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EXCHANGE PHENOMENA OF THE NUCLEONS THAT GENERATE PENETRATING SHOWERS.

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Recently G. Cocconi performed measurements on the absorption of ionizing PSPR * (which are believed to be protons) in Pb, Fe, C, and air(1). He finds that this absorption is not exponential in the condensed absorbers and that the mean range of the ionizing PSPR in the atmosphere increases with the lead thickness above the penetrating shower detector. These results are not compatible with an exponential absorption of the PSPR which was verified by Tinlot and Gregory(2) in Pb and Fe and by Wataghin(3) and Tinlot(4) in the atmosphere.

The fundamental difference between the experimental arrangement of Cocconi and those of the other authors, is that a proton of the PSPR appears as absorbed in any material situated in position Σ (fig. 1 of Cocconi's paper(1)), not only when it generates a penetrating shower, but also when it emerges from Σ in the form of a neutron.

Let us discuss Cocconi's results by taking into account the phenomenon described above. We shall make the

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* We use the abbreviation PSPR for Penetrating Shower Producing Radiation.

following hypothesis : 1) The absorption of nucleons through production of penetrating showers is exponential in all materials and is independent of their charge as well as of the altitude of the point of observation. 2) The probability of a transformation of a proton into a neutron is equal to that of the reverse process. 3) At Echo Lake (708 gr.cm^{-2}) the number of protons is already equal to the number of neutrons in the PSPR. Indeed if this were not so, Cocconi should have found a mean range for the ionizing PSPR in the atmosphere (without any absorber above the detector) smaller than the mean range of the total PSPR determined by Wataghin and Tinlot. Thus varying the absorber thickness at position Σ' in Cocconi's experiments(1) and the absorber thickness in the arrangement of Tinlot and Gregory(2) , it is only necessary to consider the exponential absorption due to the production of penetrating showers.

Following this interpretation one deduces easily from Cocconi's measurements the values of the mean range λ for the production of penetrating showers in different absorbers:

$$\lambda_{Pb} = 345 \pm 37 \quad gr.cm^{-2}$$

$$\lambda_{Fe} = 240 \pm 27 \quad "$$

$$\lambda_{C} = 100 \pm 12 \quad "$$

$$\lambda_{air} = 113 \pm 7 \quad "$$

in good agreement with the values given by other workers(2,3,4). A short calculation yields for the mean range μ for charge exchange the values

$$\mu_{Pb} = 247 \pm 61 \quad gr.cm^{-2}$$

$$\mu_{Fe} = 300 \pm 121 \quad "$$

at Echo Lake (708 gr.cm^{-2}). For μ_{C} we obtain a larger value than the preceding ones, but the statistical errors are too important to allow a precise determination. However this result is not surprising since charge exchange and ~~showers~~ shower production are competitive processes. It is then reasonable to expect that μ should increase if λ decreases. At Ithaca (1007 $gr.cm^{-2}$) μ is found to be several times larger, but a precise determination is impossible.

We see that the exchange properties of the PSPR explain Cocconi's results consistently with the results of other authors. On the other hand we wish to point out that the energy loss of the protons by ionization has been neglected, and this might not be very accurate in the case of the rather low energy events recorded by Cocconi[1] and by Tinlot and Gregory[2] . However we think that this factor could at most alter somewhat the numerical values of the mean ranges[5] .

(1) G. Cocconi, Phys.Rev. 75, 1074, 1949. We are indebted to Professor Cocconi for having communicated to us his results before publication.

(2) J.Tinlot and B.Gregory, Phys.Rev. 75, 519, 1949.

(3) G. Wataghin, Phys.Rev. 71, 453, 1947.

(4) J. Tinlot, Phys.Rev. 74, 1197, 1948.

(5) A more complete account will appear in the Anais da Acad.Bras.de Cienc.

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para Cocconi - Phys. Rev. - devolvido

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It is generally admitted that the particles constituting the PSPR * are energetic nucleons. The nucleons are believed to interact by means of exchange-forces. As a consequence, in a proton-neutron scattering, the nucleons may exchange their charges. The same should happen if the proton collides with a nucleus.[1]. This phenomenon has already been confirmed by neutron-proton scattering experiments made with the 184" cyclotron at Berkeley[2]. We should expect a similar exchange with the nucleons of cosmic radiation, in particular with the PSPR which consist at sea-level of protons and neutrons[3].

On the basis of these remarks, we shall discuss some recent experiments on ionizing PSPR by Cocconi[4] to whom we are

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* We use the abbreviations PS for Penetrating Shower and PSPR for Penetrating Shower-Producing Radiation.

deeply indebted for having communicated to us his results before publication. Cocconi's experiments consisted in measuring the absorption of the ionizing PSPR in various materials, such as lead, iron, carbon and the ~~atmosphere~~ air. He finds that the absorption is not an exponential one in the condensed absorbers. For the absorption in the atmosphere, he finds the surprising result that the mean range of the ionizing PSPR increases with the ~~absorber~~ lead thickness above the counters. This result is not compatible with an exponential absorption for the PSPR in the atmosphere, which was ~~proved~~ verified by ~~Tinlot~~ (5) Wataghin and by Tinlot (6,7)

The fundamental difference between the experimental arrangement of Cocconi and those of all the other authors, is that the PSPR must discharge one counter of each tray A and E (see figure 1 in Cocconi's paper (4)) , before giving rise to a PS detected by coincidences of the trays B, C, and D . In Cocconi's arrangement a proton of the PSPR appears as absorbed in any material situated in position Σ , not only when it gives rise to a PS but also when it emerges from Σ in the form of a neutron, even if this neutron is itself able to generate a PS. In Σ' it is only necessary to consider the absorption due to the production of PS, if we admit, in a reasonable approximation, that the ratio of ionizing to non-ionizing particles is unaltered by the absorber Σ' (see below).

