# Cosmic - Ray Underground I.

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It is assumed that only mu-mesons and their electromagnetic interactions play a rôle in the phenomena underground; pi-mesons should not be important because of the nuclear interactions they suffer.

The electromagnetic interactions are: collision processes with atomic electrons, bremsstrahlung in the nuclear (screened) Coulomb field and creation of electron-positron pairs in the nuclear Coulomb field. The first two processes depend strongly on the spin and magnetic moment of the particle in the relativistic region.

We assume that the mu-meson has spin $1/2$ and no intrinsic magnetic moment.

## Elementary processes and energy loss

For ionization and bremsstrahlung are treated by Rossi and Greisen, Rev. of Mod. Phys., 13, 240, 1941. As for the pair creation we refer to Bhabha's paper Proc. Roy. Soc. 152, 599, 1935. According to this the differential cross-section for the creation of an electron and a positron with energies between $\varepsilon, \varepsilon + d\varepsilon$ and $\varepsilon', \varepsilon' + d\varepsilon'$ respectively is given by

(1a)
$$dQ_1 = \frac{8}{\pi}(\alpha Z)^2 r_0^2 \frac{\varepsilon^2 + \varepsilon'^2 + \frac{2}{3}\varepsilon\varepsilon'}{(\varepsilon + \varepsilon')^4} \ln\left(\frac{k\,\varepsilon\varepsilon'}{(\varepsilon+\varepsilon')mc^2}\right) \ln\left(\frac{k'\xi mc^2}{\varepsilon+\varepsilon'}\right) d\varepsilon\, d\varepsilon'$$

for $mc^2 < \varepsilon, \varepsilon'$ and $\varepsilon\varepsilon'/(\varepsilon+\varepsilon') < \frac{1}{2}\gamma mc^2$

(1b)
$$dQ_2 = \frac{8}{\pi}(\alpha Z)^2 r_0^2 \frac{\varepsilon^2 + \varepsilon'^2 + \frac{2}{3}\varepsilon\varepsilon'}{(\varepsilon + \varepsilon')^4} \ln(k\gamma) \ln\left(\frac{k'\xi mc^2}{\varepsilon+\varepsilon'}\right) d\varepsilon\, d\varepsilon'$$

for $\frac{1}{2}\gamma mc^2 < \varepsilon + \varepsilon'$ and $\varepsilon, \varepsilon' < \xi mc^2$

(1c)
$$dQ_3 = \frac{8}{\pi}(\alpha Z)^2 r_0^2 \frac{(\xi mc^2)^2}{(\varepsilon + \varepsilon')^4} \ln\left(\frac{k\gamma(\varepsilon+\varepsilon')}{\xi mc^2}\right) d\varepsilon\, d\varepsilon'$$

for $\xi mc^2 < \varepsilon, \varepsilon'$ and $\varepsilon + \varepsilon' < 2\xi^2 mc^2/\gamma$

(1d)
$$dQ_4 = \frac{8}{\pi}(\alpha Z)^2 r_0^2 \frac{(\xi mc^2)^2}{(\varepsilon + \varepsilon')^4} \ln(2k\xi)\, d\varepsilon\, d\varepsilon'$$

for $2\xi^2 mc^2/\gamma < \varepsilon + \varepsilon' < \xi mc^2$

for each specified energy region. Here $\mu$ is the mass and $\xi$ the Lorentz factor of the incident charged particle, $\alpha$ the fine structure constant, $r_0$ the classical electron radius and $\gamma = 183\, Z^{-1/3}$ the shielding radius of the collided atom, $Z$ being its atomic number. $k$ and $k'$ are numerical constants $\sim 1$. The result does not depend strongly on the spin of the incident particle. For the sake of simplicity we put approximately:

(2) $$\varepsilon'\varepsilon/(\varepsilon+\varepsilon') = (\varepsilon+\varepsilon')/4$$

Then Bhabha's formulae can be reduced to:

(1a') $$dQ_1 = \frac{8}{\pi}(\alpha Z r_0)^2 \cdot \frac{7}{9}\frac{1}{(\varepsilon+\varepsilon')^2} \ln\left(\frac{k(\varepsilon+\varepsilon')}{mc^2}\right)\ln\left(\frac{k'\xi mc^2}{\varepsilon+\varepsilon'}\right) d\varepsilon\, d\varepsilon'$$

for $mc^2 < \varepsilon+\varepsilon' < 2\gamma mc^2$

(1b') $$dQ_2 = \frac{8}{\pi}(\alpha Z r_0)^2 \frac{7}{9}\frac{1}{(\varepsilon+\varepsilon')^2}\ln(k\gamma)\ln\left(\frac{k'\xi mc^2}{\varepsilon+\varepsilon'}\right) d\varepsilon\, d\varepsilon'$$

