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ON THE FLUCTUATION OF A NUCLEON CASCADE IN HOMOGENEOUS NUCLEAR MATTER AND CALCULATION OF AVERAGE NUMBERS.

PART I.

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I.—INTRODUCTION.

IN a recent publication Jánossy (1) gave the formalism for the solution of the fluctuation problem in the cases of the nucleon cascade and the electron-photon cascade. The nucleon cascade was described by means of the so-called " G " equation (see section 2). A set of " G " equations were also given for the electron photon cascade. The " G " equation does not admit any simple solution for the " G function," however the equation may be integrated numerically and the actual " distribution curve " obtained for a nucleon cascade.

The present paper is solely concerned with a number of special cases of the above for which the solutions are quite straightforward and analytical expressions easily obtained, i.e., for average numbers and the mean square deviation. Solutions for these in the case of the electron-photon cascade were given in a recent paper by Jánossy and Messel (2).

The " G " equation given by Jánossy is only valid for homogeneous nuclear matter hence the results obtained in the present paper are only applicable to this case. In a publication following this the case of inhomogeneous matter, i.e., matter in which the nucleons are grouped in nuclei, is considered.

Equation (1), given below, refers to a process in which the cross-section is a homogeneous function of primary and secondary energies. This assumption being based upon a previous publication by Heitler and Jánossy (3) in which the absorption of a nucleon cascade was considered. Throughout the work that follows the assumption is maintained and the cross-section chosen is that suggested in the note of the above paper (3), which takes into consideration the recoil nucleon.

II.—Moments of the Distribution.

The differential equation for a nucleon cascade may be given by

$$\frac{\partial}{\partial x} G(\epsilon, U, x) + a\, G(\epsilon, U, x) =$$

$$\int_0^\infty \int_0^\infty G\left(\frac{\epsilon}{\epsilon'}, U, x\right) G\left(\frac{\epsilon}{\epsilon''}, U, x\right) w(\epsilon', \epsilon'')\, d\epsilon'\, d\epsilon'' \qquad (1)$$

where

$$a = \int_0^\infty \int_0^\infty w(\epsilon', \epsilon'')\, d\epsilon'\, d\epsilon'' \qquad (1a)$$

$$G(\epsilon, U, x) = \sum_{n=0}^{\infty} \phi(\epsilon, n, x)\, U^n. \qquad (2)$$

The function $\phi(\epsilon, n, x)$ denotes the probability of finding n nucleons above a given energy ϵE_0 at a depth x. (See section IIIa for units.) These particles were given rise to by a primary of energy E_0. The initial condition that at $x = 0$ there should be one particle of energy E_0 but no other particles, yields

$$G(\epsilon, 1, x) = \sum_{n=0}^{\infty} \phi(\epsilon, n, x) = 1. \qquad (3)$$

Differentiating (2) with respect to U, it is found for $U = 1$

$$\left(\frac{\partial}{\partial U} G(\epsilon, U, x)\right)_{U=1} = \sum_{n=0} n\, \phi(\epsilon, n, x) = \overline{N}(\epsilon, x) \qquad (4)$$

where $\overline{N}(\epsilon, x)$ is the average number of particles with energies above ϵE_0 at a depth x, due to a primary of energy E_0.
Differentiating twice yields

$$\left(\frac{\partial^2}{\partial U^2} G(\epsilon, U, x)\right)_{U=1} = \sum_{n=0} n(n-1)\, \phi(\epsilon, n, x) = \overline{N(N-1)} = T(\epsilon, x) \qquad (5)$$

where $T(\epsilon, x) = \overline{N(N-1)} = \overline{N^2} - \overline{N}$ expresses the difference between the second and first moments. Upon differentiating (1) with respect to U and letting $U = 1$ we find from (3) and (4) :

$$\frac{\partial}{\partial x}\,\overline{N}(\epsilon, x) + a\,\overline{N}(\epsilon, x) =$$

$$\int_0^\infty\int_0^\infty \left\{\overline{N}\left(\frac{\epsilon}{\epsilon'}, x\right) + \overline{N}\left(\frac{\epsilon}{\epsilon''}, x\right)\right\} w(\epsilon', \epsilon'')\, d\epsilon'\, d\epsilon''. \qquad (6)$$