We shall try to discuss Cocconi's results by taking into account the phenomenon described above. We shall make the following hypotheses:

a) The absorption of nucleons by the production of PS is an exponential one in all materials and is independent of their charge.

b) The probability of a transformation of a proton into a neutron is equal to that of the reverse process.

c) At 708 $gr.cm^{-2}$ of barometric pressure (Echo Lake), the number of protons is already equal to the number of neutrons in the PSPR.

Hypothesis a) is trustworthy since it has been verified ~~by Tinlot~~ in the particular case for the absorption of the PSPR in the atmosphere; the second part of this hypothesis seems reasonable, since we can neglect the Coulomb-forces at the involved energies. Hypothesis b) follows from the symmetry of the two processes. Hypothesis c) is verified by experiment. In fact, if the equilibrium between the ionizing and the non-ionizing particles of the PSPR were not established at 708 $gr.cm^{-2}$, Cocconi should have found a mean range for the ionizing PSPR in the atmosphere (without any absorber above the counters), smaller than the mean range for the total PSPR found by Tinlot and others.

Let us suppose that a beam of fast nucleons is incident ~~incides~~ normally upon an absorber. We call P(0) and N(0) the intensities of protons and neutrons striking ~~inciding on~~ the absorber, and P(x) and N(x) the corresponding intensities at the absorber thickness x. We find readily that:

$$\left.\begin{aligned} P(x) &= \left[\frac{P(0)+N(0)}{2} + \frac{P(0)-N(0)}{2} e^{-2x/\mu}\right] e^{-x/\lambda} \\ N(x) &= \left[\frac{P(0)+N(0)}{2} - \frac{P(0)-N(0)}{2} e^{-2x/\mu}\right] e^{-x/\lambda} \end{aligned}\right\} \quad (1)$$

where we call μ the mean range of a nucleon for a charge exchange and λ the mean range for an absorption by means of generation of PS. In Cocconi's experiments equations (1) are somewhat simplified since we have N(0) = 0 because only protons striking ~~inciding~~ on the counter-tray A, above the absorber Σ, can be detected.

The mean range λ for the PS-production can be determined by the measurements with different thicknesses of the absorber Σ' maintaining the absorber thickness at Σ constant. In fact, by hypothesis c) and by equations (1), we see that in Σ', $P(x') = N(x')$, where x' is the thickness of the absorber Σ'. Then the absorption law in Σ' is:

$$P(x') = P(0)\, e^{-x'/\lambda}$$

In this way we find at Echo Lake (708 gr.cm^{-2}) with the measurements of 284 and 605 gr.cm^{-2} of lead and with the measurements of iron and carbon

$$\lambda_{Pb} = 337 \pm 36 \text{ gr.cm}^{-2}$$
$$\lambda_{Fe} = 240 \pm 27 \text{ gr.cm}^{-2}$$
$$\lambda_{C} = 100 \pm 12 \text{ gr.cm}^{-2}$$

With the same method we obtain at Ithaca (1007 gr.cm^{-2})

$$\lambda_{Pb} = 458 \pm 188 \text{ gr.cm}^{-2}$$
$$\lambda_{C} = 207 \pm 54 \text{ gr.cm}^{-2}$$

Now, we assume that in a certain material λ is the same at different altitudes. Tinlot's experiments prove this argument for the absorption of the PSPR in the atmosphere and Cocconi's results with lead are consistent with it. However, the results with carbon show a slight discrepancy which might be due to an unfavourable fluctuation. We have [redacted] reasons to suppose that [redacted] at Ithaca [redacted]. the counting rate with 110 gr.cm^{-2} of carbon should lie [redacted], somewhat lower. Summarising Cocconi's measurements of the mean ranges for the absorption by PS-production, we find that the best values are :

$$\lambda_{Pb} = 345 \pm 37 \text{ gr.cm}^{-2}$$
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Having thus determined the mean ranges for the PS-production by the ionizing PSPR in different materials, we shall calculate the mean ranges for an exchange of charge. If the absorber is situated in position Σ , the absorption law will be given by formula (1) which allows to determine μ. We find at Echo Lake :

$$\mu_{Pb} = 247 \pm 61 \text{ gr.cm}^{-2}$$
$$\mu_{Fe} = 300 \pm 121 \text{ gr.cm}^{-2}.$$

For μ_c we obtain a larger value than the preceding ones, but the statistical errors are too important ~~in order~~ to allow a precise determination. This result is not surprising, since charge exchange and PS-production are competitive processes. It is then reasonable to expect, that μ should increase if λ decreases. In particular this should be the case ~~XXXXXXXXX~~ with carbon.

At Ithaca we find that μ_{Pb} is several times larger than at Echo Lake, but we cannot determine its value accurately due to the statistical errors of the data.

The mean range μ thus obtained is only a sort of average value over the mean ranges of all the protons with different energies, but we can state that the protons have, in the mean, a larger mean range for charge exchange at 260 m than at 3260 m. We do not have much information about the charge exchange at high energies, but it is possible that we have an explanation of the phenomenon if the cross-section for the charge exchange varies rapidly with energy.*

With the absorption law (1) for absorbers located at Σ the mean range for PS-production in the atmosphere is the same

* However the order of magnitude of the cross-section seems reasonable.(1)

for different thicknesses of the absorber and has precisely the value (113 gr.cm^{-2}) obtained by Cocconi without any absorber. This value is in fair agreement with the results obtained by other authors.