for $2\gamma mc^2 < \varepsilon+\varepsilon' < \xi mc^2$

(1c') $$dQ_3 = \frac{8}{\pi}(\alpha Z r_0)^2 \frac{(\xi mc^2)^2}{(\varepsilon+\varepsilon')^2}\ln\left(\frac{k\gamma(\varepsilon+\varepsilon')}{\xi mc^2}\right) d\varepsilon\, d\varepsilon'$$

for $\xi mc^2 < \varepsilon+\varepsilon' < 2\xi^2 mc^2/\gamma$

(1d') $$dQ_4 = \frac{8}{\pi}(\alpha Z r_0)^2 \frac{(\xi mc^2)^2}{(\varepsilon+\varepsilon')^4}\ln(2k\xi)\, d\varepsilon\, d\varepsilon'$$

for $2\xi^2 mc^2/\gamma < \varepsilon+\varepsilon' < \xi mc^2$

Thus we can immediately find the cross-section for the production of a pair whose energy lies between $\varepsilon$ and $\varepsilon + d\varepsilon$ irrespective to their partition on the electron and the positron

(3a) $$dQ'_1 = \frac{8}{\pi}(\alpha Z r_0)^2 \frac{7}{9}\ln\left(\frac{k\varepsilon}{mc^2}\right)\ln\left(\frac{k'\xi mc^2}{\varepsilon}\right)\frac{d\varepsilon}{\varepsilon} \quad \text{for } mc^2 < \varepsilon < 2\gamma mc^2$$

(3b) $$dQ'_2 = \frac{8}{\pi}(\alpha Z r_0)^2 \frac{7}{9}\ln(k\gamma)\ln\left(\frac{k'\xi mc^2}{\varepsilon}\right)\frac{d\varepsilon}{\varepsilon} \quad \text{for } 2\gamma mc^2 < \varepsilon < \xi mc^2$$

(3c) $$dQ'_3 = \frac{8}{\pi}(\alpha Z r_0)^2 (\xi mc^2)^2 \ln\left(\frac{k\gamma\varepsilon}{\xi mc^2}\right)\frac{d\varepsilon}{\varepsilon} \quad \text{for } 2\xi mc^2 < \varepsilon < 2\xi^2 mc^2/\gamma$$

(3d) $$dQ'_4 = \frac{8}{\pi}(\alpha Z r_0)^2 (\xi mc^2)^2 \ln(2k\xi)\frac{d\varepsilon}{\varepsilon} \quad \text{for } 2\xi^2 mc^2/\gamma < \varepsilon < \xi mc^2$$

Multiplying each of (3) by $\varepsilon$, integrating with respect to $\varepsilon$ over each specified region and adding the results, we find that the energy loss of the incident particle by the pair creation is given by:

$$\left(-\frac{dE}{dx}\right)_{pair} \approx \frac{8}{\pi}(\alpha Z r_0)^2 \frac{N}{A}\left[\xi mc^2\left\{\frac{9}{16}\ln\gamma + 1 - \frac{m}{\mu}\ln(2\xi)\right\}\right.$$

(4) $$\left. - \frac{14}{9}\gamma mc^2\left\{\ln\xi - \ln(2\gamma) + \frac{37}{18}\right\}\right]$$

$N$ is the Loschmidt number, $A$ the atomic weight of the collided atom. Assuming $\mu = 217\,m$, $Z = 10$, $A = 20$ :

(5) $$(-dE/dx)_{pair} \cong 3{,}77.10^{-4}\left[\xi mc^2 - 24{,}8\,mc^2\log\xi\right]$$

The ionization loss (see Halpern and Hall Phys. Rev. 73, 477, 1948) may be put constant, because its logarithmic increasing will be much weakened by Fermi's density effect. Since rock is similar to C, assuming the energy of the incident particle as being $10^{11}$ eV:

(6) $$(-dE/dx)_{ion} = a = 2{,}5.10^6 \text{ eV. per g. cm}^{-2}$$

The radiation loss of a meson is given in the case where the screening is not yet completed (the complete screening sets in at about $5 \cdot 10^{11}$ eV).