Henceforth for convenience the " bars " denoting an average shall be omitted. Equation (6) may be written as

$$\frac{\partial}{\partial x} N(\epsilon, x) + a\, N(\epsilon, x) = \int_0^\infty N\left(\frac{\epsilon}{\epsilon'}, x\right) W(\epsilon')\, d\epsilon' \qquad (7)$$

with

$$W(\epsilon) = \int_0^\infty \left\{ w(\epsilon, \epsilon'') + w(\epsilon'', \epsilon)\right\} d\epsilon'' = \int_0^\infty \overline{w}(\epsilon, \epsilon'')\, d\epsilon'' \qquad (8)$$

Taking the Mellin transform of (7) with respect to ϵ gives

$$\frac{\partial}{\partial x} N(x) + a\, N_s(x) = N_s(x)\; W_s \qquad (9)$$

where

$$N_s(x) = \int_0^\infty \epsilon^{s-1} N(\epsilon, x)\, d\epsilon \qquad (10a)$$

$$W_s = \int_0^\infty \epsilon'^s\, W(\epsilon')\, d\epsilon'. \qquad (10b)$$

The solution of (9) is immediate

$$N_s(x) = N_s(o)\, e^{-a_s x} \qquad (11)$$

with

$$N_s(o) = \int_0^\infty \epsilon^{s-1} N(\epsilon, 0)\, d\epsilon = 1/s \qquad (11a)$$

and

$$a_s = a - W_s. \qquad (11b)$$

Reversing the transform we find

$$N(\epsilon, x) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{\epsilon^{-s}}{s} e^{-\alpha_s x} ds \tag{12}$$

the expression for the average numbers. The second moment is now also easily found. Differentiating (1) twice with respect to U and setting $U = 1$ yields on applying (3), (5) and (8),

$$\frac{\partial}{\partial x} T(\epsilon, x) + \alpha T(\epsilon, x) = \int_0^\infty T\left(\frac{\epsilon}{\epsilon'}, x\right) W(\epsilon') \, d\epsilon' + F(\epsilon, x) \tag{13}$$

where

$$F(\epsilon, x) = \int_0^\infty \int_0^\infty N\left(\frac{\epsilon}{\epsilon'}, x\right) N\left(\frac{\epsilon}{\epsilon''}, x\right) \overline{w}(\epsilon', \epsilon'') \, d\epsilon' \, d\epsilon'' \tag{14}$$

Taking the Mellin transform of (13) with respect to ϵ;

$$\frac{\partial}{\partial x} T_s(x) + \alpha T_s(x) = T_s(x) W_s + F_s(x) \tag{15}$$

$$T_s(x) = \int_0^\infty \epsilon^{s-1} T(\epsilon, x) \, d\epsilon \tag{15a}$$

$$F_s(x) = \int_0^\infty \epsilon^{s-1} F(\epsilon, x) \, d\epsilon \tag{15b}$$

$$W_s = \int_0^\infty \epsilon \, W(\epsilon) \, d\epsilon \tag{15c}$$

Equation (15) is an inhomogeneous differential equation whose solution is well known and may be written as

$$T_s(x) = e^{-\alpha_s x} \int_0^x e^{\alpha_s \xi} F_s(\xi) \, d\xi \tag{16}$$

On reversing the transform there results

$$T(\epsilon, x) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \epsilon^{-s} \left\{ e^{-\alpha_s x} \int_0^x e^{\alpha_s \xi} F_s(\xi)\, d\xi \right\} ds. \tag{17}$$

Upon substituting for $F_s(\xi)$, (17) becomes

$$T(\epsilon, x) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \epsilon^{-s} e^{-\alpha_s x} \int_0^x e^{\alpha_s \xi} d\xi \int_0^\infty a^{s-1}\, da \int_0^\infty \int_0^\infty \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty}$$

$$\left(\frac{a}{\epsilon'}\right)^{-r} \left(\frac{a}{\epsilon''}\right)^{-t} e^{-(\alpha_r + \alpha_t)\xi}\, \overline{w}(\epsilon', \epsilon'') \frac{dr}{r} \frac{dt}{t}\, d\epsilon'\, d\epsilon''\, ds. \tag{18}$$