Summarising, we see that the transformation of protons into neutrons explains the whole of Cocconi's results which become, in this way, consistent with results of other workers. ~~This fact seems to be an experimental evidence for the exchange character of the nuclear forces at high energies ($> 10^9$ eV) .~~

(1) For details and references see H.A.Bethe and R.F.Bacher, Rev.of Mod.Phys. 8, 83, § 15, 1936

(2) J.Hadley, C.Leith, H.York, E.Kelly and C.Wiegand, Phys.Rev. 73, 541, 1948.

(3) L.Janossy and G.D.Rochester, Phys.Rev. 64, 348, 1943
L.Janossy, "Cosmic Rays", Oxford Univ.Press, 1947, p.356.

(4) G.Cocconi, "~~Some Properties of the Cosmic-Ray Ionizing Particles that generate Penetrating Showers", to be published in the~~ Phys.Rev. 75. N. 6. (1949)

(5) ~~J.Tinlot, Phys.Rev. 74, 1197, 1948.~~
G. Wataghin P.R. 71, 453, 1947

São Paulo, February the 7th, 1949.

(6) Tinlot.

(7) H.M - - - On

However we wish to point out that we neglected in the above discussions the influence of the energy loss of the protons by ionization. This might change the numerical value of the mean range, but we do not think that our main conclusions ~~etc~~ ~~cannot~~ can be ~~appreciably influenced~~ should be modified. Indeed it can be shown that Cocconi's results cannot be

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With the absorption law (1) for absorbers located at Σ the mean range for PS - production in the atmosphere is the same for different thicknesses of the absorber and has precisely the value (113 $gr.cm^{-2}$) obtained by Cocconi without any absorber. This value is in fair agreement with the results obtained by other authors.

Summarising, we see that the transformation of protons into neutrons explains the whole of Cocconi's results which become, in this way, consistent with results of other workers. However we wish to point out that we neglected in the above discussions the energy loss of the protons by ionization. This might change the numerical values of the mean ranges but we do not think that our main comclusions should be modified. Indeed it can be shown that Cocconi's results cannot be explained on the basis of energy loss by ionization. Thus it seems that Cocconi's experiments may be interpreted as an indication for the exchange character of nuclear forces at very high energies.

(1) For details and references see H.A.Bethe and R.F.Bacher Rev.of Mod.Phys. 8, 83, §15, 1936.
(2) J.Hadley, C.Leith, H.York, E.Kelly and C.Wiegand Phys.Rev. 73, 541, 1948.
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(5) G.Wataghin, Phys.Rev. 71, 453, 1947.
(6) J.Tinlot, Phys. Rev. 74, 1197, 1948.
(7) ~~H.A.Meyer, G.Schwachheim, A.Wataghin and G.Wataghin "On penetrating showers in cosmic radiation" Phys.Rev. to be published soon.~~

São Paulo, April 28, 1949.

Apendice

Neste apendice estão expostos os métodos estatísticos usados para a determinação da curva de absorção das partículas que atingem o contador E na experiencia III e em particular para a determinação do c.l.m. para absorção das partículas produzidas localmente. Vamos mostrar como nos utilizamos, do fato que simultaneamente com as quíntuplas foram medidas quadruplas, tanto para a determinação desse c.l.m. como para melhorar a precisão com que se pode conhecer as quintuplas.

A probabilidade do contador E ser atingido pelos PS selecionados pelos telescópios (com gazolina) depende da espessura de chumbo x colocada em cima desse mesmo contador. Supomos que esta probabilidade P (x) segue uma lei do tipo:

$$P(x) = a + b\,c^{x} \qquad (1)$$

onde a, b, c, são parametros a ser determinados e $c = e^{-1/L}$. L, é o c.l.m. procurado, porque P(x) é proporcional à frequencia de quintuplas com gazolina. Queremos observar que da formula (1) obtem-se o c.l.m. corréto sòmente na aproximação que o contador E seja atingido por uma única particula do E.P. selecionado pelos telescopios. Damos a P(x) os melhores valores medidos (vide tabela abaixo).

x	P(x)
0	$N_5(0)/N_{4\,0} = 0{,}595 \pm 0{,}028$
5	$N_5(5)/N_{4\,5} = 0{,}612 \pm 0{,}028$
15	$N_5(15)/N_{4\,15} = 0{,}545 \pm 0{,}029$
20	$N_5(20)/N_{4\,20} = 0{,}554 \pm 0{,}027$
∞	$b_5/f_4 = 0{,}422 \pm 0{,}029$

onde $N_5(x)$ e $N_4(x)$ são os numeros de coincidencias quintuplas e quadruplas respectivamente medidas simultaneamente com uma dada espessura de chumbo x (com gazolina). Para $x = \infty$ a probabilidade de $P(\infty)$

será b_5/f_4 na hipotese feita que a contagem de fundo de quintuplas b_5, e que a frequencia total de quadruplas com gazolina f_4 não dependem de x.

$N_5(x)/N_4(x)$ é a frequencia de uma variavel estatística que segue a lei binomial com erro relativo $\sqrt{1/N_5(x) - 1/N_4(x)}$. O erro de B_5/f_4 será calculado porém como erro no quociente de variáveis independentes.

Calcula-se depois com o método dos minimos quadrados ponderados com aproximações sucessivas as constantes a, b, c, (e portanto L) e os seus erros.