(7) $$\left(-\frac{dE}{dx}\right)_{rad} = 4\alpha\, Z^2 r_0^2 \left(\frac{m}{\mu}\right)^2 \frac{N}{A}\,\xi\mu c^2\left[\ln\left(\frac{12\xi}{5Z^{1/3}}\right) - \frac{1}{3}\right]$$

Numerically:

(8) $$(-dE/dx)_{rad} = 3{,}45.10^{-7}\,E\left[\log\xi - 0{,}10\right] \text{ eV per g. cm}^{-2}$$

## Range of energetic mesons

The total energy loss is given by

(9) $$-(dE/dx)_{total} = (-dE/dx)_{ion} + (-dE/dx)_{rad} + (-dE/dx)_{pair}$$

Then the range of a meson with energy $E$ is given by

(10) $$x(E) = \int_0^E dE \Big/ (-dE/dx)_{total}$$

If we approximate (5) and (8) by:

(5') $$(-dE/dx)_{pair} = p E \quad eV \text{ per } g\,cm^{-2}$$

with $p = 1{,}6 \cdot 10^{-6}$

and

(8') $$(-dE/dx)_{rad} = r E \quad eV \text{ per } g \cdot cm^{-2}$$

with $r = 1.0 \cdot 10^{-6}$

(10) becomes

(11) $$x(E) = \frac{1}{p+r} \ln\left[1 + \frac{p+r}{a} E\right]$$

If the energy is so small that $(p+r)E \ll a$, it simplifies

(11') into $$x(E) \approx E/a$$

which evidently means that only ionization is effective in the lower energy region, as usually assumed. Pair creation and radiation become important at about $11''$ eV. The energy of a meson with residual range $x$ is given by solving (11).

(12) $$E = \frac{a}{p+r}\left[\exp\left\{(p+r)x\right\} - 1\right]$$

Let the integral energy spectrum $F(E)$ of mu-mesons at sea-level be known; then the function of $x$ obtained by substituting $E$ in $F(E)$ by (12) gives just the required intensity-depth relation.

## Energy spectrum of mu-mesons.

The bend of the intensity-depth curve must be attributed to the bend of the energy spectrum itself at sea-level. Such a bend does not exist in the primary spectrum because the frequency of large air showers can well be explained by extrapolating the primary spectrum at lower energy to the region of extremely high energy. It is highly improbable that the production of pi-mesons from their primaries is performed in such a way that their spectrum has a remarkable bend while the spectrum of the primaries is of a smooth form. So we must conclude that the bend is the result of the pi-mu decay. The temperature effect and the angular distribution of cosmic rays in deep regions give powerfull support for this interpretation.

The energy spectrum of mu-mesons produced by the decay of pi-mesons can be calculated neglecting ionization loss for both kinds of mesons as well as the disintegration of mu-mesons.

We further neglect, for the time being, the absorption of pi-mesons in the atmosphere.

The primaries decrease as $\exp(-l/\Lambda)$, $\Lambda$ being the mean free path for the primaries, in $g.cm^{-2}$, and produce pi-mesons with a probability proportional to $\exp(-l/\Lambda)\,dl$. The pi-meson produced at $l'\,g.cm^{-2}$ is transformed into a mu-meson at the depth $l$ with the probability

$$(13)\quad \begin{cases} 1-\left(l'/l\right)^{B/E_0} \\ B=\left(\mu_\pi c/\tau_\pi\right)\cdot z_0 = 3,4.10^{11}\ eV. \end{cases}$$

$\mu_\pi$ is the mass of the pi meson, $\tau_\pi$ its life, $E_0$ its energy, $z_0$ the height of the homogeneous atmosphere. In fact, denoting by $z'$ and $z$ the heights corresp. to the pressures $l'$ and $l$ respectiv. the probab. of survival of the pi is:

$l=\theta_0 e^{-z/z_0}$

$$e^{-\frac{1}{\gamma c\tau_\pi}(z'-z)} = e^{-\frac{1}{c\tau_\pi}\frac{\mu_\pi c^2 z_0}{E_0}\left[\ln l' + \ln l\right]} = \left(l'/l\right)^{B/E_0}$$

(see e.g. Jánossy, Mathematical relations.)

Multiplying (13) by $\exp(-l'/\Lambda)\,dl'$ and integrating over $l'$, we obtain the probability of existing a mu-meson at the depth $l$ produced somewhere by a pi-meson of energy $E_0$. The result is:

$$L(E_0)=\frac{1}{\Lambda}\int_0^l e^{-l'/\Lambda}\left[1-\left(l'/l\right)^{B/E_0}\right]dl$$

$$(14)\quad =\frac{1}{\Lambda}\left\{1-\exp(-l/\Lambda)-\left(\Lambda/l\right)^{B/E_0}\Gamma\left(B/E_0+1,\, l/\Lambda\right)\right.$$

where $\Gamma(B/E_0+1, l/\Lambda) = \int_0^{l/\Lambda} l'^{\,B/E_0} e^{-l'} dl'$

is the incomplete Gamma-function.
We now assume that the energy spectrum of $\pi$-mesons when they are produced is