Writing

$$b(r, t) = \int_0^\infty \int_0^\infty \epsilon'^{\,r} \epsilon''^{\,t}\, \overline{w}(\epsilon', \epsilon'')\, d\epsilon'\, d\epsilon'' \tag{18a}$$

and carrying out the " ξ " integration reduces (18) to

$$T(\epsilon, x) = \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} \epsilon^{-s} a^{s-t-r-1} \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} b(r, t)$$

$$\frac{e^{-\alpha_s x} - e^{-(\alpha_r + \alpha_t)x}}{\alpha_r + \alpha_t - \alpha_s} \frac{dr}{r} \frac{dt}{t}\, da\, ds. \tag{19}$$

Integrating into " a " and " s " has the effect of replacing s by $r + t$, thus

$$T(\epsilon, x) = \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \epsilon^{-(r+t)}\, b(r, t) \frac{e^{-(\alpha_{r+t})x} - e^{-\alpha_r + \omega_t)x}}{\alpha_r + \alpha_t - \alpha_{r+t}} \frac{dr}{r} \frac{dt}{t}.$$

Since

$$\frac{d}{d\epsilon} N(\epsilon, x) = -\frac{1}{2\pi i} \int_{r_0 - i\infty}^{r_0 + i\infty} \epsilon^{-(r+1)} e^{-\alpha_r x}\, dr \tag{20}$$

equation (18) may also be expressed as a four fold integral over the average number as follows :

$$T(\epsilon, x) = \int_0^x \int_0^\infty \int_0^\infty \int_0^\infty \left\{ \frac{d}{dZ} N(Z, x - \xi) \right\} N\left(\frac{\epsilon}{\epsilon''}, \xi\right) N\left(\frac{\epsilon}{\epsilon'}, \xi\right) \overline{w}(\epsilon', \epsilon'')\, d\epsilon'\, d\epsilon''\, dZ\, d\xi \qquad (21)$$

As in the case of the electron-photon cascade this form of expressing the second moment throws light upon its behaviour, however it is not suitable for calculation purposes. From (20) one may easily obtain by means of a double saddle point integration the values of the second moment ; however this quantity is not the physically significant one, instead we give the expression for the mean square deviation (henceforth shortened to " deviation "), i.e.,

$$\tau(\epsilon, x) = \overline{\delta N^2} = \frac{1}{(2\pi i)^2} \int_{r_0 + i\infty}^{r_0 - i\infty} \int_{t_0 + i\infty}^{t_0 - i\infty} \epsilon^{-(r+t)} B(r, t) \left\{ e^{-(a_r + a_t)x} - e^{-a_{r+t}x} \right\} \frac{dr}{r} \frac{dt}{t} \qquad (22)$$

where

$$\overline{\delta N^2} = \overline{N^2} - \overline{N}^2 \qquad (22a)$$

and

$$B(r, t = \left\{ \frac{b(r, t)}{a_{r+t} - a_r - a_t} - 1 \right\}.$$

III.—Numerical Evaluation of the Moments.

(*a*) Choice of cross-section.

The cross-section chosen in this paper for computational purposes has been that following from the note in the recent paper by Heitler and Jánossy (3).

Considering a process in which a nucleon of energy E_0 loses energy and also gives rise to a recoil nucleon, it is assumed that the probability for a collision to occur in a distance dx in homogeneous nuclear matter is given by

$$w(E_0\,;\,E_1, E_2)\, dE_1\, dE_2\, dx = w\left(\frac{E_1}{E_0}, \frac{E_2}{E_0}\right) \frac{dE_1\, dE_2}{E_0{}^2}\, dx$$
$$= \sigma \epsilon''^{\beta} (1 - \epsilon')^{\gamma}\, d\epsilon',\, d\epsilon''\, dx \qquad (23)$$

with a choice of $\beta = 2$, $\gamma = 1$ and the normalization factor $\sigma = 15$.

By choosing the normalization factor $\sigma = 15$ we have made our total collision probability defined by (Ia) equal one. Thus the depth "x" is measured in collision units.

It is true that the cross-section above is at best approximate, however as is apparent from (21), the "deviation" in a nucleon cascade is a process concerned with average numbers; hence even if the cross-section is out by a large factor, the deviation relative to average numbers will remain essentially of the same form.

(*b*) First moments.