Vamos agora mostrar como se pode utilizar as quadruplas para melhorar a precisão com que se pode conhecer as quintuplas, porque em muitos casos isso é de grande interesse. A frequencia de quintuplas $f_5(x)$ pode ser considerada como produto de dois fatores independentes, a probabilidade P(x) e a frequencia total de quadruplas f_4. $N_5(x)/N_4(x)$ é a melhor medida de P(x). Ao produto $N_5(x)/N_4(x).f_4$ chamamos de frequencia de quintuplas corrigida. Notamos que o erro relativo nas quintuplas corrigidas é $\sqrt{1/N_5(x) - 1/N_4(x) + 1/N_4{}^t}$ ($N_4{}^t$ é o numero total de quadruplas medidas), erro esse menor do que o erro nas quintuplas medidas diretamente.

DATA IN POISSON DISTRIBUTIONS

The data are n whole numbers, y, drawn respectively from n distributions of the type

$$e^{-\mu}\mu^{y}/y!$$

where μ ,the mean or expected value of y can be expressed as

$$\mu = h\rho$$

h being a constant-say the number of hours during which the events were counted- and ρ a rate - rate per hour, if h represent hours.

To simplify the notation, we shall not attach suffices 1, 2,...n to y, μ, ρ , etc but will use Σ only to indicate summation over the n observations, etc. Thus Σy will denote the sum of the n observed whole numbers.

The rate, ρ , is a function of k unknown parameters, denoted collectively by θ and individually by $\theta_1, \theta_2, \ldots \theta_k$, and of one or more exactly observable quantities x_1, x_2 etc. Thus, if ρ is the rate per hour of particles counted after passing through x cms of lead, we may wish to assume that

$$\rho = \exp(\alpha + \beta x)$$

Here ρ is a function of two parameters, α and β ,(β is essentially negative) and of one exactly observable quantity x, the thickness of lead.

The problem is to use the data to find efficient estimates of the parameters (i.e. estimates with the highest possible precision).

General Solution

To determine efficient estimates, we may use the method of maximal likelihood. The logarithm of the joint probability of the n observed values, y is

$$L = \Sigma y \log\rho - \Sigma h\rho + \text{terms not containing the parameters}$$

The maximal likelihood estimates, $t(t_1, t_2, \ldots t_k)$ are those which maximise the logarithm of the joint probability and are therefore the roots of k simultaneous equations of the type

$$\partial L/\partial\theta_1 = 0$$

It will be convenient to denote the estimates of θ by the corresponding latin letter t and the values of μ and ρ (when θ is replaced by t) by m and r. We shall also write

$$m_1 = \partial\mu/\partial\theta_1 \quad \text{for} \quad \theta = t$$

etc.

and

$$r_1 = \partial\rho/\partial\theta_1 \quad \text{for} \quad \theta = t$$

etc.

With these conventions of notation, we have to satisfy the k equations of the type

$$\sum y r_1/r - \sum h r_1 = 0$$

Preliminary estimates, t, can be found by any convenient method - often by adjusting a free-hand graph to the data. These preliminary estimates are inserted in the left-hand-sides of the above equations ehich will produce, instead of zero, small quantities $\epsilon_1, \epsilon_2, \ldots \epsilon_k$.

The information matrix is defined as a k x k symmetrical matrix, with terms of the type

$$\sum \mu_1 \mu_2/\mu$$

Substituting $h\rho$ for μ and the preliminary estimates for the parameters, we find the estimated information matrix

$$\{I\} = \begin{Bmatrix} \sum h r_1^2/r & \sum h r_1 r_2/r & \sum h r_1 r_3/r & \cdots \\ \sum h r_1 r_2/r & \sum h r_2^2/r & \sum h r_2 r_3/r & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{Bmatrix}$$

The reciprocal of this matrix,

$$\{V\} = \frac{1}{\{I\}} = \begin{Bmatrix} V_{11} & V_{12} & V_{13} \\ V_{12} & V_{22} & V_{23} \\ V_{13} & V_{23} & V_{33} \end{Bmatrix}$$

supplies estimates of the variances and covariances of the errors of the estimates.

We may now improve on the preliminary estimates by calculating $t_1 + \delta t_1$, $t_2 + \delta t_2$ etc, where

$$\delta t_1 = V_{11}\epsilon_1 + V_{12}\epsilon_2 + \dots + V_{1k}\epsilon_k$$
$$\delta t_2 = V_{12}\epsilon_2 + V_{22}\epsilon_2 + \dots + V_{2k}\epsilon_k$$

etc.

Readers familiar with matrix notation will recognise that these calculations can be expressed more simply as

$$\delta t = \{V\}\epsilon$$

The process may be repeated, starting afresh from the improved estimates, and will converge on the maximal likelihood solutions. It is however unnecessary to apply the process more than once, since the results of a single improvement are already efficient estimates, or, in other words, they diverge from the maximal likelihood estimates by quantities which are negligible in comparison with the standard errors.

Test of Agreement with Hypothesis

Using the improved estimates we calculate the estimated rates, r , and hence the expected numbers, $m = hr$. Then

$$\chi^2 = \sum\{(y-m)^2/m\}$$

with degrees of freedom = $n-k$,

may be used for testing the agreement between observations and hypothesis.

Exponential Rate tending to Zero.