(15) $$g(E_0)\,dE_0 = \text{const.}\; E_0^{-\gamma-1}\, dE_0$$
$$\gamma = 1.8$$

(It is also possible to take $\gamma$ as 2.0). The energy spectrum of $\mu$-mesons at sea level is then given by:

(16) $$f(E)\,dE = dE \int_E^{\left(\frac{\pi}{\mu}\right)^2 E} L(E_0)\, g(E_0)\, dE_0 / E_0$$

$\pi$ = mass of $\pi$-meson
$\mu$ = " " $\mu$- –

The low part of $f(E)$ is not correct. So for this low energy part we take the values summarized by Rossi (Rev. of Mod. Phys. 20, 537, 1948). We use Rossi's curve up to $3.10^{10}$ eV. and determine the constant in (16) in such a way that (16) coincides with Rossi's curve at that point. The integral spectrum

(17) $$F(E) = \int_E^{\infty} f(E')\, dE'$$

is thus obtained. We use $\mu_\pi = 286m$, $\tau_\pi = 10^{-8}$ sec
$\Lambda = 125$ g. cm$^{-2}$

Intensity-depth curve.

Substituting (12) into (17) we obtain the intensity.

$$\frac{\partial f}{\partial t} = -\frac{A}{t} f + Be^{-t/\Lambda}$$

$$\frac{\partial f}{\partial t} + \frac{A}{t} \cdot f = 0$$

$$\frac{df}{f} = -A \frac{dt}{t}$$

$$\log f = -A \log t$$

$$f = t^{\cancel{A}}$$

$$f = k$$

$$= \frac{\cancel{k}}{t}$$

$$f$$

$$-A \cdot t^{-A-1} = -A t^{-A-1} + Be^{-t/\Lambda}$$

$$Be^{-t/\Lambda}$$

$$\int \frac{1}{E_0^{\gamma - 1}} \quad \frac{dE_0}{E_0 \left(1 + \frac{B}{E_0}\right)^{\gamma}} \quad \frac{1}{B} \int \frac{\cancel{E_0}}{B \cdot E_0} \frac{dE_0}{\left(1 + \frac{E_0}{B}\right)} \quad \frac{B}{E_0}$$

$$\frac{1}{E_0^{\gamma + 1} \left(1 + \frac{B}{E_0}\right) E_0} = \frac{1}{E_0^{\gamma + 2} \cdot \frac{B}{E_0} \left(1 + \frac{E_0}{B}\right)} = \frac{1}{E_0^{\gamma + 1} \left(1 + \frac{E_0}{B}\right)}$$

$$\int \frac{dE_0}{E_0^{\gamma + 1} \left(1 + \frac{E_0}{B}\right)}$$

$\cancel{5/2}$

depth curve:

(18) $$I(x) = F\left(E(x)\right)$$

Effect of absorption of pi-mesons in the atmosphere.

We assume tentatively that the absorption coefficient of pi-mesons in the atmosphere is approximately so large as that of nucleons. Then the spectrum of pi-mesons satisfies the diffusion equation

(19) $$\frac{\partial g(E_0, l)}{\partial l} = -\frac{B}{E_0 l}\, g(E_0, l) - \frac{1}{\Lambda}\, g(E_0, l) + g_0(E_0, l)$$

$g(E_0, l)$ being the required spectrum and $g_0(E_0, l)$ the source function for pi-mesons. If we assume

(20) $$g_0(E_0, l)\, dE_0 = \text{const.}\, E_0^{-\gamma-1}\, dE_0 \exp(-l/\Lambda)$$

The solution of (19) is

(21) $$g(E_0, l) = \text{const.}\, l \cdot \exp(-l/\Lambda)\, E_0^{-\gamma-1} \left(1 + B/E_0\right)^{-1}$$ correct

The energy spectrum of mu-mesons at depth $l$ is then

$$f(E, l)\, dE = dE \int_E^{(K/\mu)^2 E} dE_0/E_0 \int_0^l (B/E_0 l')\, g(E_0, l')\, dl'$$

(22) $$= \text{const}\, dE \left(1 - \exp(-l/\Lambda)\right) \int_E^{(K/\mu)^2 E} E_0^{-\gamma-2} (1 + E_0/B)^{-1}\, dE_0$$

$$\cancel{\int_E^{\left(\frac{K}{\mu}\right)^2 E} E_0^{-\gamma-1} \left(1 + \frac{E_0}{B}\right)^{-1} dE_0}$$