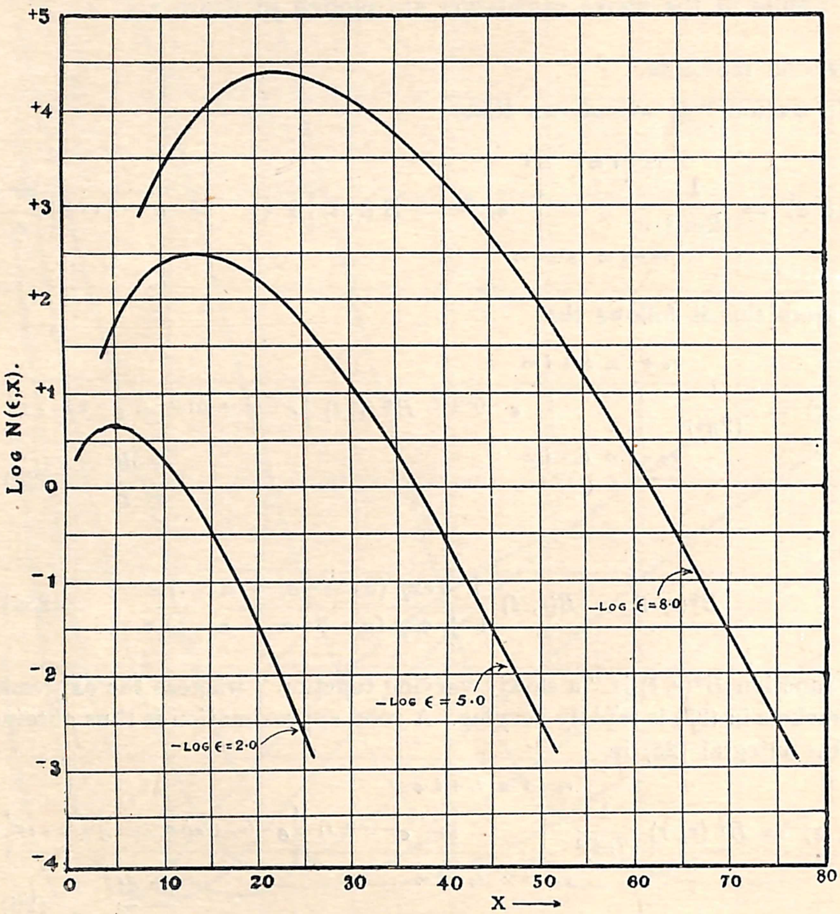

FIG. 1.

Average number of nucleons of energy greater than ϵ E_0 produced by a nucleon of energy E_0. The value of $-$ Log ϵ is attached to each curve. X is the depth in homogeneous nuclear matter.

The integral (12) may be evaluated approximately by a single saddle point integration, yielding

$$\overline{N}(\epsilon, x) = \frac{e^{-S(r_0)}}{\sqrt{-2\pi S''(r_0)}} \tag{24}$$

where

$$-S(r) = -r \ln \epsilon - \ln r - a_r x$$

$$-S''(r) = \frac{1}{r^2} - a''_r x. \tag{24a}$$

and r_0 is the saddle point.
The results of the above calculation are plotted in figure 1.

(*c*) Second moments.

In section 2 it was shown that

$$\tau(\epsilon, x) = \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 i} \epsilon^{-(r+t)} B(r, t) \left\{ e^{-(a_r - a_t)x} - e^{-a_{r+t}x} \right\} \frac{dr}{r} \frac{dt}{t}$$

and from this it follows that

$$\tau(\epsilon, x) = \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \epsilon^{-(r+t)} B^*(r, t) \left\{ e^{-(a_r + a_t)x} + e^{-a_{r+t}x} \right\} \frac{dr}{r} \frac{dt}{t} \tag{25}$$

with

$$B^*(r, t) = B(r, t) \frac{1 - exp\,(a_r + a_t - a_{r+t})\,x}{1 + exp\,(a_r + a_t - a_{r+t})\,x} \tag{25a}$$

The function $B^*(r, t)$ is " a slowly varying function " whereas the expression in brackets in (25) is rapidly varying. A good approximation is thus obtained for the integral (25) in

$$\tau(\epsilon, x) = B^*(r, r) \frac{1}{(2\pi i)^2} \int_{r_0 - i\infty}^{r_0 + i\infty} \int_{t_0 - i\infty}^{t_0 + i\infty} \epsilon^{-(r+t)} \left\{ e^{-(a_r + a_t)x} + e^{-a_{r+t}x} \right\} \frac{dr}{r} \frac{dt}{t} \tag{26}$$

hence

$$\frac{\tau(\epsilon, x)}{\overline{N}^2 + \overline{N}} = B^*(r, r). \tag{27}$$

It should be noted that the integrand in (25) is symmetrical in the variables r and t. The results of calculations carried out using the above appear in figures 2 and 3.