The hypothesis is that

$$\rho = \exp(\alpha + \beta x)$$

Hence

$$\rho_\alpha = \partial\rho/\partial\alpha = \rho$$
$$\rho_\beta = \partial\rho/\partial\beta = x\rho$$

The equations of maximal likelihood are accordingly

$$\Sigma y - \Sigma m = 0$$
$$\Sigma xy - \Sigma xm = 0$$

and the information matrix is

$$\{I\} = \begin{Bmatrix} \Sigma m & \Sigma xm \\ \Sigma xm & \Sigma x^2 m \end{Bmatrix}$$

Example of Rate tending exponentially to Zero

In order to avoid negative values of the exponential we shall take the rate in the form

$$\rho = \frac{\exp(\alpha + \beta x)}{10}$$

The x may be taken as number of cms of lead multiplied by 0,4.

cms.	x	h (hours)	y (number)	y/h	log(10y/h)
0	0	232	242	1,043	2,345
2,5	1	209	152	0,727	1,984
5	2	411	241	0,586	1,768
10	4	240	96	0,400	1,386

$$\Sigma y = 731$$

$$\Sigma xy = 1018$$

From a graph of log(10y/h) against x we obtain the preliminary estimates

$$a = 2,3$$

$$b = -0,25$$

Hence $\quad r = \{\exp(2,3 - 0,25x)\}/10$

x	2,3 - 0,25x	r	h	m (=hr)
0	2,3	0,99742	232	231,40
1	2,05	0,77679	209	162,35
2	1,8	0,60496	411	248,64
4	1,3	0,36693	240	88,063

$$\Sigma m = 730,45$$

$$\Sigma xm = 1011,88 \qquad \Sigma x^2 m = 2565,92$$

The maximal likelihood equations yield

$$\Sigma y - \Sigma m = 731 - 730,45 = +0,55 = \epsilon_a$$

$$\Sigma xy - \Sigma xm = 1018 - 1011,88 = +6,12 = \epsilon_b$$

The estimated information matrix is

$$\begin{Bmatrix} \Sigma m & \Sigma xm \\ \Sigma xm & \Sigma x^2 m \end{Bmatrix} = \begin{Bmatrix} 730,45 & 1011,88 \\ 1011,88 & 2565,92 \end{Bmatrix}$$

The covariance matrix and the discrepancies are

$$\begin{Bmatrix} V_{aa} & V_{ab} \\ V_{ab} & V_{bb} \end{Bmatrix} = \begin{Bmatrix} 0,003017 & -0,001190 \\ -0,001190 & 0,000859 \end{Bmatrix} \qquad \begin{matrix} 0,55 = \epsilon_a \\ 6,12 = \epsilon_b \end{matrix}$$

The adjustments to be made to the preliminary estimates are found by multiplying the columns of the covariance matrix by the column on the right, i.e.

$$\delta a = (0,003017)(0,55) + (-0,001190)(6,12) = -\,0,0056$$

$$(-0,001190)(0,55) + (0,000859)(6,12) = +\,0,0046$$

Hence the improved estimates are

$$a + \delta a = 2,3 - 0,0056 = 2,2944$$

$$b + \delta b = -0,25 + 0,0046 = -0,2454$$

The variance of b is estimated at 0,000859. The estimated standard error of b is accordingly

$$\sqrt{0,00859} = 0,0293$$

Hence $\qquad \beta \;\hat{=}\; -0,2454 \pm 0,0293$

To return to a scale of cms., we multiply by 4/10, giving

$$\beta = -0,0982 \pm 0,0117$$

The limits which enclose the true value with a probability of 95% are

$$\beta = -0,0982 \pm (1,96)(0,0117)$$

$$= -0,1211 \,\ldots\ldots\ldots\, -0,0753$$

The corresponding limits for mean range are

$$-1/\beta = 0,83 \,\ldots\ldots\ldots\ldots\, 1,33$$

Using the improved estimates, we recalculate the rate r, the expected number, m = hr, and finally the χ^2. At the same time we may check that $\sum m = \sum y$.

x	2,2944 -0,2454x	r	h	m (=hr)	y	$\frac{(y-m)^2}{m}$
0	2,2944	0,99183	232	230,10	242	0,615
1	2,0490	0,77601	209	162,19	152	0,640
2	1,8036	0,60714	411	249,53	241	0,292
4	1,3128	0,37166	240	89,20	96	0,518
				731,02	= 731	2,065

Hence χ^2 = 2,065

degrees of freedom = 2

The agreement with the hypothesis is satisfatory.

Exponential Rate not tending to Zero

The hypothesis is that

$$\rho \quad = \quad \exp(\alpha + \beta x) + \gamma$$

It may be noted that this is the formulation when the particles are of two kinds ("production" and "background") when the "production" particles are absorbed by lead according to the exponential law while the rate of "background" particles is not appreciably reduced by the lead. If data are available from a separate experiment in which only "background" particles are observed, the whole data for this experiment are included with x, formally, equal to infinity.

We have

$$\rho_\alpha = \partial\rho/\partial\alpha = \rho$$

$$\rho_\beta = \partial\rho/\partial\beta = x\rho$$

$$\rho_\gamma = \partial\rho/\partial\gamma = 1$$

The maximal likelihood equations are accordingly

$$\Sigma y - \Sigma m = 0$$

$$\Sigma xy - \Sigma xm = 0$$

$$\Sigma y/r - n = 0$$

and the information matrix is

$$\{I\} = \begin{Bmatrix} \Sigma m & \Sigma xm & \Sigma h \\ \Sigma xm & \Sigma x^2 m & \Sigma xh \\ \Sigma h & \Sigma xh & \Sigma h/r \end{Bmatrix}$$

