We note that both the position and the height of the maximum for homogeneous and inhomogeneous matter coincide. While the physical reason for this coincidence cannot be seen readily mathematically it can be understood as follows. Evaluating the complex integral for the average numbers with the saddle point method the maximum of the transition curves in the two cases are attained for $s \simeq 0{\cdot}65$. The saddle point conditions at the maximum are, respectively,

$$a_s = 0 \qquad \text{(homogeneous case)}$$

$$g\,(D_A\, a_s) = 0 \qquad \text{(inhomogeneous case)}$$

Since $g\,(0) = 0$ the two conditions lead to the same values of s. Furthermore we find

$$a'_s\, x_{\text{max.}} = g'\,(0)\; a'_s\, D_A\, \theta_{\text{max.}}$$

and with

$$g'\,(0) = \tfrac{2}{3} \quad x = \tfrac{2}{3}\, D_A\, \theta_{\text{max.}}$$

But in our units $\frac{2}{3}\,\theta$ is the average amount of nuclear matter traversed in an absorber θ (see e.g. ref. 4). We see, therefore, that the average amount of nuclear matter $\frac{2}{3}\,\theta_{\text{max.}}$ traversed at the maximum is equal to the actual $x_{\text{max.}}$ path traversed at the maximum in homogeneous nuclear matter. (x is now in units of nuclear diameter). Thus in the two cases the maxima are reached at equivalent thicknesses. The expressions for average number of particles at the maximum differ only in the expression for the width of the saddle, therefore, the actual numbers are not very different.

In the case of inhomogeneous matter the average number of particles found after the maximum is much longer than the corresponding numbers of particles in homogeneous nuclear matter. This behaviour can be understood easily ; in the case of inhomogeneous matter the most important contribution to the intensity arises from particles which have traversed less than average nuclear path. This circumstance has already been pointed out by Heitler and Jánossy (5).

In figs. 2, 3 and 4 we have plotted the second moments of the distribution. In fig. 3 we have also given the curves for homogeneous matter in order that the two cases may be compared. It is seen that the standard deviations in the case of inhomogeneous matter exceed those for homogeneous matter, at any rate in the region before the maximum. This point is further brought out in fig. 4 where the deviations are plotted against the logarithm of the average number for a fixed energy. The larger fluctuation is connected with the reduction of the number of independent events.

The general point of interest arising from the above is the extreme similarity of the result we obtain for the first and second moments of the nucleon and electron-photon cascade. We believe that this is not in any way accidental and that any cascade process with a homogeneous type of cross section and a multiplication ratio of 2, will lead to almost identical results.

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