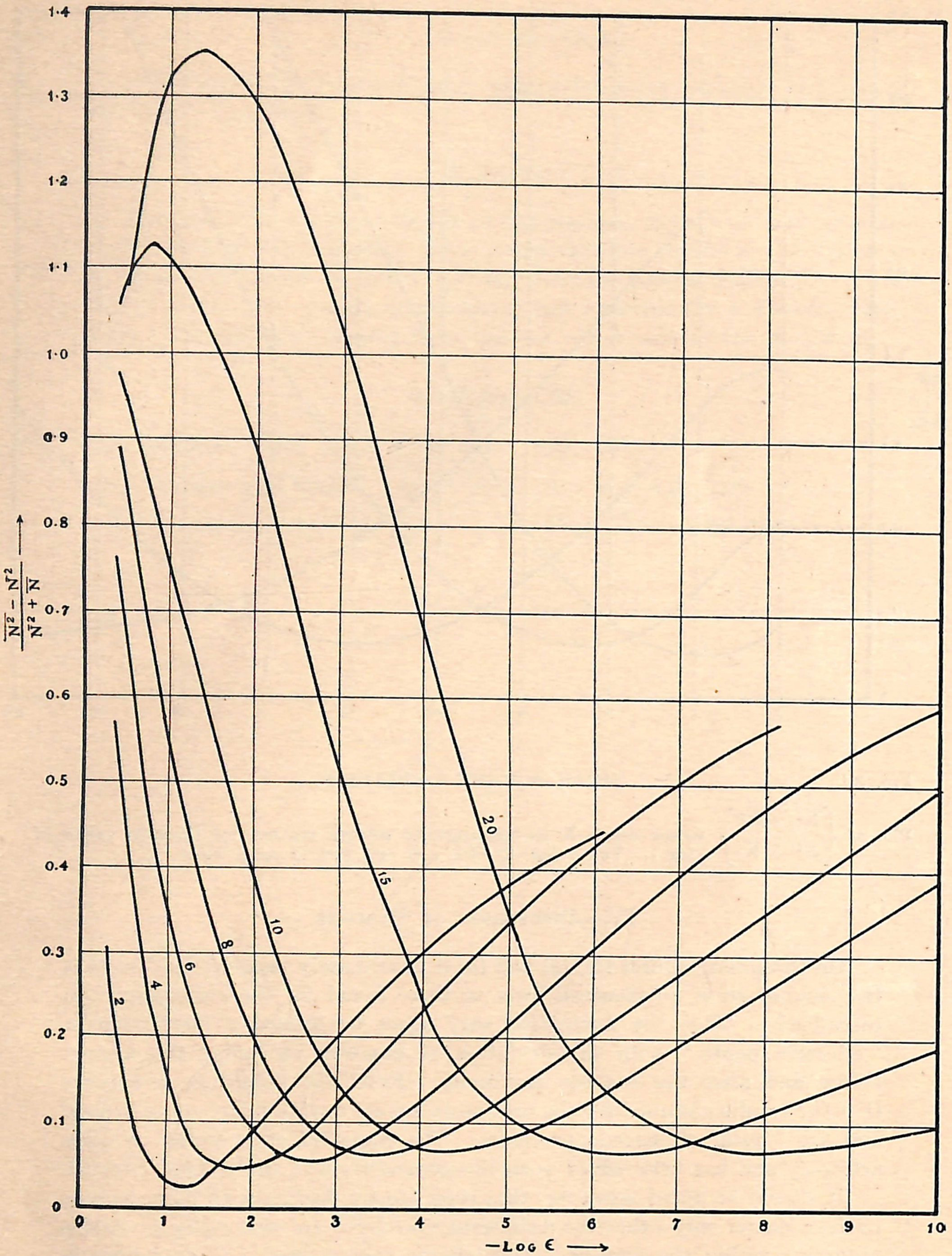

FIG. 2. MEAN SQUARE DEVIATION.

Plot of $\left\{\frac{\overline{N^2} - \bar{N}^2}{\overline{N^2} + \bar{N}}\right\}$ against log ϵ for values of constant depth which are attached to each curve.