Sum of two Exponentials

Two series of observations are made: in the first only the "background" particles are counted, their true rate being given by

$$\rho \quad = \quad \exp(\alpha + \beta x)$$

In the second, both "background" and "production" particles are counted, their combined rates being $\rho + \rho'$, where

$$\rho' \quad = \quad \exp(\alpha' + \beta' x)$$

The derivatives are

$$\partial\rho/\partial\alpha = \rho$$
$$\partial\rho/\partial\beta = x\rho$$
$$\partial\rho'/\partial\alpha' = \rho'$$
$$\partial\rho'/\partial\beta' = x\rho'$$

The equations of maximal likelihood are therefore

$$\sum \frac{yr}{r+r'} - \sum m = 0$$
$$\sum \frac{xyr}{r+r'} - \sum xm = 0$$
$$\sum \frac{yr'}{r+r'} - \sum m' = 0$$
$$\sum \frac{xyr'}{r+r'} - \sum xm' = 0$$

It is noted that in the first series,

$$\frac{r}{r+r'} = 1$$

while $\frac{r'}{r+r'}$ and m' are both zero.

Hence in the third and fourth equations, summation takes place only over the second series.

The information matrix is

$$\begin{Bmatrix} \sum \frac{mr}{r+r'} & \sum \frac{xmr}{r+r'} & \sum \frac{mr'}{r+r'} & \sum \frac{xmr'}{r+r'} \\ \sum \frac{xmr}{r+r'} & \sum \frac{x^2mr}{r+r'} & \sum \frac{xmr'}{r+r'} & \sum \frac{x^2mr'}{r+r'} \\ \sum \frac{mr'}{r+r'} & \sum \frac{xmr'}{r+r'} & \sum \frac{m'r'}{r+r'} & \sum \frac{xm'r'}{r+r'} \\ \sum \frac{xmr'}{r+r'} & \sum \frac{x^2mr'}{r+r'} & \sum \frac{xm'r'}{r+r'} & \sum \frac{x^2m'r'}{r+r'} \end{Bmatrix}$$

Notation

t = $\dfrac{\text{background rate}}{\text{gasoline rate + background rate}}$

p = proportion of (background) 4-fold & Romers coincidences which are also 5-fold coincidences

tp = $\{\exp(a+bx)\}/10$

q = proportion of (gasoline) 4 fold coincidences which are also 5-fold coincidences.

$(1-t)q$ = $\{\exp(f+gx)\}/10$

u = $tp + (1-t)q$

h = number of hours

z = number of fourfold coincidences

y = number of fivefold coincidences

S = summation over "background only" series

Σ = " " "background + gasoline" series.

Estimation

Estimate of t is given by

$$t = \frac{Sz/Sh}{\Sigma z/\Sigma h}$$

Using this value of t we have the logarithm of likelihood

$$L = S\,y\log p + S(z-y)\log(1-p) + \Sigma\,y\log u + \Sigma(z-y)\log(1-u)$$

The equations of maximal likelihood are

$$\frac{\partial L}{\partial a} = S\frac{y}{1-p} - S\frac{zp}{1-p} + t\left\{\Sigma\frac{y}{1-u}\cdot\frac{p}{u} - \Sigma\frac{zp}{1-u}\right\} = 0$$

$$\frac{\partial L}{\partial b} = S\frac{xy}{1-p} - S\frac{xzp}{1-p} + t\left\{\Sigma\frac{xy}{1-u}\cdot\frac{p}{u} - \Sigma\frac{xzp}{1-u}\right\} = 0$$

$$\frac{\partial L}{\partial f} = (1-t)\left\{\Sigma\frac{y}{1-u}\cdot\frac{q}{u} - \Sigma\frac{zq}{1-u}\right\} = 0$$

$$\frac{\partial L}{\partial g} = (1-t)\left\{\Sigma\frac{xy}{1-u}\cdot\frac{q}{u} - \Sigma\frac{xzq}{1-u}\right\} = 0$$

The Information Matrix is

$$
\begin{bmatrix}
S\frac{zp}{1-p} + t^2\sum\frac{zp}{1-u}\cdot\frac{p}{u} & S\frac{xzp}{1-p} + t^2\sum\frac{xzp}{1-u}\cdot\frac{p}{u} & t(1-t)\sum\frac{zp}{1-u}\cdot\frac{q}{u} & t(1-t)\sum\frac{xzp}{1-u}\cdot\frac{q}{u} \\
 & S\frac{x^2zp}{1-p} + t^2\sum\frac{x^2zp}{1-u}\cdot\frac{p}{u} & t(1-t)\sum\frac{xzp}{1-u}\cdot\frac{q}{u} & t(1-t)\sum\frac{x^2zp}{1-u}\cdot\frac{q}{u} \\
 & & (1-t)^2\sum\frac{zq}{1-u}\cdot\frac{q}{u} & (1-t)^2\sum\frac{xzq}{1-u}\cdot\frac{q}{u} \\
 & & (1-t)^2\sum\frac{xzq}{1-u}\cdot\frac{q}{u} & (1-t)^2\sum\frac{x^2zq}{1-u}\cdot\frac{q}{u}
\end{bmatrix}
$$

<u>Data</u>

Take x as thickness (cms) ÷ 5

	Background			Background plus gasoline		
x	y	z	h	y	z	h
0	162	267	319,91	184	309	237,50
1	123	191	237,08	185	302	252,58
3	88	151	166,66	155	284	230,16
4	76	130	142,58	189	341	287,41
Total		739 ✓	866,23 ✓		1236	1007,65