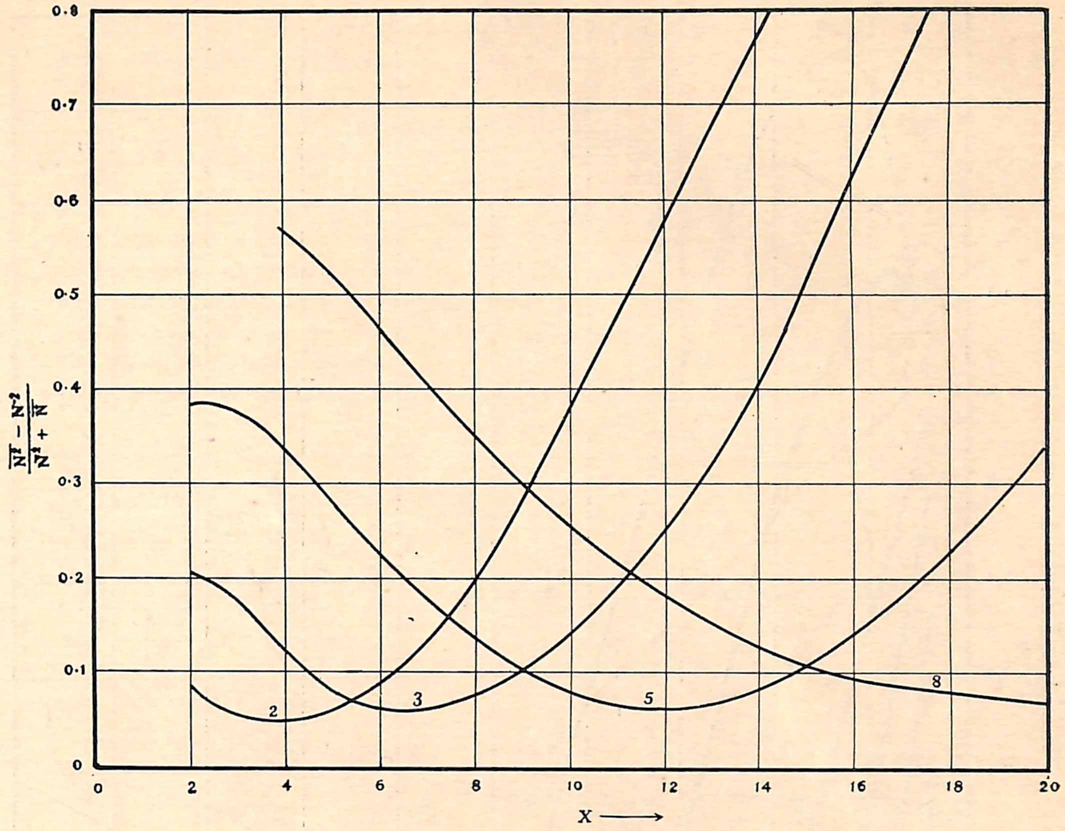

FIG. 3. MEAN SQUARE DEVIATION.

Plot of $\left\{\dfrac{\overline{N^2} - \bar{N}^2}{\overline{N^2} + \bar{N}}\right\}$ versus depth X in homogeneous nuclear matter, for constant values of $-\log \epsilon$. The value of $-\log \epsilon$ is attached to each curve.

IV.—DISCUSSION OF RESULTS.

On examining figures (1), (2) and (3) it immediately becomes evident that the results are of an identical form to those found for the electron-photon cascade (2). Thus the fluctuation may again be termed a fluctuation in " effective depth " with the deviation of numbers exceeding the normal before and after the cascade maximum. From the preceding it appears that the results obtained for the root mean square deviation are quite general and are typical of cascade processes. The choice of cross-section is quite arbitrary and has little effect upon the general form of the results obtained.

It should be noted, however, that since we are dealing with homogeneous nuclear matter only—that the deviations considered are essentially deviations occurring within the nucleus. The inhomogeneous case, i.e., deviations in matter where the nucleons are grouped in nuclei is considered in a later paper.

V.—Acknowledgments.

I am indebted to Professor Lajos Jánossy for much discussion and helpful criticism of the above.

Summary.

The first and second moments of the nucleon cascade in homogeneous nuclear matter are evaluated. It is found that the results are very similar to those previously obtained for the electron-photon cascade, i.e., the fluctuation at the cascade maximum is that expected for a Poisson distribution, but deviates greatly from this on either side of the maximum.

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