Background rate $= \frac{739}{866,23} = 0,85312$

Background + gasoline rate $= \frac{1236}{1007,65} = 1,22662$

Hence $t = \frac{0,85312}{1,22662} = 0,69550$

Preliminary estimates of a and b

x	y	z	y/z	log(10y/z)	a+bx	p	tp	
0	162 ~~264~~	267	0,607	1,803	1,85	0,63[illegible]598	0,442	1,8271
1	123	191	0,644	1,863	1,825	0,62029	0,431	1,8125
3	88	151	0,583	1,763	1,775	0,59004	0,410	1,7833
4	76	130	0,585	1,766	1,75	0,575[illegible]	0,400	1,7687

Estimates obtained graphically

$$\begin{cases} a = 1,85 \\ b = -0,025 \end{cases}$$

Preliminary estimates of f and g
Remove 4-fold and 5 fold coincidences attributable to the [illegible] background

x	y	z	tpz	y' = y−tpz	z' = (1−t)z	y'/z'	log(10y'/z')
				gasoline only.			
0	184	309	137	47	94	0,50	1,61
1	185	302	130	55	92	0,60	1,79
3	155	284	116	39	86	0,45	1,60
4	189	341	136	53	104	0,51	1,63

Estimates obtained graphically

$$\begin{cases} f = 1,74 \\ g = -0,05 \end{cases}$$

~~from~~
Computation for Background Data

$x3p$ etc

x	y	z	p	zp	$\frac{1}{1-p}$	$\frac{x}{1-p}$	$\frac{x^2}{1-p}$
0	162	267	0,63598	169,81	2,7471	0	0
1	123	191	0,62029	118,48	2,6336	2,6336	2,6336
3	88	151	0,59004	89,10	2,4393	7,3179	21,9537
4	76	130	0,575[illegible]	74,81	2,3555	9,4220	37,6880
Total	449	739		452,20			

$S\frac{y}{1-p}$ = 1162,64 $\qquad$ $S\frac{xy}{1-p}$ = 1683,98

$S\frac{zp}{1-p}$ = 1172,07 $\qquad$ $S\frac{xzp}{1-p}$ = 1668,91 $\qquad$ ~~$S\frac{x^2zp}{1-p}$~~

− 9,[illegible]43 $\qquad$ + 15,07

~~$S\frac{x^2yp}{1-p}$~~ = $\qquad$ $S\frac{x^2zp}{1-p}$ = 5087,54

Computation for Background plus Gasoline Data

x	y	z	$f+gx$	p	q exp$(f+gx)$	zp	zq	w
0	184	309	1,74	0,63598	0,56973	196,52	176,05	
1	185	302	1,69	0,62029	0,54195	187,33	163,67	
3	155	284	1,59	0,59004	0,49037	167,57	139,27	
4	189	341	1,54	0,57546	0,46646	196,23	159,06	
Total	[illegible]	1236						

473,24 ✓
469,48 $\qquad$ 421,78

414 744

1597,23

Computation for Background plus Gasoline Data

x	y	z	$f+gx$	p	q	u	p/u	q/u
0	184	309	1,74	0,63598	0,56973	0,61581	1,03275	0,92517
1	185	302	1,69	0,62029	0,54195	0,59644	1,03999	~~0,96830~~ 0,90864
3	155	284	1,59	0,59004	0,49037	0,55969	1,05423	0,87615
4	189	341	1,54	0,57546	0,46646	0,54227	1,06121	0,86020
Total		[illegible]						

x	$\frac{y}{1-u}$	$\frac{xy}{1-u}$	$\frac{zp}{1-u}$	$\frac{xzp}{1-u}$	$\frac{x^2zp}{1-u}$	$\frac{zq}{1-u}$	$\frac{xzq}{1-u}$	$\frac{x^2zq}{1-u}$
0	478,93	0	511,52	0	0	458,23	0	0
1	458,42	458,42	464,19	464,19	464,19	405,56	405,56	405,56
3	352,02	1056,06	380,57	1141,71	3425,13	316,29	948,87	2846,61
4	412,91	1651,64	428,70	~~[illegible]~~ 1714,80	6859,20	347,50	1390,00	5560,00
Total	1702,28 ✓	3166,12 ✓	1784,98	3320,70 ✓	10748,52 ✓	1527,58 ✓	2744,43 ✓	8790,17

$\sum \frac{y}{1-u}\cdot\frac{p}{u}$ = 1780,66 $\qquad$ $\sum \frac{xy}{1-u}\cdot\frac{p}{u}$ = 3342,82

$\sum \frac{zp}{1-u}$ = 1784,98 $\qquad$ $\sum \frac{xzp}{1-u}$ = 3320,70

− 4,32 $\qquad$ + 22,12

$\sum \frac{y}{1-u}\cdot\frac{q}{u}$ = 1523,24 ~~1550,59~~ ✓ $\qquad$ $\sum \frac{xy}{1-u}\cdot\frac{q}{u}$ = 2762,55 ~~2789,90~~ ✓

$\sum \frac{zq}{1-u}$ = 1527,58 $\qquad$ $\sum \frac{xzq}{1-u}$ = 2744,43

~~+ 23,01~~ − 4,34 $\qquad$ ~~+ 45,47~~ + 18,12

$$\sum \frac{3p}{1-u}\cdot\frac{p}{u} = 1867,17 \qquad \sum \frac{x3p}{1-u}\cdot\frac{p}{u} = 3506,14 \qquad \sum \frac{x^2 3p}{1-u}\cdot\frac{p}{u} = 11372,68$$

$$\sum \frac{3p}{1-u}\cdot\frac{q}{u} = \overset{1597,23\checkmark}{\cancel{[illegible]}} \qquad \sum \frac{x3p}{1-u}\cdot\frac{q}{u} = \overset{2897,16}{\cancel{2924,86}} \qquad \sum \frac{x^2 3p}{1-u}\cdot\frac{q}{u} = \overset{9322,99}{\cancel{9350,69}}$$

$$\sum \frac{3q}{1-u}\cdot\frac{q}{u} = 1368,49 \qquad \sum \frac{x3q}{1-u}\cdot\frac{q}{u} = 2395,54 \qquad \sum \frac{x^2 3q}{1-u}\cdot\frac{q}{u} = 7626,00$$

$$t^2 = 0,48372 \qquad t(1-t) = 0,21178 \qquad (1-t)^2 = 0,092720$$

Derivatives of Logarithmic Likelihood

$$\partial L/\partial a = -9,43 + (0,69550)(-4,332) = -12,43$$

$$\partial L/\partial b = +15,07 + (0,69550)(22,12) = +30,45$$

$$\partial L/\partial f = (0,30450)(-4,34) = -1,322$$

$$\partial L/\partial g = (0,30450)(18,12) = +5,518$$

Information Matrix

$$\begin{Bmatrix} 2075,3 & 3364,9 & 338,26 & 613,56 \\ 3364,9 & 10588,7 & 613,56 & 1974,42 \\ 338,26 & 613,56 & 126,89 & 222,11 \\ 613,56 & 1974,42 & 222,11 & 707,08 \end{Bmatrix}$$

Covariance Matrix

~~160,3~~

adjustments

$$\begin{Bmatrix} 0,001604 & -0,000527 & -0,004147 & 0,001384 \\ -0,000527 & 0,000370 & 0,001387 & -0,001012 \\ -0,00447 & 0,001387 & 0,001012 & -0,009142 \\ 0,00\overset{1384}{\cancel{44}} & -0,001012 & -0,009142 & 0,005912 \end{Bmatrix} \begin{matrix} -12,43 & -0,0229 \\ +30,45 & +0,0104 \\ -1,322 & +0,0420 \\ +5,518 & -0,0033 \end{matrix}$$

Adjustments to first Estimates

$a = 1,85 - 0,0229 = 1,8271$

$b = -0,025 + 0,0104 = -0,0146$

$f = 1,74 + 0,0420 = 1,7820$

$g = -0,05 - 0,0033 = -0,0533$

Variance of b = 0,000370

Standard error of b = 0,0192

Variance of g = 0,00591

Standard error of g = 0,0769

Limit of g (using 1 standard deviation) $= -0,0533 - 0,0769$

$= -0,1302$

$-1/g = 7,68$ in units of 5 cms

~~multiply~~ $= \cancel{15,3}\ 38,4$ cms.

Hence $L > 38,4$ cms

(using 1,96 s.d) $-0,0533 - 0,1507$

$= -0,2040$

$-1/g = 4,90$ (5 cms)

$L > \cancel{98}\ 24,5$ cms.

χ^2 test

Background

x	y	z	y/z
0	162	267	0,60674
1	123	191	0,643900
2	88	151	0,58278
4	76	130	0,58462
Sum = 449		739	0,607578
			0,392422
			0,23843

$\chi^2 = \dfrac{0,41712}{0,23843}$

$= 1,749$

$\phi = 3$

Background + Gasoline

ξ	y	z	y/z
−2	184	309	0,59547
−1	185	302	0,61258
+1	155	284	0,54577
+2	189	341	0,55425
	713	1236	0,576861
			0,423139
			0,24409

$\chi^2 = \dfrac{0,93949}{0,24409}$

$= 3,849$

$\phi = 3$

$P = 28\%$ approx.

Hence there is no evidence in either series that the lead reduces the proportion of 5-fold coincidences.

Weighted regression

$\Sigma \xi y = -20$ ✓ $\Sigma \xi z = 46$ ✓

$(\Sigma \xi z)(\Sigma \xi^{y}/_{z})/\Sigma z = -0,00556$ $\Sigma \xi \, {}^{y}/_{z} = -0,14925$ ✓

−20.
−19,99444

$\Sigma \xi^2 z = 3186$ ✓

$(\Sigma \xi z)^2/\Sigma z = 1,71$ ✓

3181,29

$\chi^2 = \dfrac{(0,12556)}{0,24409}$

$= 0,5145$

0,514

~~0,00166~~

160,382	−52,734	−414,744	138,363
−52,734	37,045 −14,338 133,68[illegible]	138,658	101,239
−414,744	[illegible]36,269	2823,447	−914,203
138,363	−101,239 −46,494	−914,203	501,226

Uniformity of + food rate

x	n	ratio
267	319,91	0,83461
191	237,08	0,80564
151	166,66	0,90604
130	142,58	0,91177
739	866,23	0,853122

309	[illegible]	1,2015
362	[illegible]	1,19566
224	252,10	1,[illegible]392
341	287,41	1,18646
1236	1007,65	1,226616

140,88

739

$\chi_0^2 =$ [illegible],88

1237,66

1236

$\chi_0^2 =$ 1,00

739
1236
1975

$\frac{1975}{739}$ — t 866,23 — t 1007,65 — 0

m

1975

$\frac{739}{t}$ — m 866,23 — 0

$t \left\{ \frac{1975}{1,8 \quad 739} \right.$ 866,23 — 1007,65

t 1236

1007,65

$t = \frac{0,853122}{1,226616} = 0,69551$

$1-t$ 0,30449

3364	338	613
10588	613	1974
613	126	222

[illegible]	338	613
[illegible]	10013	1974
613	222	707